PREFACE

The origin of simplicial homotopy theory coincides with the beginning of algebraic topology almost a century ago. The thread of ideas started with the work of Poincaré and continued to the middle part of the 20th century in the form of combinatorial topology. The modern period began with the introduction of the notion of complete semi-simplicial complex, or simplicial set, by Eilenberg-Zilber in 1950, and evolved into a full blown homotopy theory in the work of Kan, beginning in the 1950s, and later Quillen in the 1960s.

The theory has always been one of simplices and their incidence relations, along with methods for constructing maps and homotopies of maps within these constraints. As such, the methods and ideas are algebraic and combinatorial and, despite the deep connection with the homotopy theory of topological spaces, exist completely outside any topological context. This point of view was effectively introduced by Kan, and later encoded by Quillen in the notion of a closed model category. Simplicial homotopy theory, and more generally the homotopy theories associated to closed model categories, can then be interpreted as a purely algebraic enterprise, which has had substantial applications throughout homological algebra, algebraic geometry, number theory and algebraic K-theory. The point is that homotopy is more than the standard variational principle from topology and analysis: homotopy theories are everywhere, along with functorial methods of relating them.

This book is, however, not quite so cosmological in scope. The theory has broad applications in many areas, but it has always been quite a sharp tool within ordinary homotopy theory — it is one of the fundamental sources of positive, qualitative and structural theorems in algebraic topology. We have concentrated on giving a modern account of the basic theory here, in a form that could serve as a model for corresponding results in other areas.

This book is intended to fill an obvious and expanding gap in the literature. The last major expository pieces in this area, namely [33], [67], [61] and [18], are all more than twenty-five years old. Furthermore, none of them take into account Quillen's ideas about closed model structures, which are now part of the foundations of the subject.

We have attempted to present an account that is as linear as possible and inclusive within reason. We begin in Chapter I with elementary definitions and examples of simplicial sets and the simplicial set category **S**, classifying objects, Kan complexes and fibrations, and then proceed quickly through much of the classical theory to proofs of the fundamental organizing theorems of the subject which appear in Section 11. These theorems assert that the category of simplicial sets satisfies Quillen's axioms for a closed model category, and that the associated homotopy category is equivalent to that arising from topological spaces. They are delicate but central results, and are the basis for all that follows.

Chapter I contains the definition of a closed model category. The foundations of abstract homotopy theory, as given by Quillen, start to appear in the first section of Chapter II. The "simplicial model structure" that most of the closed model structures appearing in nature exhibit is discussed in Sections 2-7. A simplicial model structure is an enrichment of the underlying category to simplicial sets which interacts with the closed model structure, like function spaces do for simplicial sets; the category of simplicial sets with function spaces is a standard example. Simplicial model categories have a singular technical advantage which is used repeatedly, in that weak equivalences can be detected in the associated homotopy category (Section 4). There is a detection calculus for simplicial model structures which leads to homotopy theories for various algebraic and diagram theoretic settings: this is given in Sections 5-7, and includes a discussion of cofibrantly generated closed model categories in Section 6 — it may be heavy going for the novice, but homotopy theories of diagrams almost characterize work in this area over the past ten years, and are deeply implicated in much current research. The chapter closes on a much more elementary note with a description of Quillen's non-abelian derived functor theory in Section 8, and a description of proper closed model categories, homotopy cartesian diagrams and glueing and coglueing lemmas in Section 9. All subsequent chapters depend on Chapters I and II.

Chapter III is a further repository of things that are used later, although perhaps not quite so pervasively. The fundamental groupoid is defined in Chapter I and then revisited here in Section III.1. Various equivalent formulations are presented, and the resulting theory is powerful enough to show, for example, that the fundamental groupoid of the classifying space of a small category is equivalent to the free groupoid on the category, and give a quick proof of the Van Kampen theorem. The closed model structure for simplicial abelian groups and the Dold-Kan correspondence relating simplicial abelian groups to chain complexes (ie. they're effectively the same thing) are the subject of Section 2. These ideas are the basis of most applications of simplicial homotopy theory and of closed model categories in homological algebra. Section 3 contains a proof of the Hurewicz theorem: Moore-Postnikov towers are introduced here in a self-contained way, and then treated more formally in Chapter VII. Kan's Ex^{∞}-functor is a natural, combinatorial way of replacing a simplicial set up to weak equivalence by a Kan complex: we give updated proofs of its main properties in Section 4, involving some of the ideas from Section 1. The last section presents the Kan suspension, which appears later in Chapter V in connection with the loop group construction.

