Chapter IX Localization

Localization is more a way of life than any one specific result. For example, under this rubric one can include Bousfield localization with respect to a homology theory, localization with respect to a map as pioneered by Bousfield, Dror-Farjoun and elaborated on by many others, and even the formation of the stable homotopy category. We will touch on all three of these subjects, but we also have another purpose. There is a body of extremely useful techniques that we will explore and expand on. These have come to be known as Bousfield factorization, which is a kind of "trivial cofibration-fibration" factorization necessary for producing localizations, and the Bousfield-Smith cardinality argument. This latter technique arises when one is confronted with a situation where a fibration is defined to be a map which has the right lifting property with respect to some class of maps. However, for certain arguments one needs to know it is sufficient to check that the map has the right lifting property with respect to a *set* of maps. We explain both Bousfield factorization and the cardinality argument and explore the implications in a variety of contexts. In particular, we explore localization in diagram categories, with an eye towards producing a model for the stable homotopy category, and we produce a simplicial model category structure on categories of diagrams that will be useful in a later discussion of homotopy inverse limits.

The concept of localization probably has its roots in the notion of a Serre class of abelian groups and the Whitehead Theorem mod a Serre class [86, §9.6]. This result is still useful and prevalent — so prevalent, in fact, that it is often used without reference. The idea of localizing a space with respect to a homology theory appeared in Sullivan's work on the Adams conjecture [85], where there is an explicit localization of a simply connected space with respect to ordinary homology with $\mathbb{Z}[1/p]$ coefficients. Bousfield and Kan [14] gave the first categorical definition of localization with respect to homology theory and provided a localization for nilpotent spaces with respect to $H_*(\cdot, R)$, where $R = \mathbb{F}_p$ for some prime p or R a subring of the rationals. Their technique was the R-completion of space, recapitulated in Section 3 below for the case $R = \mathbb{F}_p$. It was Bousfield himself who introduced model category theoretic techniques to provide the localization of any space with respect to an arbitrary homology theory. His paper [8] has been enormously influential, as much for the methods as for the results, and it's hard to overestimate its impact. For example, the concept of localization with respect to a map and the construction of its existence, which appears in the work for Dror-Farjoun [22] and [23] is directly influenced by Bousfield's ideas. About the time Dror-Farjoun's papers were first circulating, a whole group of people began to explore these ideas, both in homotopy theory and in related algebraic subjects. The paper by Cascuberta [17] is a useful survey. One should also mention the important paper of Bousfield [11], which uses similar techniques for its basic constructions. The longest and most general work in this vein, a work that includes an exposition of the localization model category in an arbitrary cellular model category is that of Hirschhorn [42], available at this writing over the Internet. The notion of a cellular model category is one way of axiomatizing the structure necessary to make Bousfield's arguments work, based on the concept of cell complexes and inclusions of sub-complexes. We give, in Section 4, a slightly different list of hypotheses along these lines. Both systems of axioms work in any example that we know.

We emphasize, however, that Bousfield's ideas had influence outside of the area of homotopy localization. For example, Jeff Smith realized very early on that one could use these constructions to put a model category structure on the category of small diagrams of simplicial sets, so that homotopy inverse limits can be computed as total right derived functors of inverse limit. This never made it into print, and we go through the arguments in Section 5. Beyond this, there is the second author's work on the homotopy theory of simplicial presheaves [46], see also [38] as well as Joyal's result for simplicial sheaves [53]. In the context of the present discussion that work can be interpreted as follows: the category of presheaves on a Groethendieck site is a category of diagrams and there is a closed model category structure obtained by localizing with respect to a class of cofibrations determined by the topology of the underlying site.

1. Localization with respect to a map.

This section is an exposition of a technique due to Bousfield for defining localization with respect to a map in a simplicial model category. We explain the technique for the category $\mathbf{S}$ of simplicial sets. In Section 4 we will examine how this technique extends to other simplicial model categories.

In the category $\mathbf{S}$ of simplicial sets we fix a cofibration $f : A \to B$.

Definition 1.1. *A space $Z \in \mathbf{S}$ is f-local if Z is fibrant and*

$$f^* : \mathbf{Hom}(B, Z) \to \mathbf{Hom}(A, Z)$$

is a weak equivalence.

Remarks 1.2.

1) Because f is a cofibration, f^* is a fibration. Hence we could equally require that f^* be a trivial fibration.

2) The hypothesis that f be a cofibration is innocuous. Indeed, if we drop the hypothesis that f be a cofibration, we have the following observation. Factor f

as

$$A \xrightarrow{f_0} B_0 \xrightarrow{q} B$$

where f_0 is a cofibration and q is a trivial fibration. Then f^* is a weak equivalence if and only if f_0^* is a weak equivalence, because q is left inverse to a trivial cofibration.

3) If $\mathcal{F}$ is a set of cofibrations $f_\alpha : A_\alpha \to B_\alpha$ we could define a space Z to be $\mathcal{F}$-local if it is fibrant and f_α^* is a weak equivalence for all $f_\alpha \in \mathcal{F}$. However, Z would be $\mathcal{F}$-local if and only if Z were f-local where $f = \sqcup f_\alpha$. Hence we would achieve no greater generality.

We expand on the notion of f-local:

DEFINITION 1.3. *A map $q : X \to Y$ in $\mathbf{S}$ is an f-injective if q is a fibration and*

$$(q_*, f^*) : \mathbf{Hom}(B, X) \to \mathbf{Hom}(B, Y) \times_{\mathbf{Hom}(A,Y)} \mathbf{Hom}(A, X)$$

is a trivial fibration.

Since this map is a fibration by **SM7**, we are only requiring it to be weak equivalence. In light of Remark 1.2.1, Z is f-local if and only if the unique map $Z \to *$ is an f-injective.

It is convenient to have a recognition principal for f-injectives. First, if $j : C \to D$ and $q : X \to Y$ are maps in $\mathbf{S}$, let us write

$$D(j, q) = \mathbf{Hom}(D, Y) \times_{\mathbf{Hom}(C,Y)} \mathbf{Hom}(C, X).$$

Note that $D(j, q)$ is a space of diagrams; indeed an n-simplex is a commutative diagram

$$\begin{tikzcd} C \times \Delta^n \arrow[r] \arrow[d, "j\times 1"'] & X \arrow[d, "q"] \\ D \times \Delta^n \arrow[r] & Y. \end{tikzcd}$$

If L is a simplicial set, let $\#L$ denote the cardinality of the set of non-degenerate simplices in L.

LEMMA 1.4. *A morphism $q : X \to Y$ in $\mathbf{S}$ is an f-injective if and only if it is a fibration and has the right lifting property with respect to all maps*

$$A \times L \cup_{A\times K} B \times K \to B \times L$$

where $K \to L$ is a cofibration in $\mathbf{S}$ with $\#L$ finite.

PROOF: A map in $\mathbf{S}$ is a trivial fibration if and only if it has the right lifting property with respect to all cofibrations $K \to L$ with $\#L$ finite. Now use an adjunction argument. ■

Note that one could specialize to the cases

$$K = \partial\Delta^n \hookrightarrow \Delta^n = L.$$

LEMMA 1.5. *If $q : X \to Y$ is an f-injective and $K \in \mathbf{S}$, then*

$$q_* : \mathbf{Hom}(K, X) \to \mathbf{Hom}(K, Y)$$

is an f-injective.

PROOF: Since q is an f-injective

$$\mathbf{Hom}(B, X) \to D(f, q)$$

is a trivial fibration. Hence

$$\mathbf{Hom}(K, \mathbf{Hom}(B, X)) \to \mathbf{Hom}(K, D(f, q))$$

is a trivial fibration. But this is isomorphic to

$$\mathbf{Hom}(B, \mathbf{Hom}(K, X)) \to D(f, q_*).$$ ■

From this is follows that the f-local spaces and f-injectives defined above, with the usual notion of weak equivalence, form a category of fibrant objects for a homotopy theory. This is direct from the definitions, using the previous two lemmas. In particular the mapping object is supplied by, for f-local Z, $\mathbf{Hom}(\Delta^1, Z)$. This uses Lemma 1.4.

We now let $\mathbf{Loc}_f$ be the resulting homotopy category of f-local spaces obtained as the full subcategory of $\mathrm{Ho}(\mathbf{S})$ with the f-local spaces. Since every object of $\mathbf{S}$ is cofibrant and every f-local space if fibrant, this is the equivalent to the category of f-local spaces and homotopy classes of maps. We will examine the inclusion functor

$$\mathbf{Loc}_f \to \mathrm{Ho}(\mathbf{S})$$

DEFINITION 1.6. *A localization with respect to f is a functor*

$$L_f : \mathrm{Ho}(\mathbf{S}) \to \mathrm{Ho}(\mathbf{S})$$

equipped with a natural transformation $\eta_X : X \to L_f X$ so that

1) *$L_f X \in \mathbf{Loc}_f$ and the restricted functor $L_f : \mathrm{Ho}(\mathbf{S}) \to \mathbf{Loc}_f$ is left adjoint to the inclusion; and*
2) *for all X, the two morphisms $L_f \eta_X, \eta_{L_f X} : L_f X \to L_f L_f X$ are equal, to an isomorphism.*

If it exists, such a localization will be unique up to isomorphism in the homotopy category $\mathrm{Ho}(\mathbf{S})$. The existence follows from Bousfield's factorization, embodied in Proposition 1.8 below. Before stating this we need a definition of a class of cofibrations which behave much like a class of trivial cofibrations.

DEFINITION 1.7. *A cofibration $j : C \to D$ is an* f-cofibration *if the map*

$$(q_*, j^*) : \mathbf{Hom}(D, X) \to \mathbf{Hom}(D, Y) \times_{\mathbf{Hom}(C,Y)} \mathbf{Hom}(C, X)$$

is a trivial fibration for all f-injectives $q : X \to Y$. Let $\mathcal{C}_f$ be the class of f-cofibrations.

PROPOSITION 1.8 (BOUSFIELD FACTORIZATION). *Every morphism $g : X \to Y$ may be factored*

$$X \xrightarrow{j} E_f \xrightarrow{q} Y$$

where j is an f-cofibration followed by an f-injective. Furthermore, this factorization is natural in the morphism g.

We prove this below after some further preliminaries and proving the existence of the localization demanded by Definition 1.6.

First some basic properties of f-cofibrations.

LEMMA 1.9.

1) *The morphism $f : A \to B$ is an f-cofibration.*
2) *Any trivial cofibration is an f-cofibration.*
3) *Suppose $i : C \to D$ and $j : D \to E$ are cofibrations and i is an f-cofibration. Then if either of j or ji is an f-cofibration so is the other.*

PROOF: Only part 3) needs comment. We use the following fact: if one has a diagram

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[dr] & Y \arrow[d] \\ & Z \end{tikzcd}$$

and a morphism $W \to Z$, then there is a pullback square

$$\begin{tikzcd} W \times_Z X \arrow[r] \arrow[d] & X \arrow[d, "q"] \\ W \times_Z Y \arrow[r] & Y. \end{tikzcd}$$

Hence if q is a trivial fibration so is $W \times_Z X \to W \times_Z Y$. Applying this remark to the diagram

$$\begin{tikzcd}
\mathbf{Hom}(D,X) \arrow[r] \arrow[dr] & \mathbf{Hom}(D,Y) \times_{\mathbf{Hom}(C,Y)} \mathbf{Hom}(C,X) \arrow[d] \\
& \mathbf{Hom}(D,Y)
\end{tikzcd}$$

and the map $\mathbf{Hom}(E,Y) \to \mathbf{Hom}(D,Y)$ we have that if i is an f-cofibration and $q : X \to Y$ is an f-injective

$$\mathbf{Hom}(E,Y) \times_{\mathbf{Hom}(D,Y)} \mathbf{Hom}(D,X) \to \mathbf{Hom}(E,Y) \times_{\mathbf{Hom}(C,Y)} \mathbf{Hom}(C,X)$$

is a trivial fibration. Now consider the composite

$$\begin{tikzcd}
\mathbf{Hom}(E,X) \arrow[r] & \mathbf{Hom}(E,Y) \times_{\mathbf{Hom}(D,Y)} \mathbf{Hom}(D,X) \arrow[d] \\
& \mathbf{Hom}(E,Y) \times_{\mathbf{Hom}(C,Y)} \mathbf{Hom}(C,X).
\end{tikzcd}$$

If j is an f-cofibration, the first map is a trivial fibration, so the composite is and ji is an f-cofibration. The converse equally applies to show that if ji is an f-cofibration so is j. ∎

More constructive properties of f-cofibrations are given in the next result.

LEMMA 1.10.

1) *Any retract of an f-cofibration is an f-cofibration. Any coproduct of f-cofibrations is an f-cofibration.*
2) *Given a pushout diagram with j an f-cofibration*

$$\begin{tikzcd}
C \arrow[r] \arrow[d, "j"'] & C' \arrow[d, "j'"] \\
D \arrow[r] & D',
\end{tikzcd}$$

then j' is an f-cofibration.

3) *If $j : C \to D$ is an f-cofibration and $K \to L$ is any cofibration, then*

$$C \times L \cup_{C\times K} D \times K \to D \times L$$

is an f-cofibration.

Proof: Again only part 3) needs comment. First note that if $K = \phi$, we are asserting $C\times L \to D\times L$ is an f-cofibration. This requires that for all f-injectives $q : X \to Y$

$$\mathbf{Hom}(D \times L, X) \to \mathbf{Hom}(D \times L, Y) \times_{\mathbf{Hom}(C\times L,Y)} \mathbf{Hom}(C \times L, X)$$

be a trivial fibration. But this is isomorphic to

$$\begin{array}{c}\mathbf{Hom}(D, \mathbf{Hom}(L, X))\\ \Big\downarrow\\ \mathbf{Hom}(D, \mathbf{Hom}(L, Y)) \times_{\mathbf{Hom}(C,\mathbf{Hom}(L,Y))} \mathbf{Hom}(C, \mathbf{Hom}(L, X))\end{array}$$

and $\mathbf{Hom}(L, X) \to \mathbf{Hom}(L, Y)$ is an f-injective by Lemma 1.5.

For the general case consider the diagram

$$\begin{tikzcd} C \times K \arrow[r] \arrow[d] & C \times L \arrow[d, "i"] \\ D \times K \arrow[r] & C \times L \cup_{C\times K} D \times K \arrow[d, "j"] \\ & D \times L. \end{tikzcd}$$

Part 2) of this lemma says i is a f-cofibration. Since ji is a f-cofibration, the previous lemma says j is. ■

Next is the crucial lifting result.

PROPOSITION 1.11. *Given a lifting problem*

$$\begin{tikzcd} C \arrow[r] \arrow[d, "j"'] & X \arrow[d, "q"] \\ D \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where j is a f-cofibration and q is an f-injective, the lift exists and is unique up to homotopy under C and over Y.

PROOF: Consider the trivial fibration

$$\mathbf{Hom}(D, X) \to \mathbf{Hom}(D, Y) \times_{\mathbf{Hom}(C,Y)} \mathbf{Hom}(C, X) = D(j, q).$$

A diagram as above is a 0-simplex in the target. Since a trivial fibration is surjective such a lift exists. The required homotopy is adjoint to a lift in

$$\begin{tikzcd} \partial\Delta^1 \arrow[r] \arrow[d] & \mathbf{Hom}(D, X) \arrow[d] \\ \Delta^1 \arrow[r] \arrow[ur, dashed] & D(j, q). \end{tikzcd}$$

■

The next few paragraphs are devoted to the existence of localization. Define a functor $\mathcal{L} : \mathbf{S} \to \mathbf{S}$ and a natural transformation $j : X \to \mathcal{L}X$, by taking the Bousfield factorization of $X \to *$ as

$$X \xrightarrow{j} \mathcal{L}X \xrightarrow{q} *$$

where j is an f-cofibration and q is a f-injective. This is a functor since the factorization is natural. Then $\mathcal{L}X$ is f-local. The previous result implies that j has the following universal property: given a diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "j"'] & Z \\ \mathcal{L}X \arrow[ur, dashed, "\overline{g}"'] & \end{tikzcd}$$

with Z f-local, then the dotted arrow exists and is unique up to homotopy. If g is a f-cofibration, $\overline{g}$ is a homotopy equivalence.

The first result is that $\mathcal{L}(\cdot)$ passes to a functor on the homotopy category.

LEMMA 1.12. *If $\varphi : X \to Y$ is a weak equivalence, then $\mathcal{L}\varphi : \mathcal{L}X \to \mathcal{L}Y$ is a weak equivalence.*

PROOF: First of all, $\mathcal{L}$ takes trivial cofibrations to homotopy equivalences. In effect, if $i : X \to Y$ is a trivial cofibration, then i is an f-cofibration by Lemma 1.9, and so the composites

$$X \xrightarrow{j_X} \mathcal{L}X \to * \quad \text{and} \quad X \xrightarrow{j_Y i} \mathcal{L}Y \to *$$

are two factorizations of $X \to *$ as an f-injective following an f-cofibration. It follows from Proposition 1.11 that $\mathcal{L}i : \mathcal{L}X \to \mathcal{L}Y$ is a homotopy equivalence.

More generally, any weak equivalence $f : X \to Y$ has a factorization $f = p \cdot i$, where i is a trivial cofibration and p is left inverse to a trivial cofibration. Thus, $\mathcal{L}p$ is a weak equivalence, as is the map $\mathcal{L}f$. ■

Let $L_f : \mathrm{Ho}(\mathbf{S}) \to \mathrm{Ho}(\mathbf{S})$ be the functor induced by $\mathcal{L}$, and $\eta : X \to L_f X$ be induced in $\mathrm{Ho}(\mathbf{S})$ by the map $j_X : X \to \mathcal{L}X$.

