Available online at www.sciencedirect.com
SCIENCE DIRECT
Topology and its Applications 133 (2003) 69–87
TOPOLOGY AND ITS APPLICATIONS
www.elsevier.com/locate/topol

Cofibrant operads and universal E_∞ operads

R.M. Vogt

Universität Osnabrück, Fachbereich Mathematik/Informatik, Albrechtstr. 28, 49069 Osnabrück, Germany

Received 10 October 2002; received in revised form 14 February 2003

Abstract

We introduce various homotopy structures on the category of operads, which shed some light into the homotopy theoretic nature of the barconstruction $W\mathcal{B}$ of an operad, the whiskering process for operads and the Σ-freeness condition. Using the lifting property of cofibrant objects, we construct E_∞ operads $\mathcal{A}$ which are universal: any E_∞-structure lifts to an $\mathcal{A}$-structure, canonically up to homotopy through $\mathcal{A}$-structures.

1. Introduction

Operads (for a definition see [15]) were originally introduced to study iterated loop space structures [4,23,15,5] (they are already implicit in the work of Stasheff [20]). May in his work combined the operad approach with ideas of Beck [3], such as the use of the functorial twosided bar construction, which made an n-fold delooping in one step possible. The key ingredient is his approximation theorem, which compares the free $\mathcal{C}_n$-algebra $\mathcal{C}_nX$ on a connected space X with $\Omega^n\Sigma^nX$, where $\mathcal{C}_n$ is the little n-cubes operad of [4, Chapter 2, Example 5].

This approach to iterated loop space theory made homotopy invariance considerations redundant, which were in the center of the theory of Boardman and the author. To tackle homotopy invariance we introduced the bar construction $W\mathcal{B}$ for operads $\mathcal{B}$. This construction has been considered a bit mysterious in the past. In recent years it has experienced a revival, implicitly through the works of Ginzburg and Kapranov [9], Getzler and Jones [8], and Batanin [1], who used concepts of trees similar to the one in the W-construction to obtain cotriple resolutions of operads, and explicitly in the works of Markl et al. [14] and others. E.g., if $\mathcal{B}$ is a cellular operad and $C_*(\mathcal{B})$ the operad of its

E-mail address: rainer@mathematik.uni-osnabrueck.de (R.M. Vogt).

0166-8641/$ – see front matter
doi:10.1016/S0166-8641(03)00055-5

cellular chains, then Markl, Shnider, and Stasheff observed a close relationship between $\mathbf{D}(\mathbf{D}(C_*(\mathcal{B})))$ and $C_*(W\mathcal{B})$, where $\mathbf{D}$ is the dual operad construction of Ginzburg and Kapranov [14, p. 129].

In the present paper we readdress the W-construction and show that the augmentation $\varepsilon : W\mathcal{B} \to \mathcal{B}$ can be considered as a cofibrant resolution of the operad $\mathcal{B}$ with respect to a suitable homotopy structure on $\mathcal{O}pr$, the category of operads. The universal property of cofibrant objects then provides explicit examples of universal E_∞ operads.

We start with a recollection of the bar construction $W\mathcal{B}$ and its basic properties in Section 2. We then introduce a number of homotopy structures on $\mathcal{O}pr$ in Section 3. In those the weak equivalences are maps of operads which are genuine homotopy equivalences after forgetting part of the operad structure rather than weak homotopy equivalences. So they differ from the known Quillen model category structures on $\mathcal{O}pr$. Our structures make CW-approximations redundant, which are usually very big and destroy properties such as being quadratic. Apart from explaining the homotopy theoretic nature of the bar construction $W\mathcal{B}$ they shed some light into the homotopy theoretic nature of the whiskering process for operads and the Σ-freeness condition. In the final section we clarify the relationship between the W-construction and the cotriple resolution of operads mentioned above and we address the question of universal E_∞ operads and give examples.

In our early work [4] we used the language of "categories of operators in standard form" (called (topological) PROPs in [5] in reference to work of Mac Lane [11]), which precede operads and are an equivalent notion.

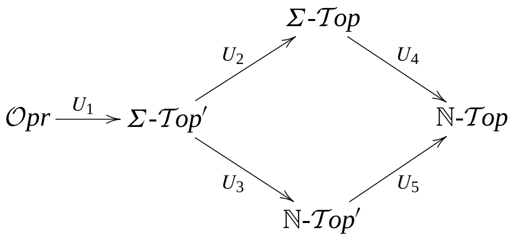
The present paper is an extended version of [25]. Since the latter has been quoted in recent publications I decided to supply the details.

2. The bar construction

The bar construction, also called W-construction, is quite formal and, for example, makes sense in the categories of spaces, simplicial Abelian groups, chain complexes, small categories, and suitable module spectra, but for the sake of an easy presentation we restrict ourselves to operads in the category $\mathcal{T}op$ of k-spaces, i.e., compactly generated spaces in the sense of [24, 5(ii)].

Consider the following diagram of categories and faithful forgetful functors.

2.1.



The objects of $\mathbb{N}$-$\mathcal{T}op$ are collections $X = \{X_n;\ n \in \mathbb{N}\}$ of topological spaces, and the morphisms $f : X \to Y$ are collections of maps $f_n : X_n \to Y_n$ (in accordance with the notation for operads we often write $X(n)$ for X_n). The category $\mathbb{N}$-$\mathcal{T}op'$ is obtained from $\mathbb{N}$-$\mathcal{T}op$ by requiring that X_1 is based and $f_1 : X_1 \to Y_1$ preserves base points. Σ-$\mathcal{T}op$ is obtained form $\mathbb{N}$-$\mathcal{T}op$ and Σ-$\mathcal{T}op'$ from $\mathbb{N}$-$\mathcal{T}op'$ by requiring that the symmetric group Σ_n acts from the right on X_n for all n and that the maps f_n are equivariant. $\mathcal{O}pr$ is the category of operads. All these categories are topologically enriched: we give $\mathbb{N}$-$\mathcal{T}op(X, Y)$ the product topology $\prod \mathcal{T}op(X_n, Y_n)$ and the morphism sets of the other categories the k-subspace topology of this product induced by the faithful forgetful functors. So the forgetful functors are continuous.

2.1. *The operad of grown trees*

A *tree* θ is a finite contractible directed planar graph except that the edges need not have vertices on both ends. Each vertex v has a finite set $\mathrm{In}(v)$ of incoming edges and exactly one outgoing edge. $\mathrm{In}(v) = \emptyset$ is allowed. Hence each tree θ has a finite set $\mathrm{In}(\theta)$ of inputs, i.e., incoming edges with no start vertices, and exactly one output, i.e., edge with no end vertex. We allow the *trivial tree* with no vertex

(directed from top to bottom).

For $X \in \mathbb{N}$-$\mathcal{T}op$ we define the *operad TX of grown trees on X* as follows. An element of $TX(n)$ is a triple (θ, f, g) consisting of a tree θ with $|\mathrm{In}\,\theta| = n$, a function f assigning to each vertex v of θ an element $x \in X_{|\mathrm{In}\,v|}$, and a bijection $g : \mathrm{In}(\theta) \to \underline{n} = \{1, 2, \ldots, n\}$. Here $|M|$ denotes the cardinality of the set M. We interprete g as the permutation which sends i to j, if j is the label of the ith input (we order the inputs from left to right). We give $TX(n)$ the obvious product topology, more precisely the function space topology, induced by the vertex labels.

We usually suppress f and g from the notation and think of an element of $TX(n)$ as a tree with vertices v labelled by $x \in X_{|\mathrm{In}\,v|}$ and inputs labelled by $1, \ldots, n$ according to g. Composition in TX

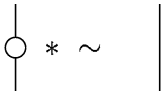
$$TX(n) \times TX(r_1) \times \cdots \times TX(r_n) \to TX(r_1 + \cdots + r_n)$$
$$(\theta; \psi_1, \ldots, \psi_n) \mapsto \varphi$$

is defined as follows: First relabel the input of ψ_i with label $k \in \underline{r_i}$ by $r_1 + \cdots + r_{i-1} + k$, then stick ψ_i with all its (new) labels onto the input of θ with label i.