Chapter IV discusses the homotopy theory, or more properly homotopy theories, for bisimplicial sets and bisimplicial abelian groups, with major applications. Basic examples and constructions, including homotopy colimits and the diagonal complex, appear in the first section. Bisimplicial abelian groups, the subject of Section 2, are effectively bicomplexes, and hence have canonical associated spectral sequences. One of the central technical results is the generalized Eilenberg-Zilber theorem, which asserts that the diagonal and total complexes of a bisimplicial abelian group are chain homotopy equivalent. Three different closed model structures for bisimplicial sets, all of which talk about the same homotopy theory, are discussed in Section 3. They are all important, and in fact used simultaneously in the proof of the Bousfield-Friedlander theorem in Section 4, which gives a much used technical criterion for detecting fibre sequences arising from maps of bisimplicial sets. There is a small technical innovation in this proof, in that the so-called π_*-Kan condition is formulated in terms of certain fibred group objects being Kan fibrations. The chapter closes in Section 4 with proofs of Quillen's "Theorem B" and the group completion theorem. These results are detection principles for fibre sequences and homology fibre sequences arising from homotopy colimits, and are fundamental for algebraic K-theory and stable homotopy theory.

From the beginning, we take the point of view that simplicial sets are usually best viewed as set-valued contravariant functors defined on a category $\mathbf{\Delta}$ of ordinal numbers. This immediately leads, for example, to an easily manipulated notion of simplicial objects in a category $\mathcal{C}$: they're just functors $\mathbf{\Delta}^{op} \to \mathcal{C}$, so that morphisms between them become natural transformations, and so on. Chapter II contains a detailed treatment of the question of when the category $s\mathcal{C}$ of simplicial objects in $\mathcal{C}$ has a simplicial model structure.

Simplicial groups is one such category, and is the subject of Chapter V. We establish, in Sections 5 and 6, the classical equivalence of homotopy theories between simplicial groups and simplicial sets having one vertex, from a modern perspective. The method can the be souped up to give the Dwyer-Kan equivalence between the homotopy theories of simplicial groupoids and simplicial sets in Section 7. The techniques involve a new description of principal G-fibrations, for simplicial groups G, as cofibrant objects in a closed model structure on the category of G-spaces, or simplicial sets with G-action (Section 2). Then the classifying space for G is the quotient by the G-action of any cofibrant model of a point in the category of G-spaces (Section 3); the classical $\overline{W}G$ construction is an example, but the proof is a bit interesting. We give a new treatment of $\overline{W}G$ as a simplicial object of universal cocycles in Section 4; one advantage of this method is that there is a completely analogous construction for simplicial groupoids, which is used for the results of Section 7. Our approach also depends on a specific closed model structure for simplicial sets with one vertex, which is given in Section 6. That same section contains a definition and proof of the main properties of the Milnor FK-construction, which is a functor taking values in simplicial groups that gives a model for loops suspension $\Omega\Sigma X$ of a given space X.

The first section of Chapter V contains a discussion of skeleta in the category of simplicial groups which is later used to show the technical (and necessary) result that the Kan loop group functor outputs cofibrant simplicial groups. Skeleta for simplicial sets first appear in a rather quick and dirty way in Section I.2. Skeleta for more general categories appear in various places: we have skeleta for simplicial groups in Chapter V, skeleta for bisimplicial sets in Section IV.3, and then skeleta for simplicial objects in more general categories later, in Section VII.1. In all cases, skeleta and coskeleta are left and right adjoints of truncation functors.

Chapters VI and VII contain material on towers of fibrations, nilpotent spaces, cosimplicial spaces and associated spectral sequences that was introduced by Bousfield and Kan in [14].

The first section of Chapter VI describes a simplicial model structure for towers, which is used in Section 2 as a context for a formal discussion of Postnikov towers. The Moore-Postnikov tower, in particular, is a tower of fibrations that is functorially associated to a space X; we show, in Sections 3-5, that the fibrations appearing in the tower are homotopy pullbacks along maps, or k-invariants, taking values in homotopy colimits of diagrams of Eilenberg-Mac Lane spaces, which diagrams are functors defined on the fundamental groupoid of X. The homotopy pullbacks can be easily refined if the space is nilpotent, as is done in Section 6. The development includes an introduction of the notion of covering system of a connected space X, which is a functor defined on the fundamental groupoid and takes values in spaces homotopy equivalent to the covering space of X.

Chapter VII contains a detailed treatment of the well known homotopy spectral sequence of a cosimplicial space. The method is to show that the category of simplicial objects in the category $\mathbf{S}^{op}$ has a Reedy model structure, along with an adequate notion of skeleta and an appropriate analogue of realization (Sections 1-3), and then reverse all the arrows to give the usual tower of fibrations from [14] in Section 5, after a general discussion of the model category structure of cosimplicial spaces in Section 4. The standard method of extracting a homotopy spectral sequence from a tower of fibrations (and the associated convergence difficulty) is explained in Section 6 and then deployed in Section 7 to produce the homotopy spectral sequence of a cosimplicial space — Bousfield-Kan p-completions appear in the examples, along with the spectral sequence for a homotopy inverse limit.