PROPOSITION 1.13. *The functor $L_f : \mathrm{Ho}(\mathbf{S}) \to \mathrm{Ho}(\mathbf{S})$ is an f-localization.*

PROOF: Since $\mathcal{L}X$ is f-local, L_f restricts to a functor

$$L_f : \mathrm{Ho}(\mathbf{S}) \to \mathbf{Loc}_f.$$

By the universal property of $\mathcal{L}X$, L_f is left adjoint to the inclusion.

If $j : X \to Y$ is an f-cofibration, then any choice of extension $\mathcal{L}j : \mathcal{L}X \to \mathcal{L}Y$ is a homotopy equivalence, by an obvious extension of the argument giving Lemma 1.12. It follow that $\mathcal{L}j_X$ is a homotopy equivalence, for any X. Also, the map $j_X : X \to \mathcal{L}X$ and the composite

$$X \xrightarrow{j_X} \mathcal{L}X \xrightarrow{j_{\mathcal{L}X}} \mathcal{L}^2 X$$

are f-cofibrations taking values in f-local spaces, and the diagram

$$\begin{tikzcd} X \arrow[r, "j_X"] \arrow[d, "j_X"'] & \mathcal{L}X \arrow[r, "j_{\mathcal{L}X}"] & \mathcal{L}^2 X \\ \mathcal{L}X \arrow[urr, "\theta"'] & & \end{tikzcd}$$

commutes, where θ could be either $\mathcal{L}j_X$ or $j_{\mathcal{L}X}$. It follows that the maps $\mathcal{L}j_X$ and $j_{\mathcal{L}X}$ are homotopic. ■

We now turn to the existence of Bousfield factorizations. To start, we must expand on Lemmas 1.9 and 1.10 and supply one more construction that preserves f-cofibrations, namely certain types of directed colimits.

Fix an infinite cardinal number β and let $\mathbf{Seq}(\beta)$ denote the well-ordered set of ordinals less than β. Then $\mathbf{Seq}(\beta)$ is a category with $\hom(s,t)$ having one element of $s \le t$ and empty otherwise.

LEMMA 1.14. *Let $C_\bullet \in \mathbf{S}$ be a diagram of spaces over $\mathbf{Seq}(\beta)$. If*

1) *for each successor ordinal $s < \beta$, $C_{s-1} \to C_s$ is an f-cofibration, and*
2) *for each limit ordinal $t < \beta$, $\varinjlim C_s = C_t$,*

then the map

$$C_0 \to \varinjlim_{s<\beta} C_s$$

is an f-cofibration.

PROOF: If K is a simplicial set, denote by $\#K$ the cardinality of the member of non-degenerate simplices of K. Notice that $D \to E$ is an f-cofibration if and only if every lifting problem

$$\begin{tikzcd} D \times L \cup_{D\times K} E \times K \arrow[r] \arrow[d] & X \arrow[d, "q"] \\ E \times L \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

can be solved, where q is an f-injective and $K \to L$ is a cofibration in $\mathbf{S}$ with $\#L$ finite. The strategy is this: we will use transfinite induction and solve the successive lifting problems

$$\begin{tikzcd} C_0 \times L \cup_{C_0\times K} C_s \times K \arrow[r] \arrow[d] & X \arrow[d, "q"] \\ C_s \times L \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{1.15}$$

in a compatible way, meaning that if $s < t$, the solution of the lifting problem for t will restrict to the solution for s. Then taking the colimit over $s < \beta$ will solve the problem for $C_0 \to \varinjlim_{s<\beta} C_s$.

So regard the lifting problem for s given in (1.15). If s is a successor ordinal, we can complete the diagram

$$\begin{tikzcd}
C_0 \times L \cup_{C_0 \times K} C_{s-1} \times K \arrow[r] \arrow[d] & C_0 \times L \cup_{C_0 \times K} C_s \times K \arrow[r] \arrow[d] & X \arrow[d] \\
C_{s-1} \times L \arrow[r] & C_{s-1} \times L \cup_{C_{s-1} \times K} C_s \times K \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

since we have a solution for $s-1$ and the left square is a pushout. Thus we need to solve a lifting problem

$$\begin{tikzcd}
C_{s-1} \times L \cup_{C_{s-1} \times K} C_s \times K \arrow[r] \arrow[d] & X \arrow[d] \\
C_s \times L \arrow[r] \arrow[ur, dashed] & Y.
\end{tikzcd}$$

But this is possible since $C_{s-1} \to C_s$ is an f-cofibration. Note that the constructed lift satisfies the compatibility requirement spelled out above.

If s is a limit ordinal, we have constructed compatible lifts, $t < s$,

$$\begin{tikzcd}
C_0 \times L \cup_{C_0 \times K} C_t \times K \arrow[r] \arrow[d] & X \arrow[d] \\
C_t \times L \arrow[r] \arrow[ur] & Y.
\end{tikzcd}$$

Taking the colimit over t yields a solution to the lifting problem for $C_0 \to \varinjlim_{t<s} C_t = C_s$ compatible with all previous lifts. ■

We now come to the factorization result. It turns on the following construction.

MAIN CONSTRUCTION 1.16.

Let $g : X \to Y$ be a map in $\mathbf{S}$, and let I be the set of morphisms

$$A \times L \cup_{A \times K} B \times K \to B \times L,$$

where $K \subseteq L$ runs over representatives for isomorphism classes of cofibrations of simplicial sets so that $\#L$ is finite. Write $j_\alpha : C_\alpha \to D_\alpha$ for a typical element in I.

Define a factorization

$$X \xrightarrow{j_1} E_1 X \xrightarrow{g_1} Y$$

of g by the pushout diagram

$$\begin{tikzcd} \bigsqcup_I C_\alpha \times D(j_\alpha, g) \arrow[r, "\epsilon"] \arrow[d] & X \arrow[d, "j_1"] \\ \bigsqcup_I D_\alpha \times D(j_\alpha, g) \arrow[r] & E_1 \end{tikzcd}$$

where

$$D(j_\alpha, g) = \mathbf{Hom}(C_\alpha, X) \times_{\mathbf{Hom}(C_\alpha, Y)} \mathbf{Hom}(D_\alpha, Y)$$

is the space of commutative squares and ϵ is induced by evaluation. Evaluation also defines a map $D_\alpha \times D(j_\alpha, g) \to Y$ and $g_1 : E_1 \to Y$ is induced by the universal property of pushouts.

Note that since every morphism in I is an f-cofibration, Lemma 1.10 implies j_1 is an f-cofibration. Equally important, note that if one has any diagram

$$\begin{tikzcd} C_\alpha \arrow[r] \arrow[d, "j_\alpha"'] & X \arrow[d, "g"] \\ D_\alpha \arrow[r, "\varphi"'] & Y \end{tikzcd}$$

where $j_\alpha \in I$, then φ factors through E_1. This is because such a diagram is a 0-simplex of $D(j_\alpha, g)$. Finally, note this construction is natural in g. ∎

Proposition 1.17. *Bousfield factorization exists.*

Proof: This is a simple transfinite variation on the small object argument. Pick an infinite cardinal $\gamma > \#B$, and choose an infinite cardinal $\beta > 2^\gamma$. Note that $\#C_\alpha < \gamma$ for all α. Fix a map $g : X \to Y$. Using recursion, we construct a diagram $\{E_s\}$ over $\mathbf{Seq}(\beta)$, with $E_0 = X$ and so that there is a map $\{g_s\} : \{E_s\} \to \{Y\}$ to the constant diagram on Y.

For $s = 0$, let $g_0\ X = E_0 \to Y$. If $s+1$ is a successor ordinal, let E_{s+1} be defined by applying the Main Construction 1.16 to $g_s : E_s \to Y$ to obtain a factorization

$$E_s \to E_1 E_s \xrightarrow{(g_s)_1} Y.$$

Then $E_{s+1} = E_1 E_s$ and $g_{s+1} = (g_s)_1$. If s is a limit ordinal, let $E_s = \varinjlim_{t<s} E_t$ and g_s the induced map. This defines $\{E_s\}$.

Define $E_f = \varinjlim_{s<\beta} E_s$. Then $g : X \to Y$ factors as

$$X \xrightarrow{j} E_f \xrightarrow{q} Y$$

where j is an f-cofibration by Lemma 1.14. To see that q is an f-injective, it is sufficient to show that any lifting problem

$$\begin{tikzcd} C_\alpha \arrow[r, "\psi"] \arrow[d, "j_\alpha"'] & E_f \arrow[d, "g"] \\ D_\alpha \arrow[r, "\varphi"'] \arrow[ur, dashed] & Y \end{tikzcd}$$

with $j_\alpha \in I$ can be solved. Since $2^{\# C_\alpha} < \beta$,

$$\hom(C_\alpha, E_f) \cong \varinjlim_{\beta} \hom(C_\alpha, E_s),$$

for otherwise C_α has too many subobjects. Thus, there is an $s < \beta$, and a factorization of ψ

$$C_\alpha \to E_s \to E_f.$$

Then the main construction implies that φ factors through E_{s+1} and the result follows. ■

The constructive part of this argument allows us to identify the class of f-cofibrations in another way. Let β be a cardinal, and recall from Definition II.7.3 that a class $\mathcal{M}$ of morphisms in a cocomplete category $\mathcal{C}$ is β-saturated if

1) coproducts and retracts of morphisms in $\mathcal{M}$ are in $\mathcal{M}$
2) given a pushout diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d, "j"'] & C' \arrow[d, "j'"] \\ D \arrow[r] & D' \end{tikzcd}$$

with j in $\mathcal{M}$, then j' is in $\mathcal{M}$.

3) If $C : \mathbf{Seq}(\beta) \to \mathcal{C}$ is a diagram over β, and
   a) for each successor ordinal s, $C_{s-1} \to C_s$ is in $\mathcal{M}$ and
   b) for each limit ordinal s, $\varinjlim_{t<s} C_t \cong C_s$,

   then $C_0 \to \varinjlim_{t<\beta} C_t$ is in $\mathcal{M}$.

COROLLARY 1.18. *The class of f-cofibrations in* $\mathbf{S}$ *is the β-saturation of the morphisms*

$$A \times L \cup_{A \times K} B \times K \to B \times L$$

with $K \to L$ a cofibration in $\mathbf{S}$ *and $\#L < \infty$, and trivial cofibrations $K \to L$ with $\#L < \infty$, and β is an infinite cardinal greater than 2^γ, where γ is an infinite cardinal larger than $\#B$.*

PROOF: Let $\mathcal{C}_f$ be this saturation. Then every morphism in $\mathcal{C}_f$ is an f-cofibration since the class of f-cofibrations is β-saturated. Also if $g : X \to Y$ is factored

$$X \xrightarrow{j} Z \xrightarrow{q} Y$$

as in the proof of 1.17, then $j \in \mathcal{C}_f$. If g is any f-cofibration, then there is a lifting

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "g"'] & Z \arrow[d, "q"] \\ Y \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

since q is an f-injective. This lifting shows g is a retract of j, hence in $\mathcal{C}_f$. ∎

We end this section with a sequence of technical lemmas on the properties of the functor $\mathcal{L}(\cdot)$. The purpose here is to have in place the structure necessary to prove the existence of the model category structure in the next section.

Recall that $\mathcal{L}(X) = \varinjlim_{s<\beta} E_s(X)$ where $E_{s+1}(X)$ is obtained from $E_s = E_s(X)$ by a pushout diagram

$$\begin{tikzcd} \coprod_{j_\alpha \in J} C_\alpha \times \mathbf{Hom}(C_\alpha, E_s) \arrow[r] \arrow[d] & E_s(X) \arrow[d] \\ \coprod_{j_\alpha \in J} D_\alpha \times \mathbf{Hom}(C_\alpha, E_s) \arrow[r] & E_{s+1}(X) \end{tikzcd} \tag{1.19}$$

Of course $E_0(X) = X$ and if s is a limit ordinal, $E_s(X) = \varinjlim_{t<s} E_t(X)$.

LEMMA 1.20. *Let λ be a cardinal larger than 2^γ, where γ is an infinite cardinal larger than $\#B$. Let $X : \mathbf{Seq}(\lambda) \to \mathbf{S}$ be λ-diagram. Suppose for each $s < \lambda$, $X_s \to X_{s+1}$ is a cofibration, and suppose for each limit ordinal $X_s = \varinjlim_{t<s} X_t$. Then the natural map*

$$\varinjlim_{s<\lambda} \mathcal{L}(X_s) \to \mathcal{L}(\varinjlim_{s<\lambda} X_s)$$

is an isomorphism.

PROOF: It is sufficient, since colimits commute, to show this statement holds for each of the functors $E_s(\cdot)$. For the same reason, it is sufficient to show, that if this statement holds for $E_t(\cdot)$ it holds for $E_{t+1}(\cdot)$. Then it will automatically hold for limit ordinals. Now, since $2^{\#C_\alpha} < \lambda$ for all α, the natural map

$$\begin{aligned}\varinjlim_{s<\lambda} \hom(C_\alpha \times \Delta^n, E_t(X_s)) \to &\hom(C_\alpha \times \Delta^n, \varinjlim_{s<\lambda} E_t X_s) \\ &\cong \hom(C_\alpha \times \Delta^n, E_t(\varinjlim_{s<\lambda} X_s))\end{aligned}$$

is an isomorphism in each degree n. It follows that the simplicial map

$$\varinjlim_{s<\lambda} \mathbf{Hom}(C_\alpha, E_t(X_s)) \to \mathbf{Hom}(C_\alpha, E_t(\varinjlim_{s<\lambda} X_s))$$

is an isomorphism. Now use the diagram (1.19) and the fact that colimits commute. ■

For the next result we need a "Reedy Lemma". Suppose we have two pushout squares

$$\begin{array}{ccc} A_1 & \longrightarrow & B_1 \\ \downarrow & & \downarrow \\ C_1 & \longrightarrow & D_1 \end{array} \qquad\qquad \begin{array}{ccc} A_2 & \longrightarrow & B_2 \\ \downarrow & & \downarrow \\ C_2 & \longrightarrow & D_2 \end{array}$$

and a map from the first square to the second.

LEMMA 1.21. *If $B_1 \to B_2$ is a cofibration and the induced map $C_1 \cup_{A_1} A_2 \to C_2$ is a fibration, then the map $D_1 \to D_2$ is a cofibration.*

PROOF: The hypotheses are what is required to show $D_1 \to D_2$ has the left lifting property with respect to all trivial fibrations. ■

LEMMA 1.22. *The functor $\mathcal{L}(\cdot)$ preserves cofibrations.*

PROOF: Suppose that $X \to Y$ is a cofibration, and presume that the induced map $E_sX \to E_sY$ is a cofibration. We will show that the map $E_{s+1}X \to E_{s+1}Y$ is a cofibration.

The diagram 1.20 for X maps to the corresponding diagram for Y. The diagram

$$\begin{tikzcd}
\bigsqcup_J C_\alpha \times \mathbf{Hom}(C_\alpha, E_sX) \arrow[r] \arrow[d] & \bigsqcup_J D_\alpha \times \mathbf{Hom}(C_\alpha, E_sX) \arrow[d] \\
\bigsqcup_J C_\alpha \times \mathbf{Hom}(C_\alpha, E_sY) \arrow[r] & \bigsqcup_J D_\alpha \times \mathbf{Hom}(C_\alpha, E_sY)
\end{tikzcd}$$

is a disjoint union of diagrams of the form

$$\begin{tikzcd}
C_\alpha \times K \arrow[r, "j_\alpha \times 1"] \arrow[d, "1 \times i"'] & D_\alpha \times K \arrow[d, "1 \times i"] \\
C_\alpha \times L \arrow[r, "j_\alpha \times 1"'] & D_\alpha \times L
\end{tikzcd}$$

which are induced by cofibrations $j_\alpha : C_\alpha \to D_\alpha$ and $i : K \to L$. Each induced map

$$C_\alpha \times L \cup_{C_\alpha \times K} D_\alpha \times K \to D_\alpha \times L$$

is plainly a cofibration. Now use Lemma 1.21. ∎

The next result is a variation on Lemma 1.20.

LEMMA 1.23. *Let λ be a cardinal number with $\lambda > \gamma$. For $X \in \mathbf{S}$, let $\{X_j\}$ be the filtered system of sub-complexes $X_j \subseteq X$ with $\#X_j \leq \lambda$. Then the natural map*

$$\varinjlim_j \mathcal{L}(X_j) \to \mathcal{L}(X)$$

is an isomorphism.

PROOF: The proof is the same as that for Lemma 1.20, once one notices that

$$\varinjlim_j \mathbf{Hom}(C_\alpha, X_j) \to \mathbf{Hom}(C_\alpha, X)$$

is an isomorphism on account of the size of C_α. ∎

Next, we have a counting argument.

LEMMA 1.24. *Let λ be any cardinal such that $\lambda \geq \beta$. If $X \in S$ has $\#X \leq 2^\lambda$, then $\#\mathcal{L}(X) \leq 2^\lambda$.*

PROOF: We show that for each ordinal $s < \beta$, $\#E_s(X) \leq 2^\lambda$. This is done by induction: it is true for $E_0(X) = X$; if s is a limit ordinal and it is true for $E_t(X)$, $t < s$, then it is true for $E_s(X)$. Finally, if it is true for s it is true for $s + 1$ by the diagram (1.19) and because

$$\#(D_\alpha \times \mathbf{Hom}(C_\alpha, E_s(X)) \leq \beta \cdot (2^\lambda)^\beta = \beta \cdot 2^{\lambda \cdot \beta} = 2^\lambda. \qquad ∎$$

Because it is a colimit, one would not expect the functor $\mathcal{L}$ to preserve inverse limits, we do have the following result. Recall that $\mathcal{L}$ preserves cofibrations and the cofibrations are inclusions.