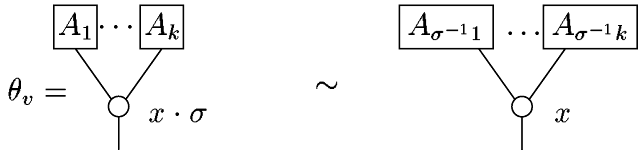
There is a right Σ_n-operation on $TX(n)$ given by $(\theta, f, g) \cdot \sigma = (\theta, f, \sigma^{-1} \circ g)$. It is easy to check that these data make TX an operad.

2.2. Relations. If $X \in \mathbb{N}$-$\mathcal{T}op'$, Σ-$\mathcal{T}op$, or Σ-$\mathcal{T}op'$ we can impose relations on TX:

(1) For $X \in \mathbb{N}$-$\mathcal{T}op'$ or Σ-$\mathcal{T}op'$ with base point $* \in X_1$ the following relation makes sense for subtrees



(2) For $X \in \Sigma\text{-}\mathcal{T}op$ or $\Sigma\text{-}\mathcal{T}op'$ we consider the following relation: Let v be a vertex of a grown tree $\theta \in TX(n)$ and θ_v the subtree (including all labels) consisting of v and all directed paths ending in v. If v has label $x \cdot \sigma$, $\sigma \in \Sigma_k$, then for subtrees



The proof of the following result is straightforward. For details see [5, p. 31ff].

2.3. Theorem. *The following functors are left adjoint to the corresponding forgetful functors*

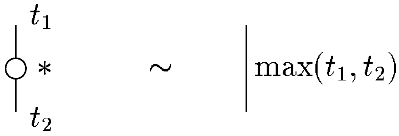
$\mathbb{N}\text{-}\mathcal{T}op \to \mathcal{O}pr$, $\quad X \mapsto TX$,
$\mathbb{N}\text{-}\mathcal{T}op' \to \mathcal{O}pr$, $\quad X \mapsto TX/$*relation* (2.2.1),
$\Sigma\text{-}\mathcal{T}op \to \mathcal{O}pr$, $\quad X \mapsto TX/$*relation* (2.2.2),
$\Sigma\text{-}\mathcal{T}op' \to \mathcal{O}pr$, $\quad X \mapsto TX/$*relations* (2.2.1), (2.2.2).

2.2. The operad of trees

The *operad $\widetilde{T}X$ of trees* is a modified version of TX. An element of $\widetilde{T}X(n)$ is a quadruple (θ, f, g, h) consisting of a grown tree (θ, f, g) and a *length function* $h: Edges\ \theta \to [0,1]$ such that the inputs and the output of θ have lengths 1. As before we suppress f, g, h from the notation. We give $\widetilde{T}X$ the obvious topology defined by the edge lengths and the vertex labels. Composition and the actions of the symmetric groups are defined as in TX; the new edges obtained via composition by sticking trees on inputs get lengths 1. These data define an operad. An element in $\widetilde{T}X$ is a non-trivial composite iff an internal edge has length 1. The operad TX can be identified with the suboperad of $\widetilde{T}X$ of all trees having only edges of lenght 1.

2.4. Relations.

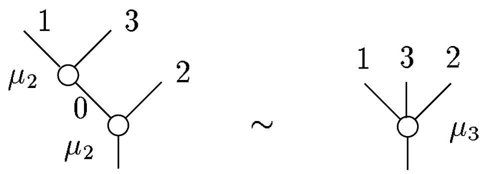
(1) Relation (2.2.1) has to be modified: for $X \in \mathbb{N}\text{-}\mathcal{T}op'$ or $\Sigma\text{-}\mathcal{T}op'$ we consider the relation ($* \in X(1)$ is the base point)



(t_1 and t_2 are the lengths of the edges).

(2) If $\mathcal{X} \in \mathcal{O}pr$, we consider the following relation: An edge of length 0 may be shrunk away by composing its vertices using the composition in $\mathcal{X}$.

2.5. Example. *Let $\mathcal{M}$ be the operad of monoids and $\mu_n \in \mathcal{M}(n)$ the n-fold multiplication. In $\widetilde{T}\mathcal{M}(3)$ we can consider the relation*



2.6. Definition. The *bar construction for operads* also called *W-construction* is the continuous functor

$$W : \mathcal{O}pr \to \mathcal{O}pr,$$
$$\mathcal{B} \mapsto \widetilde{T}\mathcal{B}/ \text{ (relations (2.2.2), (2.4.1), (2.4.2))}.$$

The unit of the adjunction

$$\Sigma\text{-}\mathcal{T}op' \rightleftarrows \mathcal{O}pr$$

extends to a continuous natural map of operads

$$\varepsilon = \varepsilon(\mathcal{B}) : W\mathcal{B} \to \mathcal{B},$$

called *augmentation*, by forgetting the length functions and composing. The counit induces a continuous section of $U_1(\varepsilon)$

$$\eta = \eta(\mathcal{B}) : U_1(\mathcal{B}) \to U_1(W\mathcal{B}),$$

which we call the *standard section* of ε.

If F_1 denotes the left adjoint of U_1, then $F_1U_1(\mathcal{B})$ can be identified with the suboperad of $W\mathcal{B}$ represented by trees having only edges of lenght 1.

2.7. Proposition. $U_1(\varepsilon) : U_1(W\mathcal{B}) \to U_1(\mathcal{B})$ *is a homotopy equivalence in* $\Sigma\text{-}\mathcal{T}op'$ *with homotopy inverse* η.

Proof. The map $h_s : U_1(W\mathcal{B}) \to U_1(W\mathcal{B})$ which replaces the lenght t of an internal edge, i.e., an edge which is neither an input nor the output, by $\max(s, t)$ defines a homotopy from the identity $(s = 0)$ to $\eta \circ U_1(\varepsilon)$ $(s = 1)$. □

3. Homotopy structures

Since $\mathcal{T}op$, the category $\mathcal{T}op^*$ of based k-spaces, and the category of G-spaces, G a discrete group, are complete and cocomplete, so are the categories $\mathcal{C} \neq \mathcal{O}pr$ in diagram 2.1. The same is true for $\mathcal{O}pr$ (we will prove this below), and we know more:

3.1. Proposition. *Each of the topologically enriched categories* $\mathcal{C}$ *of diagram* 2.1 *is topologically complete and cocomplete, i.e., all weighted limits and colimits exist* (*for definitions see* [6, 6.6]). *In particular, it is tensored and cotensored, i.e., there are continuous functors*

$$\mathcal{C} \times \mathcal{T}op \to \mathcal{C}, \qquad (X, K) \mapsto X \otimes K,$$
$$\mathcal{C} \times \mathcal{T}op^{op} \to \mathcal{C}, \qquad (X, K) \mapsto X^K,$$

and natural homeomorphisms

$$\mathcal{C}(X \otimes K, Y) \cong \mathcal{T}op\big(K, \mathcal{C}(X, Y)\big) \cong \mathcal{C}\big(X, Y^K\big).$$

Proof. Let $\mathcal{C} \neq \mathcal{O}pr$. Since $\mathcal{C}$ is complete and cocomplete, it suffices to show that $\mathcal{C}$ is tensored and cotensored [6, 6.6.16]. The cotensor X^K is the collection of function spaces $\{\mathcal{T}op(K, X_n);\ n \in \mathbb{N}\}$ in each case with the obvious action of Σ_n on $\mathcal{T}op(K, X_n)$ if $\mathcal{C} = \Sigma\text{-}\mathcal{T}op$ or $\Sigma\text{-}\mathcal{T}op'$, and the null map as base point of $\mathcal{T}op(K, X_1)$ if $\mathcal{C} = \Sigma\text{-}\mathcal{T}op'$ or $\Sigma\text{-}\mathcal{T}op'$.

For $\mathcal{C} = \mathbb{N}\text{-}\mathcal{T}op$ or $\Sigma\text{-}\mathcal{T}op$, the tensor $X \otimes K$ is the collection $\{X_n \times K;\ n \in \mathbb{N}\}$ with the trivial action on K if $\mathcal{C} = \Sigma\text{-}\mathcal{T}op$. If $\mathcal{C} = \mathbb{N}\text{-}\mathcal{T}op'$ or $\Sigma\text{-}\mathcal{T}op'$, the tensor is the collection of $X_n \times K$ for $n \neq 1$ and $X_1 \wedge (K_+)$ for $n = 1$, where $K_+ = K \cup \{*\}$ with base point $*$.