The homotopy spectral sequence of a cosimplicial space is well known to be "fringed" in the sense that the objects that appear along the diagonal in total degree 0 are sets rather than groups. Standard homological techniques therefore fail, and there can be substantial difficulty in analyzing the path components of the total space. Bousfield has created an obstruction theory to attack this problem. We give here, in Section VII.8, a special case of this theory, which deals with the question of when elements in bidegree $(0,0)$ in the E_2-term lift to path components of the total space. This particular result can be used to give a criterion for maps between mod p cohomology objects in the category of unstable algebras over the Steenrod algebra to lift to maps of p-completions.

Simplicial model structures return with a vengeance in Chapter VIII, in the context of homotopy coherence. The point of view that we take is that a homotopy coherent diagram on a catgeory I in simplicial sets is a functor $X : \mathcal{A} \to \mathbf{S}$ which is defined on a category enriched in simplicial sets and preserves the enriched structure, subject to the object $\mathcal{A}$ being a resolution of I in a suitable sense. The main results are due to Dwyer and Kan: there is a simplicial model structure on the category of simplicial functors $\mathbf{S}^{\mathcal{A}}$ (Section 1), and a large class of simplicial functors $f : \mathcal{A} \to \mathcal{B}$ which are weak equivalences induce equivalences of the homotopy categories associated to $\mathbf{S}^{\mathcal{A}}$ and $\mathbf{S}^{\mathcal{B}}$ (Section 2). Among such weak equivalences are resolutions $\mathcal{A} \to I$ — in practice, I is the category of path components of $\mathcal{A}$ and each component of $\mathcal{A}$ is contractible. A realization of a homotopy coherent diagram $X : \mathcal{A} \to \mathbf{S}$ is then nothing but a diagram $Y : I \to \mathbf{S}$ which represents X under the equivalence of homotopy categories. This approach subsumes the standard homotopy coherence phenomena, which are discussed in Section 3. We show how to promote some of these ideas to notions of homotopy coherent diagrams and realizations of same in more general simplicial model categories, including chain complexes and spectra, in the last section.

Frequently, one wants to take a given space and produce a member of a class of spaces for which homology isomorphisms are homotopy equivalences, without perturbing the homology. If the homology theory is mod p homology, the p-completion works in many but not all examples. Bousfield's mod p homology localization technique just works, for all spaces. The original approach to homology localization [8] appeared in the mid 1970's, and has since been incorporated into a more general theory of f-localization. The latter means that one constructs a minimal closed model structure in which a given map f becomes invertible in the homotopy category — in the case of homology localization the map f would be a disjoint union of maps of finite complexes which are homology isomorphisms. The theory of f-localization and the ideas underlying it are broadly applicable, and are still undergoing frequent revision in the literature. We present one of the recent versions of the theory here, in Sections 1-4 of Chapter IX. The methods of proof involve little more than aggressive cardinal counts (the cogniscenti will note that there is no mention of regular cardinals): this is where the applicability of the ideas comes from — morally, if cardinality counts are available in a model category, then it admits a theory of localization.

Some applications and related ideas are presented in the final two sections of Chapter IX. We formally introduce homotopy inverse limits in Section 5 and show that they can be recovered as inverse limits of fibrant objects within appropriate (localized) closed model structures for diagram categories, and then show that such closed model structures themselves admit f-localizations. In Section 6 we describe Bousfield's approach to localizing at a functor, and then show that it leads to the Bousfield-Friedlander model for the stable category. We also show that the stable category can be derived as a type of f-localization.

There are nine chapters in all; we use roman numerals to distinguish them. Each chapter is divided into sections, plus an introduction. Results and equations are numbered consecutively within each section. The overall referencing system for the monograph is perhaps best illustrated with an example: there is a Lemma 9.8 living in Section 9 of Chapter II — it is referred to precisely this way from within Chapter II, and as Lemma II.9.8 from outside. Similarly, the corresponding section is called Section 9 inside Chapter II and Section II.9 from without.

Despite the length of this tome, much important material has been left out: there is not a word about traditional simplicial complexes and the vast modern literature related to them (trees, Tits buildings, Quillen's work on posets); the Waldhausen subdivision is not mentioned; we don't discuss the Haussman-Husemoller theory of acyclic spaces or Quillen's plus construction; we have avoided all of the subtle aspects of categorical coherence theory, and there is very little about simplicial sheaves and presheaves. All of these topics, however, are readily available in the literature — we have tried to include a useful bibiliography.

This book should be accessible to mathematicians in the second year of graduate school or beyond, and is intended to be of interest to the research worker who wants to apply simplicial techniques, for whatever reason. We believe that it will be a useful introduction both to the theory and the current literature.

The sensitive reader may already have correctly observed that this monograph does not have the structure of a traditional text book. We have, for example, declined to assign homework in the form of exercises, preferring instead to liberally sprinkle the text with examples and remarks that are designed to provoke further thought. Everything here depends on the first two chapters; the remaining material often reflects the original nature of the project, which amounted to separately written self contained tracts on defined topics. The book achieved its current more unified state thanks to a drive to achieve consistent notation and referencing. It nevertheless remains true that somebody with a bit of experience in the area should be able to read each of the later chapters in isolation, and find an essentially complete story.

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