LEMMA 1.25. *Let X be a simplicial set and $C, D \subseteq X$ two sub-simplicial sets. Then $\mathcal{L}(C \cap D) = \mathcal{L}(C) \cap \mathcal{L}(D)$.*

PROOF: First suppose that we can show that for every ordinal s there is an equality $E_{s+1}(C \cap D) = E_{s+1}(C) \cap E_{s+1}(D)$. Then, because $E_s(\cdot) \subseteq E_{s+1}(\cdot)$, an inductive argument shows that we have the following equalities for any limit ordinal s

$$E_s(C \cap D) = \bigcup_{t<s} [E_t(C \cap D)] = \bigcup_{t<s} [E_t(C) \cap E_t(D)] = [\bigcup_{t<s} E_t(C)] \cap [\bigcup_{t<s} E_t(D)].$$

The last of these equalities uses that we have a nested sequence of inclusions, and the last listed object is $E_s(C) \cap E_s(D)$.

Now suppose that we can show $E_1(C \cap D) = E_1(C) \cap E_1(D)$. Then another inductive argument implies

$$E_{s+1}(C \cap D) = E_1(E_s(C \cap D)) = E_1(E_s(C) \cap E_s(D)) = E_{s+1}(C) \cap E_{s+1}(D).$$

Finally, to see that $E_1(C \cap D) = E_1(C) \cap E_1(D)$, note that $E_1 X_n$ has the form

$$E_1 X_n = (\bigsqcup_\alpha (D_\alpha - C_\alpha)_n) \bigsqcup X_n$$

in each simplicial degree n.

2. The closed model category structure.

We have presented Bousfield factorization as if it were a "trivial cofibration-fibration" factorization for some appropriate closed model category structure on simplicial sets. This is the case, but it requires some care to make this statement precise.

Fix a cofibration $f : A \to B$ and note that the previous section gives a functor $\mathcal{L} : \mathbf{S} \to \mathbf{S}$ equipped with a natural f-cofibration

$$j : X \to \mathcal{L}X.$$

The functor $\mathcal{L}$ preserves weak equivalences and induces the localization function L_f on the homotopy category. We now define a morphism $g : C \to D$ in $\mathbf{S}$ to be an *f-local equivalence* if the induced map

$$g^* : \mathbf{Hom}(D, X) \to \mathbf{Hom}(C, X)$$

is a weak equivalence for all f-local spaces X.

LEMMA 2.1.

1) *Any f-cofibration is an f-local equivalence.*
2) *A morphism $g : C \to D$ is an f-local equivalence if and only if $\mathcal{L}g : \mathcal{L}C \to \mathcal{L}D$ is a weak equivalence.*

PROOF: Part 1) is a consequence of Definition 1.7 with q the unique morphism $X \to *$. For part 2) consider the induced diagram, with X f-local

$$\begin{array}{ccc}
\mathbf{Hom}(\mathcal{L}D, X) & \xrightarrow{j^*} & \mathbf{Hom}(D, X) \\
{\scriptstyle \mathcal{L}g^*}\downarrow & & \downarrow{\scriptstyle g^*} \\
\mathbf{Hom}(\mathcal{L}C, X) & \xrightarrow[j^*]{} & \mathbf{Hom}(C, X).
\end{array}$$

By part 1) both maps labeled j^* are weak equivalences. If $\mathcal{L}g$ is a weak equivalence, then $\mathcal{L}g^*$ and, hence, g^* are weak equivalences, so g is a f-local equivalence. Conversely, if g^* is a weak equivalence, so is $\mathcal{L}g^*$ and

$$\mathcal{L}g^* : [\mathcal{L}D, X] \to [\mathcal{L}C, X]$$

is a bijection for all f-local X. This implies $\mathcal{L}g$ is a homotopy equivalence. To see this, set $X = \mathcal{L}C$, then there is a map $h : \mathcal{L}D \to \mathcal{L}C$ so that $\mathcal{L}g^* h = h \circ \mathcal{L}g \simeq 1_{\mathcal{L}C}$. Then set $X = \mathcal{L}D$ and compute

$$\mathcal{L}g^*(\mathcal{L}g \circ h) = \mathcal{L}g \circ h \circ \mathcal{L}g \simeq \mathcal{L}g = 1_{\mathcal{L}D} \circ \mathcal{L}g = \mathcal{L}g^*(1_{\mathcal{L}D})$$

so $\mathcal{L}g \circ h \simeq 1_{\mathcal{L}D}$. ∎

DEFINITION 2.2. *We define a morphism $g : C \to D$ to be*

1) *an f-local cofibration if it is a cofibration; and*
2) *an f-local fibration if it has the right lifting property with respect to all cofibrations which are also f-local equivalences.*

The main result of this section is:

THEOREM 2.3. *With its usual simplicial structure and the above definitions of f-local equivalence, cofibrations, and f-local fibration, the category* **S** *becomes a simplicial model category.*

REMARK 2.4. If every f-injective were an f-local fibration, this would be easy. However, this need not be the case as the following example shows. Let $f : \partial\Delta^{n+1} \to \Delta^{n+1}$ be the inclusion and consider the fibration $q : WK(\mathbb{Z}, n-1) \to K(\mathbb{Z}, n)$. Note that q is an f-injective; that is,

$$\mathbf{Hom}(\Delta^{n+1}, WK(\mathbb{Z}, n-1)) \to D(f, q)$$

is a trivial fibration. This is equivalent to the assertion that $D(f, q)$ is contractible. For this, there is a pullback diagram

$$\begin{tikzcd}
D(f,q) \arrow[r] \arrow[d] & \mathbf{Hom}(\Delta^{n+1}, K(\mathbb{Z}, n)) \arrow[d] \\
\mathbf{Hom}(\partial\Delta^{n+1}, WK(\mathbb{Z}, n-1)) \arrow[r] & \mathbf{Hom}(\partial\Delta^{n+1}, K(\mathbb{Z}, n))
\end{tikzcd}$$

so that the fibre of the induced fibration $D(f, q) \to \mathbf{Hom}(\partial\Delta^{n}, WK(\mathbb{Z}, n-1))$ is the pointed function complex

$$\mathbf{hom}_*(\Delta^{n+1}/\partial\Delta^{n+1}, K(\mathbb{Z}, n)) \simeq \Omega^{n+1}K(\mathbb{Z}, n) \simeq *.$$

Now let $* \to \partial\Delta^{n+1}$ be any vertex. Then $* \to \Delta^{n+1}$ is a trivial cofibration and $\partial\Delta^{n+1} \to \Delta^{n+1}$ is an f-local equivalence (by Lemma 2.1.1 and Lemma 1.9.1) so $* \to \partial\Delta^{n+1}$ is an f-local equivalence. But if $i : \partial\Delta^{n+1} \to K(\mathbb{Z}, n)$ is non-trivial in homotopy, there is no solution to the lifting problem

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & WK(\mathbb{Z}, n-1) \arrow[d, "q"] \\
\partial\Delta^{n+1} \arrow[r, "i"'] \arrow[ur, dashed] & K(\mathbb{Z}, n),
\end{tikzcd}$$

so q is an f-injective which is not an f-local fibration. We will have more to say about f-local fibrations after the proof of Theorem 2.3.

We start the proof of Theorem 2.3 with the following lemmas.

LEMMA 2.5. *A morphism in* $\mathbf{S}$ *is at once an* f*-local fibration and an* f*-local equivalence if and only if it is a trivial fibration.*

PROOF: If $q : X \to Y$ is a trivial fibration it has the right lifting property with respect to all cofibrations. Furthermore $\mathcal{L}q$ is a weak equivalence by Lemma 1.12. So Lemma 2.1 implies q is a f-local fibration and an f-local equivalence.

For the converse, fix $q : X \to Y$ which is an f-local fibration and an f-local equivalence. Factor q as

$$X \xrightarrow{i} Z \xrightarrow{p} Y,$$

where i is a cofibration and p is a trivial fibration. Then Lemmas 1.12 and 2.1 imply i is an f-local equivalence, so there is a solution to the lifting problem

$$\begin{tikzcd} X \arrow[r] \arrow[d, "i"'] & X \arrow[d, "q"] \\ Z \arrow[r, "p"'] \arrow[ur, dashed] & Y. \end{tikzcd}$$

Thus q is a retract of p and a trivial fibration. ∎

LEMMA 2.6. *Any morphism* $g : X \to Y$ *in* $\mathbf{S}$ *can be factored as*

$$X \xrightarrow{j} Z \xrightarrow{q} Y$$

where q *is an* f*-local fibration and* j *is at once a cofibration and an* f*-local equivalence.*

This is the heart of the matter and will be proved below. We also record:

LEMMA 2.7.

1) *The class of morphisms which are at once cofibrations and* f*-local equivalences is closed under pushouts and colimits over ordinal numbers.*
2) *Let* $j : C \to D$ *be at once a cofibration and an* f*-local equivalence, and let* $K \to L$ *be any cofibration, then*

$$D \times K \cup_{C \times K} C \times L \to D \times L$$

is a cofibration and an f*-local equivalence.*

PROOF: Observe that $j : C \to D$ is a cofibration and an f-local weak equivalence if and only if the induced map

$$j^* : \mathbf{Hom}(D, X) \to \mathbf{Hom}(C, X)$$

is a trivial fibration for all f-local spaces X. Part 1) follows immediately.

For part 2), it suffices to show that the map $j \times 1 : C \times K \to D \times K$ is a cofibration and an f-local weak equivalence if the same is true for $j : C \to D$. But the map

$$\mathbf{Hom}(D \times K, X) \xrightarrow{(j\times 1)^*} \mathbf{Hom}(C \times K, X)$$

is isomorphic to

$$\mathbf{Hom}(K, \mathbf{Hom}(D, X)) \xrightarrow{j^*} \mathbf{Hom}(K, \mathbf{Hom}(C, X)),$$

which is a trivial fibration if X is f-local and j is a cofibration and f-local weak equivalence. ■

PROOF OF THEOREM 2.3: The "trivial cofibration-fibration" factorization is Lemma 2.6. The other part of **CM5** follows from Lemma 2.5 and the axioms for **S** in its usual closed model category structure. The remaining closed model axioms are automatic. Finally, **SM7** follows from Lemma 2.7.2. ■

We now must prove Lemma 2.6. This argument that we shall give is a sequence of ideas originally due to Bousfield, and then later modified by by J. Smith and Hirschhorn. The argument that we shall give here is the iteration presented in [38]. The central device is the following:

LEMMA 2.8. *Suppose that γ is an infinite cardinal, and suppose given a diagram of simplicial set maps*

$$\begin{tikzcd} & X \arrow[d, hook, "j"] \\ A \arrow[r, hook] & Y \end{tikzcd}$$

where $\#A \leq \gamma$ and the inclusion $j : X \hookrightarrow Y$ is a weak equivalence. Then there is a subcomplex $B \subset X$ containing A such that $\#B \leq \gamma$ and the inclusion $j_B : B \cap Y \hookrightarrow B$ is a weak equivalence.

In the language of [38], Lemma 2.8 is called the "bounded cofibration property" for simplicial set cofibrations. To effect the proof, we need the following (fundamental) method of converting a map to a fibration:

LEMMA 2.9. Suppose that $f : X \to Y$ is a map of simplicial sets. Then there is a diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\sigma_f"'] & & Y \\ & Z_f \arrow[ur, "\pi_f"'] & \end{tikzcd}$$

such that π is a fibration and σ is a weak equivalence. Furthermore, this factorization preserves filtered colimits in f.

PROOF: Let $f : X \to Y$ be any simplicial set map, and use Kan's Ex^∞-construction (see III.4) to form the diagram

(2.10)
$$\begin{tikzcd} X \arrow[r, "\nu_X"] \arrow[d, "f"'] & Ex^\infty X \arrow[d, "f_*"] \\ Y \arrow[r, "\nu_Y"'] & Ex^\infty Y \end{tikzcd}$$

Note that the Ex^∞ construction commutes with filtered colimits. The next step is to use the dual of Lemma II.9.4 to convert the map $f_* : Ex^\infty X \to Ex^\infty Y$ into a fibration according to the standard classical method (which works because g_* is a map between Kan complexes). In the current context, one forms the diagram

$$\begin{tikzcd} Ex^\infty X \times_{Ex^\infty Y} \mathbf{Hom}(\Delta^1, Ex^\infty Y) \arrow[r, "\tilde{f}"] \arrow[d, "pr_L"'] & \mathbf{Hom}(\Delta^1, Ex^\infty Y) \arrow[r, "d^{1*}"] \arrow[d, "d^{0*}"] & Ex^\infty Y \\ Ex^\infty X \arrow[r, "f_*"'] & Ex^\infty Y & \end{tikzcd}$$

Write $\pi = d^{1*} \cdot \tilde{f}$, and observe that pr_L has a section σ which is induced by the map $\Delta^1 \to \Delta^0$. Then $\pi \cdot \sigma = f_*$, the map π is a fibration, and σ is a section of the trivial fibration pr_L and is therefore a weak equivalence. Write

$\tilde{Z}_f = Ex^\infty X \times_{Ex^\infty Y} \mathbf{Hom}(\Delta^1, Ex^\infty Y)$. Finally, one forms the pullback diagram

$$\begin{tikzcd} Y \times_{Ex^\infty Y} \tilde{Z}_f \arrow[r, "\nu_*"] \arrow[d, "\pi_f"'] & \tilde{Z}_f \arrow[d, "\pi"] \\ Y \arrow[r, "\nu_Y"'] & Ex^\infty Y \end{tikzcd} \tag{2.11}$$

Write $Z_f = Y \times_{Ex^\infty Y} \tilde{Z}_f$. The factorization $\pi \cdot \sigma = f_*$ determines a map from diagram (2.10) to diagram (2.11), and hence there is an induced map $\sigma_f : X \to Z_f$ such that $f = \pi_f \cdot \sigma_f$. Furthermore, π is a fibration so that π_f is a fibration and the map ν_* is a weak equivalence by properness of the closed model structure for the simplicial set category $\mathbf{S}$ (Corollary II.9.6). It follows that the map σ_f is a weak equivalence.

Note that all constructions here are natural in f ane either involve finite limits or filtered colimits, all of which commute with filtered colimits in f. ■

PROOF OF LEMMA 2.8: Say that a subcomplex B of Y is γ-bounded if $\# B \leq \gamma$. In the language of Lemma 2.9, we must show that there is a γ-bounded subcomplex $B \subset Y$ containing A such that the induced fibration π_{j_B} is trivial. Note further that Z_{j_B} is γ-bounded if B is γ-bounded.

Write $A = B_0$, and consider all diagrams of the form

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, hook] & Z_{j_{B_0}} \arrow[r] & Z_j \arrow[d, "\pi_j"] \\ \Delta^n \arrow[r] \arrow[urr, dashed, "\theta"] & B_0 \arrow[r] & Y. \end{tikzcd} \tag{2.12}$$

Observe that the lifting θ always exists, because π_j is a trivial fibration by assumption. The simplicial set Y is a filtered colimit of its γ-bounded subcomplexes B so that $Z_j = \varinjlim_B Z_{j_B}$, and there are at most γ such (solid arrow) diagrams. It follows that there is a γ-bounded subcomplex B_1 of Y such that all such solid arrow lifting problems have solutions in $Z_{j_{B_1}}$. Repeat this construction countably many times to form a sequence of γ-bounded subcomplexes $A = B_0 \subset B_1 \subset B_2 \subset \dots$, so that all lifting problems of the form (2.12) over B_i

are solved over B_{i+1}. Then $B = \cup_{i\geq 0} B_i$ is γ-bounded, and each lifting problem

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, hook] & Z_{j_B} \arrow[d, "\pi_{j_B}"] \\ \Delta^n \arrow[r] & B \end{tikzcd}$$

factors through a corresponding lifting problem over some B_i and is therefore solved over B_{i+1}. ■

LEMMA 2.13. *Let $\lambda = 2^\beta$, where $\beta > 2^\gamma$ and $\gamma > \#B$ are the choices of cardinals appearing in Proposition 1.17 Let $j : X \to Y$ be at once a cofibration and an f-trivial fibration, and suppose that A is a λ-bounded subcomplex of Y. Then there is a λ-bounded sub-complex $B \subset Y$ so that $A \subset B$ and $B \cap X \hookrightarrow B$ is an f-local equivalence.*

PROOF: We inductively define a chain of λ-bounded sub-objects $A = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq Y$ over λ, and a chain of sub-objects

$$\mathcal{L}(A) = \mathcal{L}(A_0) \subseteq X_1 \subseteq \mathcal{L}(A_1) \subseteq X_2 \subseteq \mathcal{L}(A_2) \subseteq \cdots \mathcal{L}(Y),$$

also over λ, with the property that

$$\mathcal{L}(X) \cap X_s \to X_s$$

is a weak equivalence. Then we set $B = \varinjlim_{s<\kappa} A_s$ and, by Lemmas 1.20 and 1.25,

$$\begin{aligned} \mathcal{L}(X \cap B) = \mathcal{L}(X) \cap \mathcal{L}(B) &= \varinjlim_{s<\kappa} \mathcal{L}(X) \cap X_s \\ &\to \varinjlim_{s<\kappa} X_s \cong \mathcal{L}(B) \end{aligned}$$

is a weak equivalence as required.