To prove the statement for $\mathcal{O}pr$ we apply [7, VII, 2.10]. We consider the continuous adjunction

$$T : \mathbb{N}\text{-}\mathcal{T}op \rightleftarrows \mathcal{O}pr : U$$

$\mathcal{O}pr$ is the category of algebras of the continuous monad $U \circ T$ on $\mathbb{N}\text{-}\mathcal{T}op$. By the enriched version of [12, VI.2, Ex. 2] the functor U creates all weighted limits. In particular, $\mathcal{O}pr$ is complete.

For the existence of weighted colimits it suffices to show that $U \circ T$ preserves reflexive coequalizers, i.e., coequalizers

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{h} Z$$

for which there is a morphism $t : Y \to X$ such that $f \circ t = g \circ t = \mathrm{id}_Y$ [7, VII, 2.10]. Being a left adjoint T preserves coequalizers. So it remains show that U preserves reflexive coequalizers. We show that U creates reflexive coequalizers, which is enough. Given maps f, g and t of operads, we form the coequalizer $h : Y \to Z$ in $\mathbb{N}\text{-}\mathcal{T}op$ and claim that it is the coequalizer in $\mathcal{O}pr$. We define composition in Z by

$$Z_k \times Z_{i_1} \times \cdots \times Z_{i_k} \to Z_{i_1+\cdots+i_k},$$
$$\big([y], [y_1], \ldots, [y_k]\big) \mapsto \big[y \circ (y_1 \oplus \cdots \oplus y_k)\big],$$

where $[y]$ is the element in Z represented by $y \in Y$. Since $Z_k \times Z_{i_1} \times \cdots \times Z_{i_n}$ is a quotient of $Y(k) \times Y(i_1) \times \cdots \times Y(i_k)$, it suffices to show that this map is well-defined. For $x \in X(k)$ we have to prove that

$$\big[f(x) \circ (y_1 \oplus \cdots \oplus y_k)\big] = \big[g(x) \circ (y_1 \oplus \cdots \oplus y_k)\big],$$

the argument for the other factors is the same.

$$\begin{aligned}\bigl[f(x)\circ(y_1\oplus\cdots\oplus y_k)\bigr] &= \bigl[f(x)\circ\bigl(f\circ t(y_1)\oplus\cdots\oplus f\circ t(y_k)\bigr)\bigr]\\ &= \bigl[f\bigl(x\circ\bigl(t(y_1)\oplus\cdots\oplus t(y_k)\bigr)\bigr)\bigr]\\ &= \bigl[g\bigl(x\circ\bigl(t(y_1)\oplus\cdots\oplus t(y_k)\bigr)\bigr)\bigr]\\ &= \bigl[g(x)\circ(y_1\oplus\cdots\oplus y_k)\bigr].\end{aligned}$$

Since composition in Z is defined by composing representatives, it follows that $h: Y\to Z$ is a coequalizer in $\mathcal{O}pr$. □

This proposition provides the categories with canonical cylinder functors $-\otimes I$ and path space functors $(-)^I$. Hence we have the notions of homotopy, cofibrations, fibrations and homotopy equivalences. The natural homeomorphisms of 3.1 imply, that the homotopy relation defined using cylinders coincides with the one defined using path objects, and that homotopy means homotopy through morphisms in the category in the usual sense.

3.2. Lemma.

(1) *Closed cofibrations, fibrations, and homotopy equivalences define a proper closed model structure in Quillen's sense* [18] *on Σ-$\mathcal{T}op$ and $\mathbb{N}$-$\mathcal{T}op$.*
(2) *For each of the categories of diagram* 2.1, *cofibrations and homotopy equivalences define a cofibration structure in the sense of Definition* 3.3 *below. Dually, fibrations and homotopy equivalences define a fibration structure. Moreover, all objects are fibrant and cofibrant.*

(1) follows from [22, Theorem 3] and its equivariant version, (2) is standard elementary homotopy theory.

3.3. Definition. A *cofibration category* is a category $\mathcal{C}$ with an initial object $\emptyset$ and two subcategories $\mathbf{cof}_{\mathcal{C}}$ and $\mathbf{we}_{\mathcal{C}}$, whose morphisms are called *cofibrations* and *weak equivalences* respectively. Morphisms in $\mathbf{cof}_{\mathcal{C}}\cap\mathbf{we}_{\mathcal{C}}$ are called *trivial cofibrations*. An object A is called *cofibrant*, if $\emptyset\to A$ is a cofibration, and *fibrant*, if each trivial cofibration $A\to X$ has a retraction. The following axioms hold:

(C1) Given $A\xrightarrow{f}B\xrightarrow{g}C$, if two of $f, g, g\circ f$ are in $\mathbf{we}_{\mathcal{C}}$, so is the third. Isomorphisms are trivial cofibrations.
(C2) Pushouts along cofibrations i exist.

$$\begin{array}{ccc} A & \xrightarrow{\;i\;} & B\\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar f}\\ X & \xrightarrow{\;\bar i\;} & X\cup_A B\end{array}$$

If i is a (trivial) cofibration, so is $\bar i$.
(C3) Every map factors into a cofibration followed by a weak equivalence.

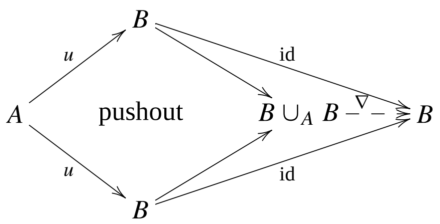
(C4) Any object X has a fibrant resolution RX, i.e., there is a trivial cofibration $e_X : X \to RX$ with RX fibrant.

We call $\mathcal{C}$ *proper*, if the following additional axiom holds.

(P) In the pushout diagram of (C2), if i is a cofibration and f a weak equivalence, then $\bar{f}$ is a weak equivalence.

3.4. Remark. Proper cofibration categories are studied extensively in [2], where they are simply called cofibration categories. Our present definition without Axiom (P) is due to Majewski [13].

Let $u : A \to B$ be a cofibration in a cofibration category. We form the pushout



We factor ∇ into a cofibration followed by a weak equivalence

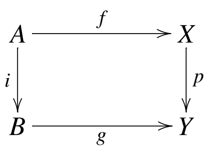
$$B \cup_A B \xrightarrow{i} C_A B \xrightarrow{p} B$$

and call the triple $(C_A B, i, p)$ a *relative cylinder of* B rel A. This construction gives rise to an *internal homotopy relation* rel A between maps $B \to X$ under A.

The proofs of the following two results in [2] do not use Axiom (P) and hence hold for our notion of cofibration category.

3.5. Proposition. *If $u : A \to B$ is a cofibration and X is fibrant, then all cylinders* rel A *define the same homotopy relation* rel A *on the set of morphisms $B \to X$ under A. Moreover, this homotopy relation is an equivalence relation* [2, II.2.2].

3.6. Lifting Lemma. *Let $\mathcal{C}$ be a cofibration category and*



a commutative diagram in $\mathcal{C}$ with p a weak equivalence between fibrant objects and i a cofibration. Then there exists a morphism $h : B \to X$ uniquely up to homotopy rel A, *such that $h \circ i = f$ and $p \circ h \simeq g$* rel A [2, II.1.1].