The objects A_s and X_s are defined recursively. Suppose $s+1$ is a successor ordinal and A_s has been defined. Then, since A_s is λ-bounded, $\mathcal{L}A_s$ is λ-bounded by Lemma 1.24. Then Lemma 2.8 implies that there is a λ-bounded sub-object $X_{s+1} \subseteq \mathcal{L}(Y)$ so that $\mathcal{L}(A_s) \subseteq X_{s+1}$ and $\mathcal{L}(X) \cap X_{s+1} \to X_{s+1}$ is a weak equivalence. Since $\mathcal{L}(Y) = \varinjlim_j \mathcal{L}(Y_j)$ where $Y_j \subseteq Y$ runs over the λ-bounded sub-objects of Y (Lemma 1.23), there is a λ-bounded sub-object A'_{s+1} so that $X_{s+1} \subseteq \mathcal{L}(A'_{s+1})$. Let $A_{s+1} = A_s \cup A'_{s+1}$. Finally, suppose s is a limit ordinal. Then set $X_s = \varinjlim_{t<s} \mathcal{L}(A_t) \cong \varinjlim_{t<s} X_t$. The object X_s is λ-bounded and $\mathcal{L}(X) \cap X_s \to X_s$ is a weak equivalence. Choose $A'_s \subseteq Y$ so that A'_s is λ-bounded and $X_s \subseteq \mathcal{L}(A'_s)$ and set $A_s = \varinjlim_{t<s} A_t \cup A'_s$. ∎

LEMMA 2.14. *Suppose that λ is the cardinal chosen in Lemma 2.13. A map $q : X \to Y$ in $\mathbf{S}$ is an f-local fibration if and only if it has the right lifting property with respect of all morphisms $j : C \to D$ which are at once cofibrations and f-local equivalences and so that $\#D \leq \lambda$.*

PROOF: The "only if" implication is clear. For the reverse implication we use a Zorn's Lemma argument. Consider an arbitrary lifting problem

$$\begin{tikzcd} C \arrow[r] \arrow[d, "j"'] & X \arrow[d, "q"] \\ D \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where j is any cofibration which is also an f-local equivalence. We must complete the dotted arrow. Define Ω to be the set of pairs (C', g) where $C \subseteq C' \subseteq D$ and $C \to C'$ is a f-local equivalence, and g solves the lifting problem

$$\begin{tikzcd} C \arrow[r] \arrow[d] & X \arrow[d, "q"] \\ C' \arrow[r] \arrow[ur, dashed, "g"] & Y. \end{tikzcd}$$

Define $(C', g) < (C'', h)$ if $C' \subseteq C''$ and $h|_{C'} = g$. Then Ω satisfies the hypotheses of Zorn's Lemma and, thus, has a maximal element (C_0, g_0). We show $C_0 = D$. Consider the new lifting problem

$$\begin{tikzcd} C_0 \arrow[r, "g_0"] \arrow[d] & X \arrow[d, "q"] \\ D \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd}$$

If $C_0 \neq D$, choose $x \in D$ so that $x \notin C_0$. By the previous lemma, there is a λ-bounded subobject $D_0 \subseteq D$ so that $x \in D_0$ and $C_0 \cap D_0 \to D_0$ is an f-local equivalence. By hypothesis, the restricted lifting problem

$$\begin{tikzcd} C_0 \arrow[r, "g_0"] \arrow[d] & X \arrow[d, "q"] \\ C_0 \cup D_0 \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

has a solution. Lemma 2.7.2 implies $C_0 \to C_0 \cup D_0$ is an f-local equivalence, so we have a contradiction to the maximality of (C_0, g_0). Thus $C_0 = D$ and the proof is complete. ■

THE PROOF OF LEMMA 2.6:

We use a Bousfield factorization. Let J be a set of maps containing one representative for each isomorphism class of cofibrations $j : C \to D$ which are f-local equivalences and so that $\#D \leq \lambda$. Define

$$\varphi = \coprod_J j : \tilde{C} = \coprod_J C \to \coprod_J D = \tilde{D}$$

and factor $g : X \to Y$ as

$$X \xrightarrow{i} E_\varphi \xrightarrow{q} Y$$

where i is the φ-cofibration of Proposition 1.17 and q is a φ-injective.

The class of φ-cofibrations is the saturation of the class of all cofibrations

$$\tilde{C} \times L \cup_{\tilde{C} \times K} \tilde{D} \times K \to \tilde{D} \times L$$

induced by φ and all inclusions $K \hookrightarrow L$ of finite simplicial sets, by Corollary 1.18. Each such map is an f-local equivalence, by Lemma 2.6.2, as well as a cofibration. The class of morphisms which are cofibrations and f-local equivalences is saturated, by Lemma 2.6.1. It follows that all φ-cofibrations are f-local equivalences. The definition of φ-injective (see 1.3) implies q has the right lifting property with respect to φ and, hence, with respect to all $j : C \to D$ of J. Hence the result follows from Lemma 2.14. ■

One can also make an argument for Lemma 2.6 directly from Lemma 2.14, by means of a transfinite small object argument.

The reader who has gotten this far will have noticed that we used Bousfield factorization twice: once to produce the functor $\mathcal{L}(\cdot)$ and once to prove Lemma 2.6. Now, local objects are produced using $\mathcal{L}(X)$ and fibrant objects by using Lemma 2.6 to factor $X \to *$ as a trivial cofibration followed by a fibration; therefore, they might be different. However, we have:

PROPOSITION 2.15. *A space $X \in \mathbf{S}$ is f-local if and only if it is fibrant in the f-local model category structure.*

PROOF: Suppose that $j : C \to D$ is a cofibration and an f-local equivalence. in particular that the induced map

$$j^* : \mathbf{Hom}(D, X) \to \mathbf{Hom}(C, X)$$

is a trivial fibration for all f-local spaces X. The map j^* is surjective in degree 0, so that the lifting problem

$$\begin{tikzcd} C \arrow[r] \arrow[d, "j"'] & X \arrow[d] \\ D \arrow[r] \arrow[ur, dashed] & * \end{tikzcd}$$

can always be solved. Every f-local space is therefore fibrant in the f-local closed model structure.

Now suppose that X is fibrant. The map $f : A \to B$ is an f-local cofibration by Lemma 1.9, and is therefore an f-local equivalence by Lemma 2.1 as well as a cofibration. Each induced map

$$A \times \Delta^n \cup_{A \times \partial\Delta^n} B \times \partial\Delta^n \to B \times \Delta^n$$

is an f-local equivalence and a cofibration, by Lemma 2.6, so that the map

$$f^* : \mathbf{Hom}(B, X) \to \mathbf{Hom}(A, X)$$

is a trivial Kan fibration. In particular, the space X is f-local. ■

3. Bousfield localization.

Let $\mathbf{S}$ be the category of simplicial sets and E_* a generalized homology theory which satisfies the limit axiom in the sense that it preserves filtered colimits. A space $Z \in \mathbf{S}$ is E_*-local if and only if Z is fibrant, and any diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "g"'] & Z \\ Y \arrow[ur, dashed] & \end{tikzcd}$$

with E_*g and isomorphism can be completed uniquely up to homotopy. The E_*-localization of a space X is a map $\eta : X \to Z$ with Z E_*-local and $E_*\eta$ an isomorphism. One easily checks that if such a localization exists, it is unique up to homotopy. The existence hinges on the existence of an appropriate Bousfield factorization and, hence, is best described under the rubric of the previous two sections.

DEFINITON 3.1. *A morphism $g : X \to Y$ in $\mathbf{S}$ is said to be*

1) *an E_*-equivalence if E_*g is an isomorphism,*
2) *an E_*-cofibration if it is a cofibration in $\mathbf{S}$, and*
3) *an E_*-fibration if it has the right lifting property with respect to all E_* trivial fibrations.*

Then one has

THEOREM 3.2. *With these definitions and its usual simplicial structure, $\mathbf{S}$ becomes a simplicial model category.*

This will be proved below.

COROLLARY 3.3.

1) *A space $Z \in \mathbf{S}$ is E_*-local if and only if it is E_*-fibrant.*
2) *Every space $X \in \mathbf{S}$ has an E_*-localization.*

PROOF: For 1) first assume Z is E_* fibrant and factor any E_*-equivalence $X \to Y$ as

$$X \xrightarrow{i} W \xrightarrow{q} Y$$

where q is a trivial fibration and i is a cofibration. Then i will be an E_*-trivial cofibration, so $\mathbf{Hom}(W, Z) \to \mathbf{Hom}(X, Z)$ is a weak equivalence. The trivial fibration q has a section $\sigma : Y \to W$ which is a trivial cofibration cofibration, so $\mathbf{Hom}(Y, Z) \to \mathbf{Hom}(W, Z)$ is a weak equivalence. It follows that Z is E_*-local. Conversely, if Z is E_*-local and $X \to Y$ is an E_*-trivial cofibration, then we need to solve any lifting problem

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Z \\ Y \arrow[ur, dashed] & \end{tikzcd}$$

Since Z is E_* local it can be solved up to homotopy. Since Z is fibrant, the original lifting problem can be solved by appropriate use of the homotopy extension property.

For 2), simply factor the unique map $X \to *$ as

$$X \xrightarrow{j} Z \xrightarrow{q} *$$

where j is a trivial E_*-cofibration and q is an E_*-fibration. (Compare the proof of Proposition 1.13.) ■

COROLLARY 3.4. *Let $j : X \to Z$ be an E_*-localization of X. Then the map j is up to homotopy, both the terminal E_*-equivalence out of X and the initial map to an E_*-local space.*

PROOF: Both clauses follow from the definition of localization. ■

The difficulty in proving Theorem 3.3 arises in verifying the "trivial cofibration-fibration" factoring axiom. This is done by using Bousfield factorization for a particular map $f : A \to B$ in $\mathbf{S}$. The next three lemmas construct and identify the map f. Choose an infinite cardinal β greater than the cardinality of E_* — that is, greater than the cardinality of the set $\bigsqcup_n E_n$.

LEMMA 3.5. *Let $g : C \to D$ be a cofibration and an E_* equivalence. Let $x \in D$ be a simplex. Then there is a sub-simplicial set $D_0 \subseteq D$ so that $x \in D$, $\#B \leq \beta$, and*

$$C \cap D_0 \to D_0$$

is an E_* equivalence.

PROOF: We use that for any simplicial set Z, the natural map

$$\varinjlim_{\alpha} E_* Z_\alpha \to E_* Z$$

is an isomorphism, where $Z_\alpha \subseteq Z$ runs over the sub-simplicial sets with finitely many non-degenerate simplices.

We recursively define a sequence (over the natural numbers) of sub-simplicial sets

$$K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq D$$

with the properties that $x \in K_0$, $\#K_s \leq \beta$, and

$$\widetilde{E}_*(K_n/K_n \cap C) \to \widetilde{E}_*(K_{n+1}/K_{n+1} \cap C)$$

is the zero map. Then we can set $D_0 = \bigcup K_n$.

For K_0, choose any sub-simplicial set with $\#K_0$ finite and $x \in K_0$. Having defined K_n, produce K_{n+1} as follows. For each $y \in \widetilde{E}_*(K_n/K_n \cap C)$ there is a finite sub-complex $Z_y \subseteq D/X$ so that y maps to zero in $\widetilde{E}_*((K_n/K_n \cap C) \cup Z_y)$. Choose a finite sub-complex $Y_y \subseteq Y$ that maps onto Z_y and let $K_{n+1} = K_n \cup \bigcup_y Y_y$. ■

LEMMA 3.6. *A morphism $q : Z \to W$ in $\mathbf{S}$ is an E_* fibration if and only if it has the right lifting property with respect to all E_*-trivial cofibrations $X \to Y$ with $\#Y \leq \beta$.*

PROOF: Use the same Zorn's lemma argument as for Lemma 2.14. ■

THE PROOF OF THEOREM 3.3: As in Lemma 2.5, a morphism is at once an E_* fibration and an E_*-equivalence if and only if it is a trivial fibration. Then all but the "trivial cofibration-fibration" factorization axiom follow immediately. For the final factorizations, let J be a set of E_*-trivial cofibrations $j_\alpha : C_\alpha \to D_\alpha$ containing one representative of each isomorphism class of E_* trivial cofibrations $j : C \to D$ with $\#D \leq \beta$. Let

$$f = \bigsqcup j_\alpha : \bigsqcup C_\alpha \to \bigsqcup D_\alpha$$

with the coproduct over J. Fix a morphism $X \to Y$ in $\mathbf{S}$ and let

$$X \to E_f \to Y$$

be the Bousfield Factorization with respect to f. Then the Mayer-Vietoris sequence and the Main Construction 1.16 implies $X \to E_f$ is an E_* trivial cofibration and, using Lemma 3.6 and arguing as in Proposition 1.17, one sees $E_f \to Y$ is an E_*-fibration.

To prove **SM7** we know it is sufficient to show that if $C \to D$ is an E_* trivial cofibration and $K \to L$ is any cofibration. Then

$$D \times K \cup_{C \times K} C \times L \to D \times L$$

is a trivial E_*-cofibration. Now use the Mayer-Vietoris sequence. ■

We give a brief review of the homotopical underpinnings of the Bousfield-Kan p-completion of spaces. The main source is of course [14], but many of the ideas here are also examined in [37].

Let **tow** be the category of towers $X = \{X_n\}$ of simplicial sets with morphisms the pro-maps

$$\hom(X, Y) = \varprojlim_n \varinjlim_k \hom_{\mathbf{S}}(X_k, Y_n).$$

Thus a morphism $X \to Y$ is an equivalence class of tower maps, meaning a "commutative ladder" of the form

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & X_{k_{s+1}} & \longrightarrow & X_{k_s} & \longrightarrow \cdots & X_{k_2} & \longrightarrow & X_{k_1} \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & Y_{n_{s+1}} & \longrightarrow & Y_{n_s} & \longrightarrow \cdots & Y_{n_2} & \longrightarrow & Y_{n_1}
\end{array}$$

where the horizontal maps are induced from the tower projections of X and Y and $\varinjlim k_s = \infty = \varinjlim n_s$.

There is a simplicial model category structure on **tow** where a pro-map $f : X \to Y$ is a weak equivalence if the induced map

$$f^* : \varinjlim H^*(X_n, \mathbb{F}_p) \longleftarrow \varinjlim H^*(Y_n, \mathbb{F}_p)$$

is an isomorphism, and $f : X \to Y$ is a cofibration if $\varprojlim X_n \to \varprojlim Y_n$ is an injection of simplicial sets. If $X \in \mathbf{S}$ is a space, the *Bousfield-Kan p-completion* of X is defined as follows: regard $X \in \mathbf{tow}$ as a constant tower, choose a weak equivalence $X \to Y$ with Y fibrant and set

$$X_p = \varprojlim Y_n.$$

Note that Y is actually a tower of fibrations. There is a map $\eta : X \to X_p$ called the completion map, and X is *p-complete* if this map is a weak equivalence. The class of p-complete spaces is closed under homotopy inverse limits and any p-complete space is $H_*(\cdot, \mathbb{F}_p)$ local. Note also that if $X \to Y$ is an $H_*(\cdot, \mathbb{F}_p)$ isomorphism, then $X_p \simeq Y_p$.

If $X \to \mathbb{F}_p^{\cdot} X$ is the Bousfield-Kan resolution of X (see Example VII.4.1) then the total tower of this cosimplicial space $\{\mathrm{Tot}_n \mathbb{F}_p^{\cdot} X\}$ (see Section VII.5) is a fibrant model for X in **tow** by [21], hence

$$X_p \simeq \mathrm{Tot}(\mathbb{F}_p^{\cdot} X).$$

This is the original definition of the completion. See also Example VII.7.18.

To analyze the homotopy type of X_p we introduce and discuss the p-completion functor on abelian groups. If A is an abelian group define

$$A_p = \varprojlim(\mathbb{Z}/p^n\mathbb{Z} \otimes A).$$

A group is p-complete if the map $A \to A_p$ is an isomorphism. We write $\mathbf{Ab}_p$ for the category of p-complete groups. Then completion is left adjoint to the inclusion functor $\mathbf{Ab}_p \to \mathbf{Ab}$. Note that the functor $A \mapsto A_p$ is neither left nor right exact, which implies that $\mathbf{Ab}_p$ is not an abelian sub-category. Nonetheless, completion has left derived functors $L_s(\cdot)_p$ and one has that $L_s(A)_p = 0$ for $s > 1$, a short exact sequence

$$0 \to \varprojlim{}^1 \operatorname{Tor}(\mathbb{Z}/p^n\mathbb{Z}, A) \to L_0(A)_p \to A_p \to 0,$$

and a natural isomorphism

$$L_1(A)_p \cong \varprojlim \operatorname{Tor}(\mathbb{Z}/p^n\mathbb{Z}, A).$$

See [39]. For example, if $\mathbb{Z}/p^\infty\mathbb{Z} = \varinjlim \mathbb{Z}/p^n\mathbb{Z}$, then $L_0(\mathbb{Z}/p^\infty\mathbb{Z})_p = 0$ and $L_1(\mathbb{Z}/p^\infty\mathbb{Z})_p \cong \mathbb{Z}_p$, the p-adic numbers. Since

$$\operatorname{Tor}(\mathbb{Z}/p^n\mathbb{Z}, A) \cong \operatorname{Hom}(\mathbb{Z}/p^n\mathbb{Z}, A)$$

one has

$$L_1(A)_p \cong \varprojlim \operatorname{Hom}(\mathbb{Z}/p^n\mathbb{Z}, A) \cong \operatorname{Hom}(\mathbb{Z}/p^\infty\mathbb{Z}, A)$$

from which is follows that

$$L_0(A)_p \cong \operatorname{Ext}(\mathbb{Z}/p^\infty\mathbb{Z}, A).$$

The smallest abelian sub-category of $\mathbf{Ab}$ containing the p-complete groups is the category of groups so that

$$\operatorname{Hom}(\mathbb{Z}[1/p], A) = 0 = \operatorname{Ext}(\mathbb{Z}[1/p], A).$$

These are known variously as Ext $-p$ *complete groups* , *p-cotorsion groups* , or *weakly p-complete groups*. See Section 4 of [44].

The functors of p-completion and its derived functors can be extended to the class of nilpotent groups, which are those groups from which the lower central series in eventually zero. See [14].

Now we consider *nilpotent spaces* X. These are connected simplicial sets X for which $\pi_1 X$ is a nilpotent group and for which $\pi_1 X$ acts nilpotently on π_n for all n. See Section VI.6 for complete definitions. The main result is the following.

THEOREM 3.7. *Let X be a nilpotent space. Then*

1) $H_*(X, \mathbb{F}_p) \to H_*(X_p, \mathbb{F}_p)$ is an isomorphism and X_p is the $H_*(\cdot, \mathbb{F}_p)$ localization of X;

2) for all $n \geq 1$ there is short exact sequence

$$0 \to L_0(\pi_n X)_p \to \pi_n X_p \to L_1(\pi_{n-1} X)_p \to 0.$$

This sequence is split, but not naturally.

Here is an outline of the proof. First one shows the result for $K(A, n)$ where A is an abelian group and $n \geq 1$. If

$$0 \to F_1 \to F_0 \to A \to 0$$

is a free resolution of A, let K be the simplicial abelian group with normalized chain complex

$$\cdots \to 0 \to F_1 \to F_0 \to \cdots \to 0$$

with F_0 in degree n. Then K is weakly equivalent to $K(A, n)$ and the simplicial abelian group K_p is a model for $K(A, n)_p$.

Next is X is nilpotent, we have, by the results of Section VI.6, a refined Postnikov tower for X; in particular, X may be written as $X \simeq \varprojlim_k X_k$ where the X_k fit into a tower and each successive stage is built by a pullback diagram

$$\begin{array}{ccc} X_k & \longrightarrow & WK(A, n_k) \\ \downarrow & & \downarrow \\ X_{k-1} & \longrightarrow & K(A, n_k + 1), \end{array}$$

and $\varinjlim_k n_k = \infty$. Thus one can proceed inductively using the "nilpotent fiber lemma" of Bousfield and Kan [14]. The necessary corollary of that result which is needed here is the following.

PROPOSITION 3.8. *Let $K(A, n) \to E \to B$ be a principal fibration, and suppose $n \geq 1$. Then*

$$K(A, n)_p \to E_p \to B_p$$

is a fibration sequence up to homotopy.

This can be proved by arguing directly that it holds for

$$K(A,n) \to WK(A,n) \to K(A,n+1)$$

and then using the Serre spectral sequence. Specifically, if we define E' by the pullback diagram

$$\begin{tikzcd} E' \arrow[r] \arrow[d] & WK(A,n)_p \arrow[d] \\ B_p \arrow[r] & K(A,n)_p, \end{tikzcd}$$

there is a map $E \to E'$ which is an $H_*(\cdot, \mathbb{F}_p)$ isomorphism by the Serre spectral sequence. Since E' is p-complete — the class of p-complete spaces is closed under homotopy inverse limits — we have $E_p \simeq E'$ and the result follows.

4. Localization in simplicial model categories.

In this section we explore localization in a more general context; namely, we examine localization with respect to a cofibration between cofibrant objects in a simplicial model category. It is possible to provide axioms so that localization exists, and to provide axioms to say when the model category structure can be localized, as we did with simplicial sets in Section 2. All of these axioms are specific and concrete except one, and that is the one which, in the case of simplicial sets, required the Bousfield-Smith cardinality argument. In the next section we give a lengthy example — we apply the technology of this section to diagram categories.

Fix a simplicial model category $\mathcal{C}$ and let $f : A \to B$ be a cofibration between cofibrant objects. The following exactly mimics Definition 1.3.

DEFINITION 4.1. *A map $q : X \to Y$ in $\mathcal{C}$ is an f-injective if q is a fibration and*

$$\mathbf{Hom}(B,X) \to \mathbf{Hom}(B,Y) \times_{\mathbf{Hom}(A,Y)} \mathbf{Hom}(A,X)$$

is a trivial fibration. An object $X \in \mathcal{C}$ is f-local if the unique map $X \to *$ to the terminal object is an f-injective.

Corresponding to f-injectives we have f-cofibrations, as in Definition 1.7.

DEFINITION 4.2. *A cofibration $j : C \to D$ is an* f-cofibration *if the induced fibration*

$$\mathbf{Hom}(D,X) \to \mathbf{Hom}(C,X) \times_{\mathbf{Hom}(C,Y)} \mathbf{Hom}(D,Y)$$

is a weak equivalence for all f-injectives $X \to Y$. Let $\mathcal{C}_f$ be the class of f-cofibrations.

We now work through some of the results of Section 1 to see what must be required of $\mathcal{C}$ in order that localization exist.

LEMMA 4.3.

1) *A morphism $q : X \to Y$ in $\mathcal{C}$ is an f-injective if and only if it is a fibration and has the right lifting property with respect to all maps*

$$A \otimes L \coprod_{A \otimes K} B \otimes K \to B \otimes L$$

where $K \to L$ is a cofibration in $\mathbf{S}$ and $\#L$ is finite.

2) *If $q : X \to Y$ is an f-injective and $K \in \mathbf{S}$, then*

$$\mathbf{hom}_{\mathcal{C}}(K, X) \to \mathbf{hom}_{\mathcal{C}}(K, Y)$$

is an f-injective.

The proofs are the same and this gives the flavor of how one must modify the statements of the previous section. Then Lemmas 1.9, 1.10, and 1.14 go through with the obvious changes.

The existence of Bousfield factorization, however, requires a "smallness" assumption. We codify this in the following definition, which is a modification of Bousfield's notion of s-definite .

DEFINITION 4.4. *Let β be a cardinal number. A β-*cofibration sequence *in $\mathcal{C}$ is a diagram $C_\bullet \in \mathcal{C}$ over $\mathbf{Seq}(\beta)$ with the properties that*

1) *for each successor ordinal $s < \beta$, $C_{s-1} \to C_s$ is a cofibration*
2) *for each limit ordinal $s < \beta$, $\varinjlim_{t<s} C_t \cong C_s$.*

REMARK 4.5. A cofibrant β-sequence in the sense of Section II.9 is a β-cofibration sequence $C_\bullet$ for which every object C_s is cofibrant.

DEFINITION 4.6. *An object $A \in \mathcal{C}$ is cofibration definite if there is a cardinal α such that for every cardinal $\beta \geq \alpha$ and every β-cofibration sequence $C_\bullet$, the natural map*

$$\varinjlim_{s<\kappa} \hom_{\mathcal{C}}(A, C_s) \to \hom_{\mathcal{C}}(A, \varinjlim_{s<\kappa} C_s) \tag{4.7}$$

is a bijection.

EXAMPLES 4.8. In the following simplicial model categories, every object is cofibration-definite:

1) Simplicial sets $\mathbf{S}$ and pointed simplicial sets $\mathbf{S}_*$.

2) Simplicial groups, simplicial abelian groups, simplicial algebras, simplicial lie algebras, and so on. The same applies to the graded analogs.

3) Compactly generated Hausdorff spaces. Notice that the map (4.7) is not, in this category, an isomorphism for an arbitrary colimit. However, cofibrations are closed inclusions so $\varinjlim_{s<\kappa} C_s \cong \bigcup_s C_s$. Thus we need only take α greater than the cardinality $2^{\#A}$ of the power set of A.

4) The category S^I of I diagrams in simplicial sets S. Here I is small category.

PROPOSITION 4.9 (BOUSFIELD FACTORIZATION). *Let $\mathcal{C}$ be a simplicial model category satisfying the additional hypotheses that*

1) *every object of $\mathcal{C}$ is cofibration definite; and*
2) *there is a set $\mathcal{F}$ of trivial cofibrations so that a morphism in $\mathcal{C}$ is a fibration if and only if it has the right lifting property with respect to all morphisms in $\mathcal{F}$.*

Then any morphism $g : X \to Y$ in $\mathcal{C}$ can be factored naturally as

$$X \xrightarrow{j} Z \xrightarrow{q} Y.$$

where j is an f-cofibration and q is an f-injective.

The proof follows the Main Construction 1.16 and Proposition 1.17. To choose β, notice that for all $L \in \mathbf{S}$ with $\#L$ finite and $C \to D$ in $\mathcal{F}$, the functors $\hom_{\mathcal{C}}(B \otimes L, \ )$ and $\hom_{\mathcal{C}}(D, \ )$ will commute with β-cofibration sequences for some large enough β. Note than in all the examples given in 4.8, the closed model category structure is cofibrantly generated (see Definition II.6.6) which implies 4.9.2.

In the presence of Bousfield factorization, one also has localization, provided one takes a little care. Let $\mathbf{Loc}_f \subseteq \mathrm{Ho}(\mathcal{C})$ be the full sub-category with objects the f-local objects. An *f-localization* will be an idempotent functor $L_f : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C})$ that restricts to a functor $L_f : \mathrm{Ho}(\mathcal{C}) \to \mathbf{Loc}_f$ left adjoint to the inclusion. If it exists it is unique.

PROPOSITION 4.10. *If Bousfield factorization holds in $\mathcal{C}$, f-localization exists.*

PROOF: If every object in $\mathcal{C}$ is cofibrant so that the notion of homotopy makes sense, we can proceed as in 1.11–1.13. If not, we must modify the procedure somewhat.

If $X \in \mathcal{C}$, choose a trivial fibration $Y \to X$ with Y cofibrant. Y is unique up to unique homotopy equivalence over X. Define $\mathcal{L}Y$ by factoring $Y \to *$ as

$$Y \xrightarrow{j} \mathcal{L}Y \xrightarrow{q} *$$

where j is an f-cofibration and q is an f-fibration. Then $L_f X = \mathcal{L}Y$. The reader is invited to modify 1.11–1.13 to complete the proof.

Notice that if $\mathcal{C}$ has natural factorizations (see Section II.5),' then $\mathcal{L}Y$ can be chosen naturally in X. ■

We now turn to the model category structure. As in the previous proof we have a functor $\mathcal{L} : \mathcal{C} \to \mathcal{C}$ and a natural f-cofibration $X \to \mathcal{L}X$ so that $\mathcal{L}X$ is f-local — although $\mathcal{L}X$ is not a model for the f-localization of X unless X is cofibrant. But see Lemma 4.18 below.

LEMMA 4.11. *Suppose that for all cofibrant $C \in \mathcal{C}$ and all cofibrations $U \to V$ of $\mathcal{C}$, the function $\hom_{\mathcal{C}}(C, U) \to \hom_{\mathcal{C}}(D, V)$ is an injection. Then the functor $\mathcal{L} : \mathcal{C} \to \mathcal{C}$ preserves cofibrations.*

PROOF: When recapitulating the proof of Lemma 1.22, one finds that one needs to know that for all $j_\alpha : C_\alpha \to D_\alpha$ in J, the map of simplicial sets

$$\mathbf{Hom}_{\mathcal{C}}(C_\alpha, E_s X) \to \mathbf{Hom}_{\mathcal{C}}(D_\alpha, E_s Y)$$

is a cofibration; that is, we need it to be an inclusion. This is equivalent to the assertion that for all n,

$$\hom_{\mathcal{C}}(C_\alpha \otimes \Delta^n, E_s X) \to \hom_{\mathcal{C}}(C_\alpha \otimes \Delta^n, E_s Y)$$

is an injection. Since, by induction, $E_s X \to E_s Y$ is a cofibration and $C_\alpha \otimes \Delta^n$ is cofibrant, this follows from the hypothesis. ■

REMARK 4.12. The hypothesis of Lemma 4.11 is relatively innocuous. It is satisfied, for example, by all the simplicial model categories listed in Example 4.8.

DEFINITION 4.13. *A morphism $C \to D$ between cofibrant objects is an f-local equivalence if $\mathbf{Hom}_{\mathcal{C}}(D, X) \to \mathbf{Hom}_{\mathcal{C}}(C, X)$ is a weak equivalence for all f-local objects X of $\mathcal{C}$. More generally, a morphism $g : C \to D$ is an f-local equivalence if there is a diagram*

(4.14)
$$\begin{tikzcd} C_0 \arrow[r, "g_0"] \arrow[d, "p"'] & C_0 \arrow[d, "q"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

where p and q are weak equivalences, C_0 and D_0 are cofibrant, and $g_0 : C_0 \to D_0$ is a f-local equivalence.

A diagram such as (4.14) will be called a *cofibrant approximation* to g.

REMARK 4.15. If $g_0 : C_0 \to D_0$ is an f-local equivalence for some cofibrant approximation to g, it is for all. To see this, choose a cofibrant approximation

$$\begin{tikzcd} C_1 \arrow[r, "g_1"] \arrow[d] & D_1 \arrow[d] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

with $D_1 \to D$ a trivial fibration and $C_1 \to C \times_D D_1$ a trivial fibration. Then any cofibrant approximation (4.14) lifts to a diagram

$$\begin{tikzcd} C_0 \arrow[r, "g_0"] \arrow[d] & D_0 \arrow[d] \\ C_1 \arrow[r, "g_1"'] & D_1 \end{tikzcd}$$

and, hence, for an f-local $X \in \mathcal{C}$ there is a diagram

$$\begin{tikzcd} \mathbf{Hom}_{\mathcal{C}}(D_1, X) \arrow[r, "g_1^*"] \arrow[d] & \mathbf{Hom}_{\mathcal{C}}(\mathcal{C}_1, X) \arrow[d] \\ \mathbf{Hom}_{\mathcal{C}}(D_0, X) \arrow[r, "g_0^*"'] & \mathbf{Hom}_{\mathcal{C}}(D_1, X). \end{tikzcd}$$

The vertical maps are weak equivalences by **SM7** and the claim follows.

We'd now like to define f-local cofibrations in $\mathcal{C}$ to be cofibrations in $\mathcal{C}$, and f-local fibrations to be those morphisms with the appropriate lifting property. Without a further hypothesis, however, we cannot prove a crucial lemma; namely, it will not be clear that the class of morphisms which are at once cofibrations and f-local equivalences is closed under pushouts. Thus:

DEFINITION 4.16. *A closed model category $\mathcal{C}$ is rigid on cofibrations if*

1) *given any diagram*

$$\begin{tikzcd} D_1 \arrow[d] & C_1 \arrow[l, "j_1"'] \arrow[r] \arrow[d] & X_1 \arrow[d] \\ D_2 & C_2 \arrow[l, "j_2"] \arrow[r] & X_2 \end{tikzcd}$$

with j_1 and j_2 cofibrations and all vertical maps weak equivalences, then the induced map on pushouts $D_1 \cup_{C_1} X_1 \to D_2 \cup_{C_2} X_2$ is a weak equivalence.

2) *Let κ be an ordinal number and let $X_\bullet \to Y_\bullet$ be a morphism of diagrams over $\mathbf{Seq}(\kappa)$ such that for each $s < \kappa$, $X_s \to Y_s$ is a weak equivalence, the maps $X_s \to X_{s+1}$, $Y_s \to Y_{s+1}$ are cofibrations and if s is a limit ordinal, $\varinjlim_{t<s} X_t = X_s$ and $\varinjlim_{t<s} Y_t = Y_s$. Then the induced map*

$$\varinjlim_{s<\kappa} X_s \to \varinjlim_{s<\kappa} Y_s$$

is a weak equivalence.

Before exploring the implications of this definition for localized closed model categories, we give some examples.

EXAMPLES 4.17.

1) Any closed model category in which every object is cofibrant is rigid on cofibrations (Lemma II.9.8, Corollary II.9.25).

2) Lemma II.9.21 says that any *proper* closed model category satisfies 4.16.1. In many cases, 4.16.2 is a triviality. Thus, for example, compactly generated Hausdorff spaces in the usual structure is rigid on cofibrations. And so are the categories of simplicial algebras, simplicial commutative algebras, simplicial Lie algebras, simplicial groups, simplicial abelian groups and their graded analogs.

3) Let $\mathcal{C}$ be a closed model category which is rigid on cofibrations and let $\mathcal{C}^I$ be the category of diagrams in $\mathcal{C}$ for some small category I. Suppose $\mathcal{C}^I$ has a closed model category structure where a morphism of diagrams $X_\bullet \to Y_\bullet$ is a weak equivalence if and only if $X_i \to Y_i$ is a weak equivalence for all $i \in I$, and if a morphism $X_\bullet \to Y_\bullet$ is a cofibration, then $X_i \to Y_i$ is a cofibration. (This applies to $\mathbf{S}^I$ in its usual structure (Example II.6.9), for example.) Then $\mathcal{C}^I$ is rigid on cofibrations. This follows from the fact that colimits in $\mathcal{C}^I$ are formed level-wise; that is, for any diagram of objects $X_\bullet$ in $\mathcal{C}^I$,

$$(\varinjlim X_\bullet)_i \cong \varinjlim (X_i).$$

The crucial implication of this property of the rigidity hypothesis is the following.