On the categories of diagram 2.1 we now have an internal homotopy relation rel A arising from the cofibration category structures of Lemma 3.2(2) and the usual one arising from the cylinder functor. We show that the two agree:

3.7. Proposition. *Let $u : A \to B$ be a cofibration in any of the categories of diagram* 2.1. *Then the pushout*

$$\begin{array}{ccc} A \otimes I & \xrightarrow{u \otimes I} & B \otimes I \\ \downarrow & & \downarrow \\ A \otimes * = A & \longrightarrow & C_A B \end{array}$$

with the natural maps $B \cup_A B \to C_A B \to B$ is a cylinder of B rel *A in the internal homotopy structure.*

Proof. By [19] $u \otimes I$ and $(B \cup B) \cup_{(A \cup A)} A \otimes I \cong B \otimes S^0 \cup_{A \otimes S^0} A \otimes I \to B \otimes I$ are cofibrations. Hence the induced map $B \cup_A B \to C_A B$ is a cofibration and the induced map $C_A B \to B$ a homotopy equivalence by [2, II.1.2]. $\square$

For the remainder of the section let $\mathcal{C}$ and $\mathcal{D}$ be two categories of diagram 2.1 linked by a forgetful functor (we allow $\mathrm{Id}_{\mathcal{C}}$)

$$U : \mathcal{C} \to \mathcal{D}.$$

Adopting the terminology of relative homological algebra we define

3.8. Definition. A morphism f in $\mathcal{C}$ is called

(1) a *$\mathcal{D}$-fibration* respectively a *$\mathcal{D}$-equivalence*, if $U(f)$ is a fibration respectively a homotopy equivalence in $\mathcal{D}$,
(2) a *trivial $\mathcal{D}$-fibration*, if it is a $\mathcal{D}$-fibration and a $\mathcal{D}$-equivalence,
(3) a *$\mathcal{D}$-cofibration*, if it has the left lifting property (LLP) for all trivial $\mathcal{D}$-fibrations, and *trivial cofibration*, if it is a $\mathcal{D}$-cofibration and a $\mathcal{D}$-equivalence.

3.9. Warning. Not all cofibrations in $\mathcal{C}$ are $\mathcal{C}$-cofibrations. If $\mathcal{C} = \mathbb{N}\text{-}\mathcal{T}op$ or $\Sigma\text{-}\mathcal{T}op$ the closed cofibrations are precisely the $\mathcal{C}$-cofibrations. If $\mathcal{C} = \mathbb{N}\text{-}\mathcal{T}op'$ or $\Sigma\text{-}\mathcal{T}op'$, closed cofibrations of *well-pointed* objects are $\mathcal{C}$-cofibrations, but there might be more. (Recall that a space is *well-pointed* if the inclusion of the base point is a closed cofibration. A collection $X = \{X_n;\ n \in \mathbb{N}\}$ will be called *well-pointed* if X_1 is well-pointed.)

In each category $\mathcal{C}$ of diagram 2.1 the objects are $\mathcal{C}$-cofibrant, because trivial $\mathcal{C}$-fibrations in $\mathcal{C}$ have sections.

Since $\mathcal{D}$-cofibrations are defined by a LLP, we obtain

3.10. Lemma. *The class of $\mathcal{D}$-cofibrations in $\mathcal{C}$ is closed under pushouts, arbitrary sums, sequential colimits, and retracts in the category of morphisms.*

3.11. Lemma. *Let $F : \mathcal{D} \to \mathcal{C}$ be left adjoint to U and let $V : \mathcal{D} \to \mathcal{E}$ be another forgetful functor of diagram* 2.1. *Then*

(1) *U and F preserve the homotopy relation and hence homotopy equivalences.*

(2) *Every fibration in $\mathcal{C}$ is a $\mathcal{D}$-fibration, every $\mathcal{D}$-cofibration in $\mathcal{C}$ is a cofibration.*
(3) *If f is an $\mathcal{E}$-cofibration in $\mathcal{D}$, then $F(f)$ is an $\mathcal{E}$-cofibration in $\mathcal{C}$.*

Proof. Since (F, U) is an enriched adjoint pair, F preserves tensors, hence cylinders and the homotopy relation, and U preserves cotensors, hence path objects and the homotopy relation. Passage to adjoints shows that U preserves fibrations. Cofibrations are precisely those morphisms which have the LLP for all morphisms $Z^{i_0} : Z^I \to Z$, induced by the inclusion $i_0 : \{0\} \to I$. Since Z^{i_0} is a trivial $\mathcal{D}$-fibration, each $\mathcal{D}$-cofibration is a cofibration. Passage to adjoints implies statement (3). □

3.12. Lemma. *Let $i : K \to L$ be a closed cofibration in $\mathcal{T}op$ and $j : A \to B$ a $\mathcal{D}$-cofibration in $\mathcal{C}$. Then*

$$(j, i) : A \otimes L \cup_{A\otimes K} B \otimes K \to B \otimes L$$

is a $\mathcal{D}$-cofibration in $\mathcal{C}$.

Proof. First let $U \neq \mathrm{Id}_{\mathcal{O}pr}$. If $p : X \to Y$ is a trivial $\mathcal{D}$-fibration, then so is

$$p^i : X^L \to Y^L \times_{Y^K} X^K$$

by the k-space version of [21, Theorem 10] and its equivariant analogue, because U preserves cotensors and limits. Hence the adjoint diagram of

$$\begin{array}{ccc} A \otimes L \cup_{A\otimes K} B \otimes K & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ B \otimes L & \longrightarrow & Y \end{array}$$

has a filler $\hat{h} : B \to X^L$ whose adjoint $H : B \otimes L \to X$ is the required filler of the given diagram.

If $U = \mathrm{Id}_{\mathcal{O}pr}$ we replace [21, Theorem 10] in the argument by [19, Corollary 2.8 and Add. 3.6]. □

3.13. Relative Lifting Lemma. *Given a commutative diagram in $\mathcal{C}$*

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & Y \end{array}$$

with j a $\mathcal{D}$-cofibration and p a $\mathcal{D}$-equivalence, then there exists a morphism $h : B \to X$ uniquely up to homotopy rel A, *such that $h \circ j = f$ and $p \circ h \simeq g$* rel A.

Proof. Using the mapping path space $P(p)$ of p we factor p

$$p : X \xrightarrow{s} P(p) \xrightarrow{r} Y$$

into a homotopy equivalence s and a fibration r. Observe that s admits a retraction $q : P(p) \to X$ such that $q \circ s = \mathrm{id}_X$ and $s \circ q \simeq \mathrm{id}_{P(p)} \operatorname{rel} X$. Since s is also a $\mathcal{D}$-equivalence and r a $\mathcal{D}$-fibration, the latter is a trivial $\mathcal{D}$-fibration. So there is a morphism $k : B \to P(p)$ such that $r \circ k = g$ and $k \circ j = s \circ f$. The morphism $h = q \circ k : B \to X$ satisfies $h \circ j = q \circ s \circ f = f$ and $p \circ h = r \circ s \circ q \circ k \simeq r \circ k = g \operatorname{rel} A$. Suppose h' is a second such h, consider the diagram

$$\begin{array}{ccc} A \otimes I \cup_{A \otimes \partial I} B \otimes \partial I & \xrightarrow{F} & X \\ \downarrow{\scriptstyle (j,i)} & & \downarrow{\scriptstyle p} \\ B \otimes I & \xrightarrow{G} & Y \end{array}$$

where G is composed of the two homotopies $p \circ h \simeq g \simeq p \circ h' \operatorname{rel} A$ and F is defined by the constant homotopy on f and the morphisms h and h'. Since (j, i) is a $\mathcal{D}$-cofibration by 3.12, the above argument gives a filler $H : B \otimes I \to X$, which is a homotopy rel A from h to h'. □

3.14. Corollary. *If $j : A \to B$ is a $\mathcal{D}$-cofibration and a $\mathcal{D}$-equivalence there is a retraction $r : B \to A$ such that $r \circ j = \mathrm{id}_A$ and $j \circ r \simeq \mathrm{id}_B \operatorname{rel} A$. In particular, all objects are $\mathcal{D}$-fibrant in the sense of* 3.3.

3.15. Corollary. *If f is a $\mathcal{D}$-cofibration and $\mathcal{D}$-equivalence, then any pushout of f is so.*

Proof. Let $\overline{f}$ be a pushout of f. By 3.10 it remains to show that $\overline{f}$ is a $\mathcal{D}$-equivalence. By 3.11 and 3.14, f is a cofibration and homotopy equivalence. Since $\mathcal{C}$ with cofibrations and homotopy equivalences is a cofibration category (see 3.2), $\overline{f}$ is a homotopy equivalence, hence a $\mathcal{D}$-equivalence. □

3.16. Proposition.

(1) *If $\mathcal{C} \neq \mathcal{O}pr$ and $\mathcal{D} = \Sigma$-$\mathcal{T}op$ or $\mathbb{N}$-$\mathcal{T}op$, then ($\mathcal{C}$, $\mathcal{D}$-cofibrations, $\mathcal{D}$-equivalences) is a cofibration category with all objects fibrant.*
(2) *If $\mathcal{C} \neq \mathcal{O}pr$ and $\mathcal{D} = \mathbb{N}$-$\mathcal{T}op'$ or Σ-$\mathcal{T}op'$, the same holds for the full subcategories of well-pointed objects* (*recall the definition from* 3.9).

Proof. So far we have verified all axioms except of (C3). So let us consider a morphism

$$f : M \to X$$

in $\mathcal{C}$. The pair (X, f) is an object in the under category $M/\mathcal{C}$, and we have a forgetful functor

$$U_M : M/\mathcal{C} \to \mathcal{C} \to \mathcal{D}, \qquad (X, f) \mapsto U(X)$$

with a left adjoint

$$F_M : \mathcal{D} \to \mathcal{C} \to M/\mathcal{C}, \qquad Y \mapsto M \cup F(Y).$$

Let $T_M = U_M \circ F_M$ denote the associated monad on $\mathcal{D}$. The Godement resolution of (X, f) is the map of simplicial objects in $M/\mathcal{C}$

$$\varepsilon : B_\bullet(X, f) \to (X, f)_\bullet$$

where $(X, f)_\bullet$ is the constant simplicial object and

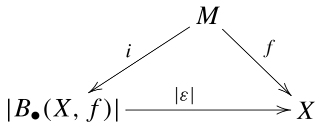
$$B_n(X, f) = F_M \circ T_M^n \circ U_M(X, f).$$

The simplicial structure maps and the simplicial map ε are induced by the adjunction maps of the pair (F_M, U_M). Moreover, $U_M(\varepsilon)$ has a natural section

$$\eta : U_M(X, f)_\bullet \to U_M B_\bullet(X, f),$$

and there is a simplicial homotopy $\eta \circ U_M(\varepsilon) \simeq \mathrm{id}$.

We take the usual topological realization and obtain a candidate for the factorization axiom



If $\mathcal{C}$ is one of the equivariant cases, we have an induced Σ_k-action on the kth space $|B_\bullet(X, f)|(k) = |B_\bullet(X, f)(k)|$ of the collection $|B_\bullet(X, f)|$. In the based cases $|B_\bullet(X, f)|(1)$ has a natural base point given by the base point in $B_0(X, f)(1) = (M \cup FUX)(1)$.

We have $U_M(|B_\bullet(X, f)|) = |U_M(B_\bullet(X, f))|$, because the realization is formed in $\mathbb{N}$-$\mathcal{T}op$. Since the realization commutes with products, $U_M|\varepsilon|$ is a homotopy equivalence in $\mathcal{D}$. Hence $|\varepsilon|$, considered as morphism in $\mathcal{C}$, is a $\mathcal{D}$-equivalence.

Let $|B_\bullet(X, f)|^{(n)}$ denote the n-skeleton of $|B_\bullet(X, f)|$. The canonical morphism

$$M \to \left|B_\bullet(X, f)\right|^{(0)} = B_0(X, f) = M \cup FUX$$

is a $\mathcal{C}$-cofibration by 3.10 and 3.11, because UX is $\mathcal{C}$-cofibrant.

It remains to show that $|B_\bullet(X, f)|^{(n-1)} \to |B_\bullet(X, f)|^{(n)}$ is a $\mathcal{D}$-cofibration.

Let $i : sB_n(X, f) \to B_n(X, f)$ denote the subobject of degenerate elements. Then $|B_\bullet(X, f)|^{(n)}$ is obtained from $|B_\bullet(X, f)|^{(n-1)}$ by attaching $B_n(X, f) \times \Delta^n$ along $sB_n(X, f) \times \Delta^n \cup B_n(X, f) \times \partial\Delta^n$ in $\mathbb{N}$-$\mathcal{T}op$, where Δ^n is the standard n-simplex. In view of 3.12 it suffices to show that i is a $\mathcal{D}$-cofibration. Each degeneracy s_i is of the form $F_M(s_i')$ with $s_i' : T_M^{n-1} \circ U_M(X, f) \to T_M^n \circ U_M(X, f)$. Let $j : \bigcup T_M^{n-1} \circ U_M(X, f) \to T_M^n \circ U_M(X, f)$ be the subobject defined by the s_i', so that $i = F_M(j)$. Since these subobjects are maps whose domains are iterated pushouts and F_M preserves pushouts, it suffices to show that j is a $\mathcal{D}$-cofibration in $\mathcal{D}$.

Each s_i' is a closed cofibration, and, by Lillig's union theorem for cofibrations [10] and its equivariant analogue [5, App. 2.7], j is a closed cofibration and hence a $\mathcal{D}$-cofibration in $\mathcal{D}$ if $\mathcal{D}$ is Σ-$\mathcal{T}op$ or $\mathbb{N}$-$\mathcal{T}op$ by [22, Proposition 1] and its equivariant version. If $\mathcal{D} = \Sigma$-$\mathcal{T}op'$ or $\mathbb{N}$-$\mathcal{T}op'$ the same argument applies to the spaces in all grades except of grade 1. Direct inspection shows that

$$M(1) \to \left|B_\bullet(X, f)\right|(1) \to X(1)$$

is the reduced mapping cylinder construction if $U = \mathrm{Id}_{\Sigma\text{-}\mathcal{T}op'}$ or U_3 and the unreduced one in the other cases with base point from $M(1)$, if a base point is required. If $|B_\bullet(X, f)|(1)$ is the unreduced mapping cylinder, then $M(1) \to |B_\bullet(X, f)|(1)$ is $\mathcal{D}$-cofibrant by [22, Proposition 1]. The same is true for the reduced mapping cylinder by [22, Proposition 9], provided M and X are well-pointed. □

If $\mathcal{C} = \mathcal{O}pr$, our result is not quite as nice as Proposition 3.16, because we do not know whether the pushout of well-pointed operads along a $\mathcal{D}$-cofibration is well-pointed. But our result is good enough for all practical purposes.

3.17. Proposition. *Let $f : M \to X$ be a morphism in $\mathcal{O}pr$ and M be well-pointed.*

(1) *If $\mathcal{D} = \mathbb{N}\text{-}\mathcal{T}op$ or $\Sigma\text{-}\mathcal{T}op$, then f factors into a $\mathcal{D}$-cofibration followed by a $\mathcal{D}$-equivalence.*
(2) *If $\mathcal{D} = \mathbb{N}\text{-}\mathcal{T}op'$ or $\Sigma\text{-}\mathcal{T}op'$ and X is well-pointed, the same holds.*

Proof. We consider the internal realization in $\mathcal{O}pr$

$$\left|B_\bullet(X, f)\right|_{\mathcal{O}pr} = \bigcup_{n \geqslant 0} B_n(X, f) \otimes \Delta^n/\sim$$

with the usual relations. By the argument of [17, 4.4] the internal realization coincides with the usual one so that $|B_\bullet(X, f)|_{\mathcal{O}pr} \cong |B_\bullet(X, f)|$. We now apply the argument of the proof of 3.16 to this internal realization. In particular,

$$M \to \left|B_\bullet(X, f)\right|^{(0)}_{\mathcal{O}pr} = M \cup FUX$$

is a $\mathcal{D}$-cofibration.

To ensure that each s_i' is a $\mathcal{D}$-cofibration in $\mathcal{D}$ we need to know that $Y \to U_M(M \cup FY)$ is a $\mathcal{D}$-cofibration for $Y \in \mathcal{D}$, and that T_M preserves $\mathcal{D}$-cofibrations.

If $\mathcal{D}$ is unbased, $Y \to UFY$ is a closed cofibration and hence a $\mathcal{D}$-cofibration. In the based cases induction over the number of internal edges in the tree description of FY shows that the same is true provided Y is well-pointed (relation 2.2.1 makes this extra condition necessary). Moreover, the induction also shows that UFY is well-pointed. Since M is well-pointed, the inclusion $UFY \to U(M \cup FY)$ is a $\mathcal{D}$-cofibration. This follows by a similar induction using the tree description of a sum of operads (e.g., see [5, (2.15)(i),(ii),(iii)]). Again we find that $U(M \cup FY)$ is well-pointed.