LEMMA 4.18. *Assume the hypotheses of Proposition 4.9, and suppose the simplicial closed model category $\mathcal{C}$ is rigid on cofibrations. Then the functor $\mathcal{L} : \mathcal{C} \to \mathcal{C}$ preserves all weak equivalences.*

PROOF: In advance we know only that $\mathcal{L}$ preserves weak equivalences between cofibrant objects. Fix any weak equivalence $X \to Y$. We will modify the proof of Proposition 1.17 to produce a diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \mathcal{L}'(X) \\ \downarrow & & \downarrow \\ Y & \xrightarrow[\eta_Y]{} & \mathcal{L}'(Y) \end{array}$$

where η_X and η_Y are f-cofibrations and $\mathcal{L}'(X) \to \mathcal{L}'(Y)$ is a weak equivalence between f-local objects. The result will follow.

As in the proof of Proposition 1.17 we will produce, for each ordinal number s, an object E'_sX, functorial in X, so that there is an f-cofibration $E'_sX \to E'_{s+1}X$, if s is a limit ordinal, then

$$\varinjlim_{t<s} E'_tX = E'_sX,$$

and, for s large enough, E'_sX is f-local. Then $\mathcal{L}'(X) = E'_sX$ for s large enough. However, we will modify the construction so that $E'_sX \to E'_sY$ is a weak equivalence for all s.

As before, set $E'_0X = X$. Then $E'_0X \to E'_0Y$ is a weak equivalence. Suppose E'_sX has been defined and $E'_sX \to E'_sY$ is a weak equivalence. Since $\mathcal{C}$ is cofibrantly generated, there is a natural square

$$\begin{array}{ccc} E'_sX & \longrightarrow & \overline{E}_sX \\ \downarrow & & \downarrow \\ E'_sY & \longrightarrow & \overline{E}_sY \end{array}$$

where the horizontal morphisms are trivial cofibrations and $\overline{E}_sX$ and $\overline{E}_sY$ are fibrant. Now apply the Main Construction 1.16 to $\overline{E}_sX$ to define $E_{s+1}X$; thus, there is a pushout diagram

$$\begin{array}{ccc} \coprod_I C_\alpha \otimes \mathbf{Hom}(C_\alpha, \overline{E}_sX) & \longrightarrow & \overline{E}_sX \\ \downarrow & & \downarrow \\ \coprod_I D_\alpha \otimes \mathbf{Hom}(C_\alpha, \overline{E}_sX) & \longrightarrow & E'_{s+1}X. \end{array}$$

Now, since $\overline{E}_s X \to \overline{E}_s Y$ is a weak equivalence between fibrant objects and C_α is cofibrant,

$$C_\alpha \otimes \mathbf{Hom}(C_\alpha, \overline{E}_s X) \to C_\alpha \otimes \mathbf{Hom}(D_\alpha, \overline{E}_s Y)$$

is a weak equivalence. Similarly with D_α in place of C_α. Thus Condition 4.16.1 implies $E'_{s+1}X \to E'_{s+1}Y$ is a weak equivalence. Since any trivial cofibration is an f-cofibration, one has as in the proof of Proposition 1.17, that $E'_s X \to E'_{s+1} Y$ is an f-cofibration.

If s is a limit ordinal, define $E'_s X = \varinjlim_{t<s} E'_t X$. Then Condition 4.16.2 implies $E'_s X \to E'_s Y$ is a weak equivalence. The remainder of the proof is accomplished as in Proposition 1.17. ∎

COROLLARY 4.19. *Assume the hypotheses of Lemma 4.18. Then a morphism $C \to D$ in $\mathcal{C}$ is an f-local equivalence if and only if $\mathcal{L}(C) \to \mathcal{L}(D)$ is a weak equivalence.*

PROOF: If C and D are cofibrant, this can be proved as in Lemma 2.1.2. For the general case, $g : C \to D$ is an f-local equivalence if and only if $g_0 : C_0 \to D_0$ is an f-local equivalence for some cofibrant approximation for g. Now consider the diagram

$$\begin{tikzcd} \mathcal{L}(C_0) \arrow[r, "\mathcal{L}g_0"] \arrow[d] & \mathcal{L}(D_0) \arrow[d] \\ \mathcal{L}(C) \arrow[r, "\mathcal{L}g"'] & \mathcal{L}(D). \end{tikzcd}$$

By Lemma 4.18, the vertical maps are weak equivalences; hence $\mathcal{L}g_0$ is a weak equivalence if and only if $\mathcal{L}g$ is a weak equivalence. ∎

COROLLARY 4.20. *Assume the hypotheses of Lemma 4.18.*

1) *The class of morphisms in $\mathcal{C}$ which are at once cofibrations and f-local equivalences is closed under pushouts and colimits over ordinal numbers.*
2) *Let $j : C \to D$ be a morphism in $\mathcal{C}$ that is at once a cofibration and an f-local equivalence, and $K \to L$ be any cofibration in $\mathbf{S}$. Then if $\mathcal{L}(\cdot)$ preserves cofibrations*

$$D \otimes K \cup_{C \otimes K} C \otimes L \to D \otimes L$$

is a cofibration and an f-local equivalence.

PROOF: Consider the maps

$$D \overset{j}{\leftarrow} C \to X$$

where j is a cofibration and an f-local equivalence. We need to show that the cofibration $X \to D \cup_C X$ is an f-local equivalence. If D, C, and X are cofibrant, then one can use the proof of Lemma 2.7.1. If not, find a diagram

$$\begin{tikzcd} D_0 \arrow[d] & C_0 \arrow[l, "j_0"'] \arrow[r] \arrow[d] & X_0 \arrow[d] \\ D & C \arrow[l, "j"] \arrow[r] & X \end{tikzcd}$$

where the vertical maps are weak equivalences, j_0 is a cofibration and D_0, C_0, and X_0 are cofibrant. Then j_0 is a f-local equivalence by Lemma 4.18 and Corollary 4.19. Hence $X_0 \to D_0 \cup_{C_0} X_0$ is an f-local equivalence. By 4.16.1 there is a diagram

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & D_0 \cup_{C_0} X_0 \arrow[d] \\ X \arrow[r] & D \cup_C X \end{tikzcd}$$

with the vertical maps weak equivalences. Hence $X \to D \cup_C X$ is an f-local equivalence.

A similar argument using 4.16.2 handles the case of colimits over ordinal numbers. Part 2) follows from part 1) exactly as in the proof of Lemma 2.7.2. ■

HYPOTHESES/DEFINITION 4.21. We now recapitulate. We have a simplicial model category $\mathcal{C}$ such that

1) $\mathcal{C}$ is cofibrantly generated;
2) every object of $\mathcal{C}$ is cofibration definite;
3) $\mathcal{C}$ is rigid on cofibrations.

Then Lemma 4.18, Corollary 4.19 and Corollary 4.20.1 hold. If, in addition, we have

4) for all cofibrant objects $C \in \mathcal{C}$ and all cofibrations $U \to V$, the functor $\hom_{\mathcal{C}}(C, U) \to \hom(C, V)$ is an injection,

then $\mathcal{L}(\cdot)$ preserves cofibrations and Corollary 4.20.2 holds. Examples of $\mathcal{C}$ satisfying hypotheses 1)–4) are given in 4.17.

We now define a morphism in $\mathcal{C}$ to be an *f-local cofibration* if it is a cofibration and and *f-local fibration* if it has the right lifting property with respect to all morphisms which are at once f-local equivalences and cofibrations. The notions of f-local equivalence, f-local cofibration, and f-local fibration together will be called the f-local structure.

We now add one more hypothesis of a less concrete nature:

5) there is a set J of morphisms $j : C \to D$ in $\mathcal{C}$ so that

   i) every element of J is a cofibration and an f-local equivalence; and
   ii) a morphism $X \to Y$ in $\mathcal{C}$ is an f-local fibration if and only if it has the right lifting property with respect to all elements of J.

Proposition 4.22. *Let $\mathcal{C}$ be a simplicial model category satisfying 4.21.1–4. Then with the f-local structure defined above and the same simplicial structure, $\mathcal{C}$ becomes a simplicial model category provided*

(∗) *every morphism $X \to Y$ can be factored*

$$X \xrightarrow{j} Z \xrightarrow{q} Y$$

where q is an f-local fibration and j is at once a cofibration and an f-local equivalence.

Proof: This goes exactly as in the proof of Theorem 2.3. The analog of Lemma 2.5 is easy in the presence of Corollary 4.19 (use the same proof as in 2.5) and we have assumed Lemma 2.6. Axiom **SM7** follows from Corollary 4.20.2.■

In order to apply this result, we will need to verify the factorization hypothesis. In practice this is done as in the proof of Lemma 2.6. To be specific, we have

Proposition 4.23. *Suppose that $\mathcal{C}$ is a simplicial model category satisfying 4.21.1–3 and 4.21.5. Then every morphism $X \to Y$ can be factored*

$$X \xrightarrow{j} Z \xrightarrow{q} Y$$

where q is an f-local fibration and j is at once a cofibration and an f-local equivalence.

5. Localization in diagram categories.

Suppose that I is a simplicial category, and form the category $\mathbf{S}^I$ of simplicial functors, as in Section VIII.1. This means in particular that I has a set of objects, and the morphism objects between objects i and j of I consist of simplicial sets $\mathbf{Hom}(i,j)$, so that a simplicial functor $X : I \to \mathbf{S}$ consists of a simplicial set-valued function $i \mapsto X_I$ defined on $\mathrm{Ob}(I)$ and simplicial set maps $X : \mathbf{Hom}(i,j) \to \mathbf{Hom}(X_i, X_j)$ which respect composition and identities in the obvious sense. Recall that a morphism $f : X \to Y$ of $\mathbf{S}^I$ is a collection of simplicial set maps $f : X_i \to Y_i$, $i \in \mathrm{Ob}(I)$, which behaves like a natural transformation in each simplicial degree. Recall further that there is a natural simplicial structure on $\mathbf{S}^I$ for which $X \otimes K$ is defined for $X \in S^I$ and simplicial sets K by the formula

$$(X \otimes K)_i = X_i \times K,$$

and for which $\mathbf{Hom}(X,Y)$ has n-simplices of the form::

$$\mathbf{Hom}(X,Y)_n = \hom_{\mathbf{S}^I}(X \otimes \Delta^n, Y).$$

It has been shown (in various places: Remark VIII.1.6, Section IV.3.1, Example II.7.11) that there is a simplicial model structure on $\mathbf{S}^I$ for which a map $f : X \to Y$ is a fibration (respectively weak equivalence) if all the component maps $f : X_i \to Y_i$ are fibrations (respectively weak equivalences) of simplicial sets. This result, which we often call the *pointwise fibration structure* is one of the key components of all our discussions of homotopy coherence, but a sensible discussion of localization in this context requires the introduction of a different structure:

Proposition 5.1 (The pointwise cofibration structure). *With the simplicial structure defined above, $\mathbf{S}^I$ has a simplicial model category structure where a morphism $f : X \to Y$ is*

1) *a weak equivalence if each $f_i : X_i \to Y_i$ is a weak equivalence;*
2) *a cofibration if each $f_i : X_i \to Y_i$ is a cofibration (that is, an injection), and*
3) *a fibration if it has the right lifting property with respect to all trivial cofibrations.*

It helps to know that the functor

$$G_i : \mathbf{S}^I \to \mathbf{S} \tag{5.2}$$

given by $G_i X = X_i$ has both a left and right adjoint. The left adjoint is given by

$$(L_i Y)(j) = Y \otimes \mathbf{Hom}_I(i,j) \tag{5.3}$$

and the right adjoint given by

$$(R_i Y)(j) = \hom_{\mathcal{C}}(\mathbf{Hom}_I(j,i), Y). \tag{5.4}$$

Note that in the trivial case where $\mathbf{Hom}_I(i,j) = \hom_I(i,j)$ one has $(L_i Y)(j) = \bigsqcup_{i \to j} Y$ and $(R_i Y)(j) = \prod_{j \to i} Y$.

PROOF OF PROPOSITION 5.1: Assume first that we can prove **CM5**. Then **CM1–CM3** are clear and we need only prove the "cofibration-trivial fibration" half of **CM4** to get a closed model category. For this consider a lifting problem in $\mathbf{S}^I$

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where i is a cofibration and p is a trivial fibration. Form the diagram

$$\begin{tikzcd} X \arrow[rr, "="] \arrow[d, "\bar{\imath}"'] & & X \arrow[d, "p"] \\ X \cup_A B \arrow[r, "j"] & Z \arrow[r, "q"] \arrow[ur, dashed] & Y \end{tikzcd}$$

where we have used **CM5** to factor $X \sqcup_A B \to Y$ as a cofibration j by a trivial fibration q. Then $q(j\bar{\imath}) = p$ is a weak equivalence and q is a weak equivalence, so ji is a trivial cofibration and the indicated lift exists. The composite $B \to X \sqcup_A B \to Z \to X$ solves the original lifting problem. Finally, Axiom **SM7** follows from the corresponding property for simplicial sets.

This leaves **CM5**. For this we use Bousfield factorization twice. The argument is completed in Lemma 5.6, once we prove a preliminary lemma. ■

If $X \in \mathbf{S}^I$ let $\#X$ be the cardinality of the simplices of $\bigsqcup_i Xi$.

LEMMA 5.5. *There is a cardinal number β so that a morphism $q : X \to Y$ is a fibration in the level-wise cofibration structure if and only if it has the right lifting property with respect to all trivial cofibrations $j : A \to B$ with $\#B \leq \beta$.*

PROOF: Let β be a fixed infinite cardinal greater than the cardinality of $\bigsqcup_{(i,j)} \mathbf{Hom}_I(i,j)$, where (i,j) runs over all pairs of objects in I. The claim is that if $j : A \to B$ is any trivial cofibration, an $x \in Bi$ is a simplex, there is a sub-object $C \subseteq B$ so that $x \in C$, $\#C \leq \beta$, and $A \cap C \to C$ is a trivial cofibration. Then the Zorn's lemma argument of Lemma 2.14 applies to finish the result.

To prove the claim, one constructs objects $C_n \subseteq B$ so that $x \in C_0$, $\#C \leq \beta$, $C_n \subseteq C_{n+1}$ and $\pi_k(|C_n|, |C_n \cap A|) \to \pi_k(|C_{n+1}|, |C_{n+1} \cap A|)$ is trivial for all $k \geq 0$ (Note that $\pi_0(P,Q)$ is the quotient set $\pi_0 P/\pi_0 Q$.) and all choices of basepoint. Then $C = \bigcup_n C_n$. To get C_n, choose C_0 to be any sub-object with $x \in C_0$ and $\#C_0 \leq \beta$. This is possible; choose $x \in C_0(i) \subseteq Bi$ where $C_0(i)$ has only finitely non-degenerate simplices and let C_0 be the image of the induced map $L_i C_0(i) \to B$ where L_i is as in (5.3). Having produced C_n, proceed as follows. For each i and each k and any choice of basepoint in $C_n(i) \cap A$, let $z \in \pi_k(|C_n(i)|, |C_n(i) \cap A|)$. There is a sub-complex $D_z^{(i)} \subseteq B(i)$ with finite many-degenerate simplices so that z is trivial in $\pi_k(|C_n(i) \cup D_z(i)|, |(C_n(i) \cup D_z(i)) \cap A(i)|)$. This is because $A \to B$ is a trivial cofibration. Let $D_z =$ image of $L_i D_z(i) \to B$ and let $C_{n+1} = C_n \cup (\bigcup_z D_z)$. ■

LEMMA 5.6. *Axiom* **CM5** *holds for the level-wise cofibration structure on* $\mathbf{S}^I$.

PROOF: For the "trivial cofibration-fibration" half of this axiom, note that trivial cofibrations are closed under pushouts and colimits over ordinal numbers. Thus we need only perform the Bousfield factorization with respect to

$$f = \bigsqcup_J j : \bigsqcup_J A \to \bigsqcup_J B$$

where J runs over a set of isomorphism representatives for all trivial cofibrations $j : A \to B$ with $\#B \leq \beta$. The argument can be completed as in Proposition 1.17.

For the "cofibration-trivial fibration" half of this axiom, we make the following observation: a morphism $q : X \to Y$ is a trivial fibration if and only if it is a fibration and $q_i : Xi \to Yi$ is a trivial fibration for all i. This follows from the fact that the functors $L_i : \mathbf{S} \to \mathbf{S}^I$ preserve trivial cofibrations. Hence $q : X \to Y$ is a trivial fibration if and only if it has the right lifting property with respect to the set of maps J' of the form $A \subset F_i\Delta^n$. Since $\#L_i\Delta^n \leq \beta$, we can perform the Bousfield factorization with respect to $g = \bigsqcup_{J'} j : \bigsqcup_{J'} C \to \bigsqcup_{J'} D$ to finish the argument. ■

The level-wise cofibration structure on $\mathbf{S}^I$ is particularly suited to localization, even though the exact nature of the fibrations is mysterious. We take some paragraphs to explain.

Fix a level-wise cofibration $f : A \to B$ in $\mathbf{S}^I$.

PROPOSITION 5.7. *Localization with respect to f exists in $\mathbf{S}^I$; that is, there is a functor $\mathcal{L} : \mathbf{S}^I \to \mathbf{S}^I$ and a natural f-cofibration $X \to \mathcal{L}X$ so that $\mathcal{L}X$ is f-local and induces a localization functor*

$$L_f : \mathrm{Ho}(\mathbf{S}^I) \to \mathbf{Loc}_f.$$

PROOF: This follows from Propositions 4.9 and 4.10. ■

Note that the analogous statement holds in the level-wise fibration structure, although one must take some care with Proposition 4.9 as not every object is cofibrant.

PROPOSITION 5.8. *The functor $\mathcal{L} : \mathbf{S}^I \to \mathbf{S}^I$ has the following properties:*

1) *if $C \to D$ is a level-wise cofibration, so is $\mathcal{L}C \to \mathcal{L}D$*
2) *there is a cardinal λ so that for all $C \in \mathbf{S}^I$*

$$\varinjlim_{\alpha} \mathcal{L}(C_\alpha) \to \mathcal{L}(C)$$

   is an isomorphism where C_α runs over the sub-objects of C with $\#C_\alpha \leq \lambda$.
3) *If C_1, C_2 are sub-objects of some $D \in \mathbf{S}^I$, then $\mathcal{L}(C_1 \cap C_2) \cong \mathcal{L}(C_1) \cap \mathcal{L}(C_2)$.*

PROOF: Part 1) follows from Lemma 4.11, Part 2) is proved as in Lemma 1.23, and Part 3) as in Lemma 1.25. ■

PROPOSITION 5.9. *The category $\mathbf{S}^I$ is rigid on level-wise cofibrations, and the functor $\mathcal{L} : \mathbf{S}^I \to \mathbf{S}^I$ preserves all weak equivalences.*

PROOF: The category $\mathbf{S}^I$ is proper since the simplicial set category is proper, so we may apply Lemmas II.9.21 and 4.18. ■

The analogous statement holds for the level-wise fibration structure. We'd now like to produce the localization model category structure. We define a morphism $q : X \to Y$ in $\mathbf{S}^I$ to be an f-local fibration if it has the right lifting property with respect to all level-wise cofibrations $j : C \to D$ which are f-local equivalences.

PROPOSITION 5.10. *There is an infinite cardinal ξ so that a morphism in $\mathbf{S}^I$ is an f-local fibration if and only if it has the right lifting property with respect to all level-wise cofibrations $j : C \to D$ which are f-local equivalences and for which $\#D \leq \xi$.*

PROOF: In the presence of Proposition 5.8, the Bousfield-Smith cardinality argument pertains. See Lemmas 2.13–2.14. ■

The next result says that the level-wise cofibrations structure localizes. We have a level-wise cofibration $f : A \to B$ in $\mathbf{S}^I$.

PROPOSITION 5.11. *The category $\mathbf{S}^I$, with its simplicial structure, has an f-local simplicial category structure where a morphism is a*

1) *weak equivalence if it is an f-local equivalence;*
2) *cofibration if it is a level-wise cofibration;*
3) *fibration if it is an f-local fibration.*

PROOF: Using Proposition 5.10, apply Propositions 4.22 and 4.23. ■

A consequence of Proposition 5.11 is that one can also localize the level-wise fibration closed model category structure on $\mathbf{S}^I$. For this one would require a cofibration $f : A \to B$ between cofibrant objects in this structure. We append the adjective "modified" to all the constructions; therefore, we have modified f-local equivalences and modified f-local fibrations. As noted above, the analogs of Propositions 5.7 and 5.9 hold. Note that a cofibration in the level-wise fibration structure is a level-wise cofibration.

LEMMA 5.12.

1) If an object X in $\mathbf{S}^I$ is f-local, it is modified f-local.

2) If a morphism $C \to D$ is cofibration in the level-wise fibration structure and a modified f-local equivalence, it is a level-wise cofibration and an f-local equivalence.

3) If a morphism $X \to Y$ is an f-local fibration, it is a modified f-local fibration.

PROOF: Part 1) follows from the definition of f-local and the fact that a fibration in the level-wise cofibration structure is a level-wise fibration. Part 2) is a consequence of Part 1) and the definition of f-local equivalence. Part 3) follows from Part 2) by a lifting argument. ■

We now have the following consequence of Proposition 5.11.

COROLLARY 5.13. *The category $\mathbf{S}^I$, with its simplicial structure has an f-local simplicial category structure, where a morphism is*

1) *a weak equivalence if it is an f-local equivalence;*
2) *a cofibration if it is a cofibration in the level-wise fibration structure;*
3) *a fibration if it is a modified f-local fibration.*

PROOF: We apply Proposition 4.22. To verify the "trivial cofibration-fibration" factorization we use Proposition 5.11. Fix a morphism: $X \to Y$. Factor g as

$$X \xrightarrow{j} Z \xrightarrow{q} Y$$

where j is a level-wise cofibration and an f-local equivalence, and q is an f-local fibration in the sense of Proposition 5.11. Now factor j as

$$X \xrightarrow{i} Z' \xrightarrow{p} Y$$

where p is a level-wise trivial fibration and i is a cofibration in the level-wise fibration structure. Then $j = pi$ is an f-local equivalence, so i is an f-local equivalence. Hence we need only verify $qp : Z' \to Y$ is a modified f-local fibration. However, any level-wise trivial fibration will be a modified f-local fibration, because it will have the right lifting property with respect to all cofibrations in the level-wise fibration structure. Finally q is a modified f-local fibration by Lemma 5.12.3. ■

We close this section with some comments on homotopy inverse limits. Let I be a discrete category. The constant diagram functor $\mathbf{S} \to \mathbf{S}^I$ sending $X \in \mathbf{S}$ to the diagram $Xi = X$ and all the maps the identity sends cofibrations to level-wise cofibrations and preserves weak equivalences. Thus the inverse limit functor

$$\varprojlim = \varprojlim_I : \mathbf{S}^I \to \mathbf{S}$$

preserves weak equivalences between fibrant objects (in the level-wise cofibration structure) and hence there is a total right derived functor

$$\underset{\longleftarrow I}{\mathrm{holim}} = \mathbf{R}\varprojlim_I : \mathrm{Ho}(\mathbf{S}^I) \to \mathrm{Ho}(\mathbf{S}).$$

Thus if $X \in \mathbf{S}^I$ one chooses a level-wise trivial cofibration $X \to Y$ to a fibrant object. Then $\varprojlim_I Y$ is a model for $\underset{\longleftarrow}{\mathrm{holim}}_I X$.

Observe that there is a much more concrete description in the case of towers — see Definition VI.1.7.

PROPOSITION 5.14. *Let $f : A \to B$ be a cofibration in $\mathbf{S}$ and suppose $X \in \mathbf{S}^I$ is an I-diagram so that Xi is f-local for all $i \in I$. Then $\underset{\longleftarrow}{\mathrm{holim}}_I X$ is f-local.*

PROOF: We work with the level-wise cofibration structure on $\mathbf{S}^I$ and the localization in $\mathbf{S}^I$ provided by Proposition 5.7.

Let $G : \mathbf{S}^I \to \mathbf{S}$ be the functor $G_i Z = Zi$ and let L_i and R_i be its right and left adjoints respectively. Let $c : \mathbf{S} \to \mathbf{S}^I$ be the constant diagram functor. Let g be the coproduct in $\mathbf{S}^I$ of the maps

$$L_i f : L_i A \to L_i B \qquad i \in I$$

and

$$cf : cA \to cB.$$

Then g is a level-wise cofibration. Choose a g-cofibration $X \to Y$ with Y g-local. Then we claim that for each i, Y_i is f-local and the maps $X_i \to Y_i$ are f-cofibrations. If this is the case $\operatorname{holim} X = \varprojlim Y$, since Y is fibrant and $\varprojlim Y$ is f-local since

$$f^* : \mathbf{Hom}(B, \varprojlim Y) \to \mathbf{Hom}(A, \varprojlim Y)$$

is isomorphic

$$cf^* : \mathbf{Hom}(cB, Y) \to \mathbf{Hom}(cA, Y)$$

which is a weak equivalence.

To see that each Y_i is local, note that

$$f^* : \mathbf{Hom}(B, Y_i) \to \mathbf{Hom}(A, Y_i)$$

is isomorphic to

$$L_i f^* : \mathbf{Hom}(L_i B, Y) \to \mathbf{Hom}(L_i A, Y)$$

which is a weak equivalence.

To see that $X_i \to Y_i$ is an f-cofibration, we prove that if $E \to D$ is an f-injective, then $R_i E \to R_i D$ is a g-injective. Then

$$\mathbf{Hom}(Y_i, E) \to \mathbf{Hom}(Y_i, D) \times_{\mathbf{Hom}(X_i, D)} \mathbf{Hom}(X_i, E)$$

is isomorphic to

$$\mathbf{Hom}(Y, R_i E) \to \mathbf{Hom}(Y, R_i D) \times_{\mathbf{Hom}(BX, R_i D)} \mathbf{Hom}(X, R_i E)$$

which will be a weak equivalence.

Now consider

$$\mathbf{Hom}(L_j B, R_i E) \to \mathbf{Hom}(L_j B, R_i D) \times_{\mathbf{Hom}(L_j A, R_i D)} \mathbf{Hom}(L_j A, R_i E). \tag{5.15}$$

This is isomorphic to

$$\mathbf{Hom}((L_j B)_i, E) \to \mathbf{Hom}((L_j B)_i, D) \times_{\mathbf{Hom}((L_j A)_i, D)} \mathbf{Hom}((L_j A)_i, E). \tag{5.16}$$

But $(L_j A)_i \to (L_j A)_i$ is the map

$$\bigsqcup f : \bigsqcup_{j \to i} A \to \bigsqcup_{j \to i} B$$

which is an f-cofibration. So (5.16) is a weak equivalence and, hence (5.15) is a weak equivalence. The same argument shows

$$\mathbf{Hom}(cB, R_i E) \to \mathbf{Hom}(cB, R_i D) \times_{\mathbf{Hom}(cA, R_i D)} \mathbf{Hom}(cA, R_i E)$$

is a weak equivalence, and we have proved $R_i E \to R_i D$ is a g-injective. ∎

6. A model for the stable homotopy category.

We have so far concentrated on the concept of localization at a cofibration or set of cofibrations. This is not the only method for constructing localization theories: there are also theories which arise from classes of cofibrations satisfying certain axioms — this is the approach taken in [38], and it completely subsumes the material presented here in cases where the ambient closed model category admits cardinality arguments. There is another, older approach, which involves localizing at a sufficiently well-behaved functor $Q : \mathcal{C} \to \mathcal{C}$ from a proper simplicial model category to itself. This technique is due to Bousfield and Friedlander [12], and is the basis for their approach to constructing the stable homotopy category. We shall present this method here.

Suppose that $\mathcal{C}$ is a proper closed (simplicial) model category, and let $Q : \mathcal{C} \to \mathcal{C}$ be a functor. Suppose further that there is a natural transformation $\eta_X : X \to Q(X)$ from the identity functor on $\mathcal{C}$ to Q. Say that a map $f : X \to Y$ is a *Q-weak equivalence* if the induced map $Q(f) : Q(X) \to Q(Y)$ is a weak equivalence of $\mathcal{C}$, and say that a map is a *Q-fibration* if it has the right lifting property with respect to all cofibrations of $\mathcal{C}$ which are Q-weak equivalences.

We shall require that the functor Q and the natural map η together satisfy the following list of properties:

A1 The functor Q preserves weak equivalences.

A2 The maps $\eta_{QX}, Q(\eta_X) : Q(X) \to Q^2(X)$ are weak equivalences of $\mathcal{C}$.

A3 The class of Q-weak equivalences is closed under pullback along Q-fibrations: given maps
$$B \xrightarrow{g} Y \xleftarrow{p} X$$
with p a Q-fibration and g a Q-weak equivalence, the induced map $g_* : B \times_Y X \to X$ is a Q-weak equivalence. Dually, the class of Q-weak equivalences is closed under pushout along cofibrations of $\mathcal{C}$.

The first major result of this section is the following:

THEOREM 6.1. *Suppose that $\mathcal{C}$ is a proper closed model category, $Q : \mathcal{C} \to \mathcal{C}$ is a functor, and $\eta_X : X \to Q(X)$, $x \in \mathrm{Ob}(\mathcal{C})$, is a natural transformation, all satisfying the properties* **A1**–**A3** *above. Then there is a closed model structure on $\mathcal{C}$ for which the weak equivalences are the Q-weak equivalences, the cofibrations are the cofibrations of $\mathcal{C}$, and the fibrations are the Q-fibrations.*

We shall refer to the closed model structure of Theorem 6.1 as the *Q-structure* on the category $\mathcal{C}$. Note first that this result implies that there is an idempotent functor $G : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C})$ in the sense of Adams, and hence an associated categorical localization theory (see [8, p.135]):

COROLLARY 6.2. *Suppose that the conditions for Theorem 6.1 hold. Then there is a functor $G : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C})$ and a natural transformation $j : 1_{Ho(\mathcal{C})} \to G$ such that $j_{GX} = G(j_X)$, and j_{GX} is an isomorphism of $Ho(\mathcal{C})$ for all X.*

PROOF: It suffices to work on the level of objects of $\mathcal{C}$ which are fibrant and cofibrant for the original closed model structure on $\mathcal{C}$, so that the morphisms of $\mathrm{Ho}(\mathcal{C})$ can be identified with homotopy classes of maps.

Choose a map $j_X : X \to GX$ such that j_X is a cofibration and a Q-equivalence, and GX is Q-fibrant. Every trivial cofibration of $\mathcal{C}$ is a Q-equivalence, by **A1**, so GX is fibrant in the original structure on $\mathcal{C}$.

If $f : X \to Y$ is a map such that Y is Q-fibrant, then there is an extension $f_* : GX \to Y$ such that $f_* \cdot j_X = f$. Furthermore, the choice of f_* is independent up to homotopy of the homotopy class of f. It follows that any map $f : X \to Y$ determines an extension $Gf : GX \to GY$ such that $Gf \cdot j_X = j_Y \cdot f$, and this construction determines a functor $G : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C})$, and a natural transformation $j : 1_{\mathrm{Ho}(\mathcal{C})} \to G$. The maps j_{GX} and Gj_X are two extensions of a common map $X \to G^2X$, so they coincide up to homotopy, and the map $Gj_X : GX \to G^2X$ is a Q-equivalence between objects which are Q-fibrant and cofibrant, so Gj_X is a homotopy equivalence. ∎

The proof of Theorem 6.1 will be given in a series of lemmas.

LEMMA 6.3. *Assume the conditions of Theorem 6.1. Then a map $f : X \to Y$ is a fibration and a weak equivalence of $\mathcal{C}$ if and only if it is a Q-equivalence and a Q-fibration.*

PROOF: If f is a fibration and a weak equivalence of $\mathcal{C}$, it is a Q-equivalence by **A1** and has the right lifting property with respect to all cofibrations, so it's a Q-fibration.

Suppose that f is a Q-equivalence and a Q-fibration, and take a factorization $f = p \cdot i$, where where $i : X \to Z$ is a cofibration and $p : Z \to Y$ is a trivial fibration in $\mathcal{C}$. Then i is a Q-equivalence by **A1**, so the lifting θ exists in the diagram

$$\begin{tikzcd} X \arrow[r, "="] \arrow[d, "i"'] & X \arrow[d, "f"] \\ Z \arrow[r, "p"'] \arrow[ur, "\theta"] & Y \end{tikzcd}$$

and so f is a retract of p. ∎

LEMMA 6.4. *Suppose that $f : X \to Y$ is a fibration of $\mathcal{C}$, and that $\eta_X : X \to QX$ and $\eta_Y : Y \to QY$ are weak equivalences of $\mathcal{C}$. Then f is a Q-fibration.*

Proof: Start with a lifting problem

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r, "\beta"'] \arrow[ur, dashed] & Y \end{tikzcd}$$

where i is a cofibration and a Q-equivalence. We have to show that the dotted arrow exists.

There is a diagram

$$\begin{tikzcd} QA \arrow[rr, "Q\alpha"] \arrow[dr, "j_\alpha"'] \arrow[dd, "Qi"'] & & QX \arrow[dd, "Qf"] \\ & Z \arrow[ur, "p_\alpha"'] \arrow[dd, "\pi"] & \\ QB \arrow[rr] \arrow[dr, "j_\beta"'] & & QY \\ & W \arrow[ur, "p_\beta"'] & \end{tikzcd}$$

where j_α and j_β are trivial cofibrations of $\mathcal{C}$ and p_α and p_β are fibrations. To see this, find the factorization $Q\beta = p_\beta j_\beta$, and then factorize the induced map $QA \to W \times_{QY} QX$ as a trivial cofibration j_α followed by fibration. There is an induced diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & Z \times_{QX} X \arrow[r] \arrow[d, "\pi_*"'] & X \arrow[d, "f"] \\ B \arrow[r] & W \times_{QY} Y \arrow[r] & Y \end{tikzcd}$$

such that the top horizontal composite is α and the bottom composite is β.

We will show that the conditions on f and i imply that the map π_* is a weak equivalence of $\mathcal{C}$. This suffices, for then π_* has a factorization $\pi_* = q \cdot j$ where q

is a trivial fibration and j is a trivial cofibration, and there is a diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[dd, "i"'] & Z \times_{QX} X \arrow[r] \arrow[d, "j"'] & X \arrow[dd, "f"] \\
 & U \arrow[ur, "\omega"] \arrow[d, "q"] & \\
B \arrow[ur, "\theta"] \arrow[r] & W \times_{QY} Y \arrow[r] & Y
\end{tikzcd}$$

so that $\omega \cdot \theta$ is the desired lift.