Finally, given a $\mathcal{D}$-cofibration $B \subset Y$ in $\mathcal{D}$ (of well-pointed objects if $\mathcal{D}$ is based) and a well-pointed operad M, induction over the number of vertices which are not in B in the tree descriptions shows that

$$U(M \cup FB) \to U(M \cup FY)$$

is a $\mathcal{D}$-cofibration.

We now proceed as in the proof of 3.16 using Lillig's union theorem and the observation that any cofibration is also a based cofibration. □

3.18. Corollary. ($\mathcal{O}pr$, $\mathcal{D}$*-cofibrations*, $\mathcal{D}$*-equivalences*) *satisfies all axioms of a cofibration category except of possibly the factorization axiom* (C3), *which is replaced by Proposition* 3.17. *All operads are* $\mathcal{D}$*-fibrant.*

3.19. Definition. A $\mathcal{D}$*-cofibrant resolution* of X in $\mathcal{C}$ is a $\mathcal{D}$-cofibrant object QX together with a $\mathcal{D}$-equivalence $\varepsilon_X : QX \to X$.

Since $QX := |B_\bullet(X, \emptyset \to X)| \to X$ is a $\mathcal{D}$-cofibrant resolution, we get

3.20. Corollary. *Given a forgetful functor* $U : \mathcal{C} \to \mathcal{D}$ *of diagram* 2.1, *then*

(1) *if* $\mathcal{C} \neq \mathcal{O}pr$, *there is a functorial* $\mathcal{D}$*-cofibrant resolution* $\varepsilon_X : QX \to X$ *for each* X *in* $\mathcal{C}$,
(2) *if* $\mathcal{C} = \mathcal{O}pr$ *and* $\mathcal{D} = \mathbb{N}\text{-}\mathcal{T}op$ *or* $\Sigma\text{-}\mathcal{T}op$, *each operad has a functorial* $\mathcal{D}$*-cofibrant resolution,*
(3) *if* $\mathcal{C} = \mathcal{O}pr$ *and* $\mathcal{D} = \mathbb{N}\text{-}\mathcal{T}op'$ *or* $\Sigma\text{-}\mathcal{T}op'$ *each well-pointed operad has a functorial* $\mathcal{D}$*-cofibrant resolution.*

An inspection of $QX = |B_\bullet(X, \emptyset \to X)|$ shows

3.21.

(0) *If* $U = \mathrm{Id}_{\mathcal{C}}$, *then* $QX = X$.
(1) *If* $U = U_2$ *or* U_5, *then* $(QX)_n = X_n$ *for* $n \neq 1$ *and* $(QX)_1 = X_I$, *the mapping cylinder* $(I \cup X_1)/\!\sim$ *of* $\{*\} \to X_1$ *with* $1 \sim *$ *and base point* $0 \in I$.
(2) *If* $U = U_3$ *or* U_4, *then* $(QX)_n = X_n$ *for* $n = 0, 1$ *and* $(QX)_n = B(X, \Sigma_n, \Sigma_n)$, *the two sided barconstruction, for* $n \geqslant 2$. *Recall that there is a* Σ_n*-equivariant homeomorphism* $B(X_n, \Sigma_n, \Sigma_n) \cong E\Sigma_n \times X_n$ *with diagonal* Σ_n*-action on* $E\Sigma_n \times X_n$.
(3) *For* $U : \Sigma\text{-}\mathcal{T}op' \to \mathbb{N}\text{-}\mathcal{T}op$ *the* $\mathcal{D}$*-cofibrant resolution* QX *is a combination of* (1) *and* (2).
(4) *If* $U = U_1$ *and* $\mathcal{B}$ *is a well-pointed operad, then* $Q\mathcal{B}$ *is the cotriple resolution of* $\mathcal{B}$ *associated with the adjoint pair* (F_1, U_1), *which we mentioned in the introduction* (*e.g., see* [1, p. 88]).

3.22. Remark. We now have various notions of homotopy in $\mathcal{C}$. Let $C_\emptyset A$ be a cylinder object of A with respect to the $\mathcal{D}$-structure (in $\mathcal{O}pr$ we have to assume that A is well-pointed to ensure the existence of $C_\emptyset A$). The Relative Lifting Lemma applied to

$$\begin{array}{ccc} A \cup A & \longrightarrow & A \otimes I \\ \downarrow & & \downarrow \\ C_\emptyset A & \longrightarrow & A \end{array}$$

shows that homotopic morphisms are $\mathcal{D}$-homotopic. Hence the standard homotopy relation in $\mathcal{C}$ is finer than the $\mathcal{D}$- homotopy relation.

4. Universal E_∞ operads

4.1. Theorem. *Let* $U : \Sigma\text{-}\mathcal{T}op' \to \mathcal{D}$ *be a forgetful functor of diagram* 2.1. *Let* $\mathcal{B}$ *be a well-pointed operad such that* $U_1(\mathcal{B})$ *is* $\mathcal{D}$*-cofibrant in* $\Sigma\text{-}\mathcal{T}op'$. *Then*

$$\varepsilon(\mathcal{B}) : W\mathcal{B} \to \mathcal{B}$$

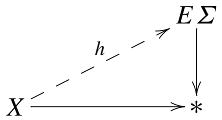
is a $\mathcal{D}$*-cofibrant resolution of* $\mathcal{B}$. *In particular,* $W\mathcal{B}$ *is homotopy equivalent in* $\mathcal{O}pr$ *to the cotriple resolution* $Q\mathcal{B}$ *of* $\mathcal{B}$ 3.21(4).

Before we prove the theorem let us characterize $\mathcal{D}$-cofibrant objects in the categories $\mathcal{C} \neq \mathcal{O}pr$.

4.2. Proposition.

(1) *An object* X *in* $\Sigma\text{-}\mathcal{T}op'$ *or in* $\mathbb{N}\text{-}\mathcal{T}op'$ *is* $(\Sigma\text{-}\mathcal{T}op)$*-respectively* $(\mathbb{N}\text{-}\mathcal{T}op)$*-cofibrant iff* X_1 *is well-pointed.*
(2) *An object* X *in* $\Sigma\text{-}\mathcal{T}op$ *is* $(\mathbb{N}\text{-}\mathcal{T}op)$*-cofibrant iff* X_n *is a numerable principal* Σ_n*-space for* $n \geqslant 2$.
(3) X *in* $\Sigma\text{-}\mathcal{T}op'$ *is* $(\mathbb{N}\text{-}\mathcal{T}op)$*-cofibrant iff* X_1 *is well-pointed and* X_n *is a numerable principal* Σ_n*-space for* $n \geqslant 2$. *Such* X *are also* $(\mathbb{N}\text{-}\mathcal{T}op')$*-cofibrant.*

Proof. (1) follows from [22, Proposition 1]. (3) is a consequence of (1) and (2). So let X be in $\Sigma\text{-}\mathcal{T}op$. Recall that X_n is a numerable principal Σ_n-space iff there is an equivariant classifying map $X_n \to E\Sigma_n$. Let $E\Sigma$ denote the collection $\{E\Sigma_n;\ n \in \mathbb{N}\}$ with $E\Sigma_1 = E\Sigma_0 = *$, and let X be $(\mathbb{N}\text{-}\mathcal{T}op)$-cofibrant. Since each $E\Sigma_n \to *$ is a trivial fibration in $\mathcal{T}op$, there is a lift h



producing classifying maps $h_n : X_n \to E\Sigma_n$. Conversely, classifying maps h_n define a section (h, id)

$$X \xrightarrow{(h,\mathrm{id})} E\Sigma \times X \to X$$

of the projection. Hence X is $(\mathbb{N}\text{-}\mathcal{T}op)$-cofibrant, being a retract of $E\Sigma \times X$, which is $(\mathbb{N}\text{-}\mathcal{T}op)$-cofibrant by 3.21. $\square$

Proof of 4.1. Define the r-skeleton $W^r\mathcal{B}$ of $W\mathcal{B}$ to be the suboperad generated by those elements which can be represented by trees with at most r internal edges. Then $W^0\mathcal{B} = F_1 \circ U_1(\mathcal{B})$, where F_1 is left adjoint to U_1. By 3.11 $W^0\mathcal{B}$ is $\mathcal{D}$-cofibrant. Since $W\mathcal{B} = \operatorname{colim}_r W^r\mathcal{B}$, it remains to show that $W^{r-1}\mathcal{B} \subset W^r\mathcal{B}$ is a $\mathcal{D}$-cofibration.

Let λ be an abstract planar tree with r internal edges and n inputs as described in Section 2. The space M_λ of elements in $\widetilde{T}U_1(\mathcal{B})$ with underlying tree λ is of the form

$$M_\lambda \cong I^r \times \prod_j \mathcal{B}(n_j)^{m_j} \times \Sigma_n$$

if λ has m_j vertices with n_j inputs. Here I^r codifies the lengths of the internal edges, $\prod_j \mathcal{B}(n_j)^{m_j}$ codifies the vertex labels and Σ_n the input labels.

Let Λ be the set of all trees which can be obtained from λ by iterated application of relation 2.2.2. We call Λ the *shape orbit* of λ. We have a group G_Λ acting on $M_\Lambda := \bigcup_{\lambda\in\Lambda} M_\lambda$, given as follows: Σ_r permutes the coordinates of I^r, Σ_{m_j} and $(\Sigma_{n_j})^{m_j}$ act on $\mathcal{B}(n_j)^{m_j}$ by permuting factors respectively by the right action of Σ_{n_j} on $\mathcal{B}(n_j)$, Σ_n acts on Σ_n by composition on the right. Let G_λ denote the subgroup of G_Λ generated by all $g \in G_\Lambda$ which map M_λ into itself and for which the labelled trees A and $g(A)$ are related by a single application of relation 2.2.2.

A labelled tree $A \in M_\lambda$ represents an element in $W^{r-1}\mathcal{B}$ iff

(1) some vertex is an identity (relation 2.4.1 applies),
(2) some internal edge has length 0 (relation 2.4.2 applies),
(3) some internal edge has length 1 (then A decomposes into smaller trees).

The subspace $N_\lambda \subset M_\lambda$ consisting of all labelled trees satisfying one of these conditions is G_λ-invariant. Note that the orbit spaces N_λ/G_λ and M_λ/G_λ have right Σ_n-actions, defined by (see Section 2)

$$[\theta, f, g, h]\cdot \pi = \big[\theta, f, \pi^{-1}\circ g, h\big].$$

We consider N_λ/G_λ and M_λ/G_λ as objects in $\Sigma\text{-}\mathcal{T}op'$, consisting of the base point in grade 1 and the spaces N_λ/G_λ respectively M_λ/G_λ in grade n, all other grades being empty. By construction, $W^r\mathcal{B}$ may be identified with the following pushout in $\mathcal{O}pr$

$$\begin{array}{ccc} \coprod_\lambda F_1(N_\lambda/G_\lambda) & \longrightarrow & \coprod_\lambda F_1(M_\lambda/G_\lambda) \\ \downarrow & & \downarrow \\ W^{r-1}\mathcal{B} & \longrightarrow & W^r\mathcal{B} \end{array}$$

where λ runs through a complete set of representatives of shape orbits of trees with r internal edges.

By 3.10 and 3.11 we have to show that $N_\lambda/G_\lambda \to M_\lambda/G_\lambda$ is a $\mathcal{D}$-cofibration.

To combine the G_λ-action with the Σ_n-action we decompose:

$$M_\lambda = \bigcup_{\sigma\in\Sigma_n} P_{\lambda,\sigma}, \quad \text{where } P_{\lambda,\sigma} \cong I^r \times \prod_j \mathcal{B}(n_j)^{m_j} \times \sigma.$$

An element $g \in G_\lambda$ maps $P_{\lambda,\sigma}$ to $P_{\lambda,\tau}$ with $\tau = \sigma \circ p(g^{-1})$, where $p : G_\lambda \to \Sigma_n$ is the homomorphism sending g to its left action on the input labels. Put $P_\lambda = P_{\lambda,\mathrm{id}}$ and $Q_\lambda = N_\lambda \cap P_\lambda$. Define a G_λ-action on P_λ by

$$G_\lambda \times P_\lambda \to M_\lambda \to P_\lambda$$

where the first map is the restriction of the G_λ-action on M_λ and the second is induced by the homeomorphisms $P_{\lambda,\sigma} \cong P_\lambda$ which forget the input labels. In particular $A \in P_\lambda$ and $g(A)\cdot p(g^{-1}) \in P_{\lambda,p(g^{-1})}$ are related by 2.2.2.

Let $q : X \to Y$ be a trivial U-fibration in $\Sigma\text{-}\mathcal{T}op'$. Consider a commutative diagram

$$\begin{array}{ccc} N_\lambda/G_\lambda & \xrightarrow{\bar{u}} & X_n \\ \downarrow & \overset{\bar{h}}{\nearrow} & \downarrow{\scriptstyle q} \\ M_\lambda/G_\lambda & \xrightarrow{\bar{v}} & Y_n \end{array}$$

Define a G_λ-action on X_n by $g \cdot x = x \cdot p(g^{-1})$ and similarly on Y_n. The diagram induces a G_λ-equivariant commutative square

$$\begin{array}{ccc} Q_\lambda & \xrightarrow{u} & X_n \\ {\scriptstyle j}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle q} \\ P_\lambda & \xrightarrow{v} & Y_n \end{array}$$

It suffices to construct a G_λ-equivariant filler h to obtain the required Σ_n-equivariant filler $\bar{h}$.

If $U = U_2$, the filler h exists by the equivariant version of [22, Proposition 1], because $Q_\lambda \to P_\lambda$ is a closed G_λ-equivariant cofibration (see [5, App. 2]).

If $U = U_3$ or $U_5 \circ U_3$, then each $\mathcal{B}(k)$ is a numerable principal Σ_k-space by assumption, and we observe that P_λ and Q_λ are numerable principal G_λ-spaces. In this case q is an ordinary trivial fibration. The based cases do not cause problems because Q_λ and P_λ are well-pointed.

Let $w : P_\lambda \to EG_\lambda$ be a classifying map. We obtain a G_λ-equivariant commutative diagram

$$\begin{array}{ccc} Q_\lambda & \xrightarrow{(w\circ j, u)} & EG_\lambda \times X_n \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle (\mathrm{id}, q)} \\ P_\lambda & \xrightarrow{(w, v)} & EG_\lambda \times Y_n \end{array}$$

But $(\mathrm{id}, q) : EG_\lambda \times X_n \to EG_\lambda \times Y_n$ is a trivial fibration in the category of G_λ-spaces. Hence the last diagram has a filler.

This completes the proof of the first part of the theorem.

Since $\mathcal{B}$ is well-pointed, both $W\mathcal{B}$ and $Q\mathcal{B}$ are $\Sigma\text{-}\mathcal{T}op'$-cofibrant resolutions of $\mathcal{B}$. Hence they are homotopy equivalent by the Relative Lifting Lemma. □

Our results allow the construction of universal E_∞ operads.

4.3. Definition. An operad $\mathcal{B}$ is called an E_∞ *operad* if the unique morphism $\mathcal{B} \to \mathcal{C}om$ into the operad of commutative monoids is an $(\mathbb{N}\text{-}\mathcal{T}op)$-equivalence, i.e., if each space $\mathcal{B}(n)$ is contractible.

An E_∞ operad $\mathcal{B}$ is called *universal* if for any E_∞ operad $\mathcal{C}$ there is a map of operads $\mathcal{B} \to \mathcal{C}$, i.e., any $\mathcal{C}$-structure can be pulled back to a $\mathcal{B}$-structure.

Observe that our notion of an E_∞ operad differs from the one in [15] in so far as we do not require Σ-freeness. In particular, $\mathcal{C}om$ is E_∞.

4.4. Proposition. *Let $\mathcal{B}$ be an E_∞ operad and let $\mathcal{Q} \to \mathcal{C}om$ be an $(\mathbb{N}\text{-}\mathcal{T}op)$-cofibrant resolution of $\mathcal{C}om$. Then there is a functor of operads $\mathcal{Q} \to \mathcal{B}$ uniquely up to homotopy in $\mathcal{O}pr$ which makes*

$$\begin{array}{ccc} \mathcal{Q} & \longrightarrow & \mathcal{B} \\ & \searrow \quad \swarrow & \\ & \mathcal{C}om & \end{array}$$

commute. In particular, $\mathcal{Q}$ is universal. Any two such resolutions $\mathcal{Q}$ are homotopy equivalent in $\mathcal{O}pr$.

Proof. Apply the Relative Lifting Lemma. □

We know how to construct such resolutions. Starting with any operad $\mathcal{B}$ we first whisker $\mathcal{B}(1)$ as in 3.21(1) to obtain an operad $\mathcal{B}'$ such that $U_3 \circ U_1(\mathcal{B}')$ is an $(\mathbb{N}\text{-}\mathcal{T}op)$-cofibrant resolution of $U_3 \circ U_1(\mathcal{B})$ (cf. [4, p. 1120]). The composition in $\mathcal{B}'$ is the one in $\mathcal{B}$ for elements in $\mathcal{B}$, and for the new elements $t \in I$ we define

$$\begin{aligned} & f \circ t = f && \text{if } f \notin I, \\ & f \circ (f_1 \oplus \cdots \oplus t \oplus \cdots \oplus f_n) && \\ & \quad = f \circ (f_1 \oplus \cdots \oplus \mathrm{id} \oplus \cdots \oplus f_n) && \text{if } f \notin I, \\ & t \circ f = f \circ t = \max(f, t) && \text{if } f \in I. \end{aligned}$$

In a second step we replace a well-pointed operad $\mathcal{B}$ (such as $\mathcal{B}'$) by the operad $\overline{\mathcal{B}} = \mathcal{B} \times \Gamma$, where Γ is the topological realization of the Barratt–Eccles operad, i.e., $\Gamma(n) = E\Sigma_n$ (see [16, §4] for an explicit description). The projection $\overline{\mathcal{B}} \to \mathcal{B}$ is an $(\mathbb{N}\text{-}\mathcal{T}op)$-cofibrant resolution of $U_1(\mathcal{B})$.

Finally, from Theorem 4.1, we obtain

4.5. Proposition.

(1) *If $\mathcal{B}$ is any operad, then $W\overline{\mathcal{B}'}$ is an $(\mathbb{N}\text{-}\mathcal{T}op)$-cofibrant resolution of $\mathcal{B}$.*
(2) *If $\mathcal{B}$ is a well-pointed operad, then $W\overline{\mathcal{B}}$ is an $(\mathbb{N}\text{-}\mathcal{T}op)$-cofibrant resolution of $\mathcal{B}$.*
(3) *If $\mathcal{B}$ is a well-pointed operad such that each $\mathcal{B}(n)$ is a numerable principal Σ_n-space, then $W\mathcal{B}$ is an $(\mathbb{N}\text{-}\mathcal{T}op)$-cofibrant resolution of $\mathcal{B}$.*

4.6. Examples of universal E_∞-operads.

(1) *Let Γ be the Barratt–Eccles operad. Then $\Gamma = \overline{\mathcal{C}om}$ and $W\Gamma$ is universal.*
(2) *Let $\mathcal{Q}_\infty$ be the infinite little cubes operad of* [4]. *$\mathcal{Q}_\infty$ is well-pointed* (*see the proof of* [5, (2.50)]). *Each space $\mathcal{Q}_\infty(n)$ is a numerable principal Σ_n-space, because there is a Σ_n-equivariant map to the configuration space $F(\mathbb{R}^\infty, n)$ which is a numerable principal Σ_n-space. Hence $W\mathcal{Q}_\infty$ is universal.*
(3) *Let $\mathcal{L}$ be the linear isometry operad of* [4]. *$\mathcal{L}$ is well-pointed and each space $\mathcal{L}(n)$ is a numerable principal Σ_n-space* (*this follows from* [7, *Proposition* 1.4 *and Lemma* 1.7, p. 199]). *Hence $W\mathcal{L}$ is universal.*

Acknowledgements

I want to thank C. Berger, Z. Fiedorowicz and the referee for helpful suggestions concerning the presentation and the Deutsche Forschungsgemeinschaft for support during the preparation of the paper.

References

[1] M.A. Batanin, Homotopy coherent category theory and A_∞-structures in monoidal categories, J. Pure Appl. Algebra 123 (1998) 67–103.
[2] H.J. Baues, Algebraic Homotopy, Cambridge University Press, Cambridge, 1989.
[3] J. Beck, On H-spaces and infinite loop spaces, in: Lecture Notes in Math., Vol. 99, Springer-Verlag, Berlin, 1969, pp. 139–153.
[4] J.M. Boardman, R.M. Vogt, Homotopy-everything H-spaces, Bull. Amer. Math. Soc. 74 (1968) 1117–1122.
[5] J.M. Boardman, R.M. Vogt, Homotopy Invariant Structures on Topological Spaces, in: Lecture Notes in Math., Vol. 347, Springer-Verlag, Berlin, 1973.
[6] F. Borceux, Handbook of Categorial Algebra, Vol. II, in: Encyclopedia of Mathematics and Its Applications, Cambridge University Press, Cambridge, 1994.
[7] A.D. Elmendorf, I. Kriz, M.A. Mandell, J.P. May, Rings, Modules, and Algebras in Stable Homotopy Theory, in: Amer. Math. Soc. Surveys and Monographs, Vol. 47, American Mathematical Society, Providence, RI, 1997.
[8] E. Getzler, J.D.S. Jones, Operads, homotopy algebra, and iterated integrals for double loop spaces, Preprint.
[9] V. Ginzburg, M.M. Kapranov, Koszul duality for operads, Duke Math. J. 76 (1994) 202–273.
[10] J. Lillig, A union theorem for cofibrations, Arch. Math. 24 (1973) 410–415.
[11] S. Mac Lane, Categorical algebra, Bull. Amer. Math. Soc. 71 (1965) 40–106.
[12] S. Mac Lane, Categories for the Working Mathematician, Springer-Verlag, Berlin, 1971.
[13] M. Majewski, Tame homotopy theory via polynomial forms I, Diplomarbeit, FU Berlin, 1988.
[14] M. Markl, S. Shnider, J. Stasheff, Operads in Algebra, Topology and Physics, in: Amer. Math. Soc. Surveys and Monographs, Vol. 96, American Mathematical Society, Providence, RI, 2002.
[15] J.P. May, The Geometry of Iterated Loop Spaces, in: Lecture Notes in Math., Vol. 171, Springer-Verlag, Berlin, 1972.
[16] J.P. May, E_∞ spaces, group completion, and permutative categories, Lecture Notes London Math. Soc. 11 (1974) 61–92.
[17] J.E. McClure, R. Schwänzl, R.M. Vogt, $THH(R) \cong R \otimes S^1$ for E_∞ ring spectra, J. Pure Appl. Algebra 140 (1999) 23–32.
[18] D.G. Quillen, Homotopical Algebra, in: Lecture Notes in Math., Vol. 43, Springer-Verlag, Berlin, 1967.
[19] R. Schwänzl, R.M. Vogt, Strong cofibrations and fibrations in enriched categories, Arch. Math. 79 (2002) 449–462.
[20] J.D. Stasheff, Homotopy associativity of H-spaces I, II, Trans. Amer. Math. Soc. 108 (1963) 275–312.
[21] A. Strøm, Note on cofibrations II, Math. Scand. 22 (1968) 130–142.
[22] A. Strøm, The homotopy category is a homotopy category, Arch. Math. 23 (1972) 435–441.
[23] R.M. Vogt, Categories of operators and H-spaces, Thesis, University of Warwick, 1968.
[24] R.M. Vogt, Convenient categories of topological spaces for homotopy theory, Arch. Math. 22 (1971) 545–555.
[25] R.M. Vogt, Cofibrant operads and universal E_∞ operads, Preprint E99-005, 81–89, http://www.mathematik.uni-bielefeld.de/preprints/index99.html.