There is a final diagram

$$\begin{tikzcd}
QA \arrow[r, "j_\alpha"] \arrow[d, "Qi"'] & Z \arrow[d, "\pi"'] & Z \times_{QX} X \arrow[l, "pr"'] \arrow[d, "\pi_*"] \\
QB \arrow[r, "j_\beta"'] & W & W \times_{QY} Y \arrow[l, "pr"]
\end{tikzcd}$$

The map Qi is a weak equivalence since i is a Q-equivalence, so that π is a weak equivalence. The maps η_X and η_Y are weak equivalences by assumption and p_α and p_β are fibrations, so the maps labelled pr are weak equivalences by the properness assumption on $\mathcal{C}$. It follows that π_* is a weak equivalence of $\mathcal{C}$. ■

LEMMA 6.5. *Any map $f : QX \to QY$ has a factorization $f = p \cdot i$ where $p : Z \to QY$ is a Q-fibration and $i : QX \to Z$ is a cofibration and a Q-equivalence.*

PROOF: Take a factorization $f = p \cdot i$ where $p : Z \to QY$ is a fibration and $i : QX \to Z$ is a cofibration and a weak equivalence. Applying the functor Q to this diagram, one sees that η_{QX} is a weak equivalence by **A2**, and i is a weak equivalence so that Qi is a weak equivalence by **A1**. It follows that $\eta_Z : Z \to QZ$ is a weak equivalence, as is η_{QY}, so that p is a Q-fibration by Lemma 6.4. ■

LEMMA 6.6. *Any map $f : X \to Y$ of $\mathcal{C}$ has a factorization $f = q \cdot j$ where $q : Z \to Y$ is a Q-fibration and $j : X \to Z$ is a cofibration and a Q-equivalence.*

PROOF: Take the factorization $f = p \cdot i$ of Lemma 6.5 and pull $\eta_Y : Y \to QY$ along p to give a diagram

$$\begin{tikzcd} X \arrow[r, "i_*"] \arrow[d, "\eta_X"'] & Z \times_{QY} Y \arrow[r, "p_*"] \arrow[d, "\eta_*"'] & Y \arrow[d, "\eta_Y"] \\ QX \arrow[r, "i"'] & Z \arrow[r, "p"'] & QY, \end{tikzcd}$$

where $p_* \cdot i_* = f$. Then η_* is a Q-equivalence by **A2** and **A3**, as are both η_X and i. It follows that i_* is a Q-equivalence.

Now factorize i_* in $\mathcal{C}$ as $i_* = \pi \cdot j$, where j is a cofibration and π is a trivial fibration. Then π is a Q-fibration by Lemma 6.3, so that $q = p_* \cdot \pi$ is a Q-fibration. ■

PROOF OF THEOREM 6.1: The closed model axioms **CM1**–**CM3** are trivial to verify. One part of the factorization axiom is Lemma 6.6. The other is a consequence of Lemma 6.3, as is **CM4**. ■

REMARK 6.7. Observe that if $\mathcal{C}$ has a simplicial model structure in addition to the conditions of Theorem 6.1, then the Q-structure on $\mathcal{C}$ is a simplicial model structure, essentially for free. Axiom **SM7** is a consequence of the fact that the two structures have the same cofibrations, and every trivial cofibration of $\mathcal{C}$ is a Q-equivalence.

Similarly, the full statement of **A3** (we've only used the fibration part of it up to now) implies that the Q-structure is proper.

Here's the second major result:

THEOREM 6.8. *Suppose that the proper closed model category $\mathcal{C}$ and the functor Q together satisfy the conditions for Theorem 6.1. Then a map $f : X \to Y$ is a Q-fibration if and only if it is a fibration and the square*

(6.9)
$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & QX \arrow[d, "Qf"] \\ Y \arrow[r, "\eta_Y"'] & QY \end{tikzcd}$$

is a homotopy cartesian diagram of $\mathcal{C}$.

PROOF: Suppose that the diagram (6.9) is homotopy cartesian and that $f : X \to Y$ is a fibration of $\mathcal{C}$. We will show that the map f is a retract of a Q-fibration.

Factorize Qf as $Qf = p \cdot i$, where $p : Z \to QY$ is a fibration of $\mathcal{C}$ and $i : QX \to Z$ is a trivial cofibration. Then, as in the proof of Lemma 6.5, the map p is a Q-fibration, so that the induced map $p_* : Y \times_{QY} Z \to Y$ is a Q-fibration. The induced map $i_* : X \to Y \times_{QY} Z$ is a weak equivalence of $\mathcal{C}$, since $\mathcal{C}$ is proper, and so it has a factorization $i_* = \pi \cdot j$, where $j : X \to W$ is a trivial cofibration and $\pi : W \to Y \times_{QY} Z$ of $\mathcal{C}$ is a trivial fibration of $\mathcal{C}$. Then the composite $p_*\pi : W \to Y$ is a Q-fibration by Lemma 6.3, and the lifting exists in the diagram

$$\begin{tikzcd} X \arrow[r, "="] \arrow[d, "j"'] & X \arrow[d, "f"] \\ W \arrow[r, "p_*\pi"'] \arrow[ur] & Y \end{tikzcd}$$

since f is a fibration. It follows that f is a retract of $p_*\pi$.

For the converse, suppose that $f : X \to Y$ is a Q-fibration. We show that f is a retract of a fibration g for which the diagram

$$\begin{tikzcd} Z \arrow[r, "\eta_Z"] \arrow[d, "g"'] & QZ \arrow[d, "Qg"] \\ W \arrow[r, "\eta_W"'] & QW \end{tikzcd} \tag{6.10}$$

is homotopy cartesian in $\mathcal{C}$.

Observe that the class of maps g which are fibrations of $\mathcal{C}$ and for which the diagram (6.10) is homotopy cartesian is closed under composition, and includes all trivial fibrations of $\mathcal{C}$ by **A1**. It suffices therefore, with respect to the construction giving the first part of the proof, to show that the map p_* is a candidate for one of these maps g.

The component square diagrams in

$$\begin{tikzcd} Y \times_{QY} Z \arrow[r, "\eta_*"] \arrow[d] & Z \arrow[r, "\eta_Z"] \arrow[d, "p"'] & QZ \arrow[d, "Qp"] \\ Y \arrow[r, "\eta_Y"'] & QY \arrow[r, "\eta_{QY}"'] & Q^2Y \end{tikzcd}$$

are homotopy cartesian since η_Z and η_{QY} are weak equivalences (see the proof of Lemma 6.5), so that the composite square is homotopy cartesian. This composite coincides with the composite of the squares

$$\begin{tikzcd} Y \times_{QY} Z \arrow[r, "\eta"] \arrow[d, "p_*"'] & Q(Y \times_{QY} Z) \arrow[r, "Q\eta_*"] \arrow[d, "Qp_*"'] & QZ \arrow[d, "Qp"] \\ Y \arrow[r, "\eta_Y"'] & QY \arrow[r, "Q\eta_Y"'] & QQY \end{tikzcd} \tag{6.11}$$

The map η_* is a Q-equivalence by **A3**, so that $Q\eta_*$ is a weak equivalence, as is $Q\eta_Y$. It follows that the square on the left in (6.11) is homotopy cartesian, but this is what we had to prove. ∎

COROLLARY 6.12. *Suppose that the proper closed model category $\mathcal{C}$ and the functor Q together satisfy the conditions for Theorem 6.1. Then an object X of $\mathcal{C}$ is Q-fibrant if and only if it is fibrant in $\mathcal{C}$ and the map $\eta_X : X \to QX$ is a weak equivalence of $\mathcal{C}$.*

The principal application of this theory is the Bousfield-Friedlander construction of the stable homotopy category.

A *spectrum* X (or rather a spectrum object in simplicial sets) consists of pointed simplicial sets X^n, $n \geq 0$, together with pointed simplicial set maps $\sigma : S^1 \wedge X^n \to X^{n+1}$, which we call *bonding maps*. Here, $S^1 = \Delta^1/\partial\Delta^1$ is the simplicial circle. A *map of spectra* $f : X \to Y$ consists of pointed simplicial set maps $f : X^n \to Y^n$, $n \geq 0$, which respect structure in the sense that all diagrams

$$\begin{tikzcd} S^1 \wedge X^n \arrow[r, "\sigma"] \arrow[d, "S^1 \wedge f"'] & X^{n+1} \arrow[d, "f"] \\ S^1 \wedge Y^n \arrow[r, "\sigma"'] & Y^{n+1} \end{tikzcd}$$

commute. The resulting category of spectra will be denoted by **Spt**.

A map $f : X \to Y$ of spectra is said to be a *strict weak equivalence* (respectively *strict fibration*) if all of the component maps $f : X^n \to Y^n$ are weak equivalences (respectively fibrations) of simplicial sets. A map $j : A \to B$ of spectra is said to be a *cofibration* if the following two conditions are satisfied:

(1) the map $i : A^0 \to B^0$ is a cofibration (or monomorphism) of simplicial sets, and

(2) each induced map

$$S^1 \wedge B^n \cup_{S^1 \wedge A^n} A^{n+1} \to B^{n+1}$$

is a cofibration of simplicial sets.

PROPOSITION 6.13. *With the definitions of strict fibration, strict weak equivalence, and cofibration given above, the category* **Spt** *satisfies the axioms for a proper simplicial model category.*

It follows from the definition of cofibration that a spectrum X is cofibrant if and only if the bonding maps $S^1 \wedge X^n \to X^{n+1}$ are all cofibrations. Notice further that if $i : A \to B$ is a cofibration of spectra, then all of the maps $i : A^n \to B^n$ are cofibrations of simplicial sets, but the converse may not be true. We usually say that a map $j : C \to D$ of spectra for which all the maps $j : C^n \to D^n$ are cofibrations is a *pointwise cofibration.* In the same terminology, strict fibrations are pointwise fibrations and strict weak equivalences are pointwise weak equivalences.

REMARK 6.14. To see the simplicial structure for **Spt** a little more clearly, note that if X is a spectrum and K is a pointed simplicial set, then there is a spectrum $X \wedge K$ with $(X \wedge K)^n = X^n \wedge K$, and having bonding maps of the form

$$S^1 \wedge X^n \wedge K \xrightarrow{\sigma \wedge K} X^{n+1} \wedge K.$$

Then, for an arbitrary simplicial set L and a spectrum X, $X \otimes L = X \wedge L_+$, where $L_+ = L \sqcup *$ is L with a disjoint base point attached. Dually, the pointed function complex spectrum $\mathbf{hom}_*(K, X)$ (denoted by $\mathbf{hom}_{\mathbf{S}_*}(K, X)$ in Section II.2) is the spectrum with

$$\mathbf{hom}_*(K, X)^n = \mathbf{hom}_*(K, X^n),$$

with bonding map $S^1 \wedge \mathbf{hom}_*(K, X^n) \to \mathbf{hom}_*(K, X^{n+1})$ adjoint to the composite

$$S^1 \wedge \mathbf{hom}_*(K, X^n) \wedge K \xrightarrow{S^1 \wedge ev} S^1 \wedge X^n \xrightarrow{\sigma} X^{n+1}$$

It follows that the evaluation maps $ev : \mathbf{hom}_*(K, X^n) \wedge K \to X^n$ determine a map of spectra $ev : \mathbf{hom}_*(K, X) \wedge K \to X$.

Given a spectrum X, there is a spectrum ΣX having $(\Sigma X)^n = S^1 \wedge X^n$ and with bonding maps

$$S^1 \wedge S^1 \wedge X^n \xrightarrow{S^1 \wedge \sigma} S^1 \wedge X^{n+1}.$$

This spectrum ΣX *is not* the suspension object $X \wedge S^1$ arising from the simplicial structure on **Spt**: the two differ by a twist of circle smash factors — this fact is an avatar of one of the standard dangerous bends in the foundations of stable homotopy theory. We say that ΣX is the *fake suspension spectrum* of X.

Similarly, there is a *fake loop spectrum* ΩY, which does not coincide with the function complex object $\mathbf{hom}_*(S^1, Y)$. In effect, $(\Omega Y)^n = \mathbf{hom}_*(S^1, Y^n)$, but the bonding map is the composite

$$S^1 \wedge \mathbf{hom}_*(S^1, Y^n) \underset{\cong}{\xrightarrow{\tau}} \mathbf{hom}_*(S^1, Y^n) \wedge S^1 \xrightarrow{ev} Y^n \xrightarrow{\sigma_*} \mathbf{hom}_*(S^1, Y^{n+1}),$$

where $\sigma_* : Y^n \to \mathbf{hom}_*(S^1, Y^{n+1})$ is adjoint to the composite

$$Y^n \wedge S^1 \underset{\cong}{\xrightarrow{\tau}} S^1 \wedge Y^n \xrightarrow{\sigma} Y^{n+1}.$$

The fake loop construction is right adjoint to the fake suspension.

The closed model structure determined by Proposition 6.13 is usually called the *strict closed model structure* on the category of spectra. The associated homotopy category is not yet the stable category. We stabilize by localizing at a suitable functor Q.

The construction of the functor $Q : \mathbf{Spt} \to \mathbf{Spt}$ requires a natural strictly fibrant model. There are both geometric and combinatorial ways of producing such an object, corresponding to the two standard methods of functorially replacing simplicial set by a Kan complex. For the geometric method, observe that there is a natural map

$$S|K| \wedge S|L| \to S|K \wedge L|$$

for each pair of pointed simplicial sets K and L. Given a spectrum X, the bonding maps $\sigma : S^1 \wedge X^n \to X^{n+1}$ therefore induce composites

$$S^1 \wedge S|X^n| \xrightarrow{\eta \wedge 1} S|S^1| \wedge S|S^n| \to S|S^1 \wedge X^n| \xrightarrow{S|\sigma|} S|X^{n+1}|,$$

so that applying realization and singular functors levelwise to a spectrum X determines a spectrum $S|X|$. It's not hard to see that the canonical weak equivalences $\eta : X^n \to S|X^n|$ collectively determine a natural strict equivalence of spectra $\eta : X \to S|X|$. Of course, $S|X|$ is strictly fibrant. Kan's Ex^∞ construction can be goaded into performing a similar service: the natural composite

$$K \times Ex^\infty L \xrightarrow{\nu \wedge 1} Ex^\infty K \times Ex^\infty L \cong Ex^\infty(K \times L)$$

induces a natural map

$$\nu_* : K \wedge Ex^\infty L \to Ex^\infty(K \wedge L)$$

which, in the presence of a spectrum X determines composites

$$S^1 \wedge Ex^\infty X^n \to Ex^\infty(S^1 \wedge X^n) \xrightarrow{Ex^\infty \sigma} Ex^\infty X^{n+1}.$$

Furthermore, the canonical maps $\nu : X^n \to Ex^\infty X^n$ together determine a natural strict equivalence $\nu : X \to Ex^\infty X$, taking values in a strictly fibrant spectrum. Observe that $Ex^\infty X$ and $S|X|$ are naturally strictly equivalent.

The indices in spectra can be shifted at will: if X is a spectrum and n is an integer, there is a spectrum $X[n]$ with

$$X[n]^k = \begin{cases} X^{n+k} & \text{if } n+k \geq 0, \text{ and} \\ * & \text{if } n+k < 0. \end{cases}$$

Shifting indices is functorial, and is cumulative in the sense that there are canonical natural isomorphisms $X[n][k] \cong X[n+k]$.

The bonding maps $\sigma : S^1 \wedge X^n \to X^{n+1}$ of a fixed spectrum X determine a map of spectra $\sigma : \Sigma X \to X[1]$. This map is natural in X, and has a natural adjoint $\sigma_* : X \to \Omega X[1]$, which is defined levelwise by the (twisted) adjoints $\sigma_* : X^n \to \mathbf{hom}_*(S^1, X^{n+1})$ of the maps $\sigma : S^1 \wedge X^n \to X^{n+1}$. This construction can be repeated, to form an inductive system of maps

$$X \xrightarrow{\sigma_*} \Omega X[1] \xrightarrow{\Omega\sigma_*[1]} \Omega^2 X[2] \xrightarrow{\Omega^2\sigma_*[2]} \dots.$$

Write $\Omega^\infty X = \varinjlim_n \Omega^n X[n]$

The functor $Q : \mathbf{Spt} \to \mathbf{Spt}$ is defined for spectra X by $QX = \Omega^\infty \operatorname{Ex}^\infty X$, and there is a natural map $\eta_X : X \to QX$ given by the composite

$$X \xrightarrow{\nu} Ex^\infty X \xrightarrow{\tau} \Omega^\infty Ex^\infty X.$$

where $\tau : Y \to \Omega^\infty Y$ denotes the canonical map to the colimit.

Say that a map $f : X \to Y$ of spectra is a *stable equivalence* if it induces a strict equivalence $Qf : QX \to QY$. The map $p : Z \to W$ is said to be a *stable fibration* if it has the right lifting property with respect to all maps which are cofibrations and weak equivalences.

THEOREM 6.15. *With these definitions, the category* **Spt** *of spectra, together with cofibrations, stable equivalences and stable fibrations satisfies the axioms for a proper simplicial model category.*

PROOF: We need only verify that the functor Q satisfies axioms **A1**–**A3**. Of these, **A1** is clear, and **A2** is a consequence of the observation that

$$\Omega^\infty \tau_Y = \tau_{\Omega^\infty Y} : \Omega^\infty Y \to \Omega^\infty \Omega^\infty Y$$

is an isomorphism by a cofinality argument. The functor Q preserves strict fibrations and pullbacks, giving the pullback part of **A3**. Recall that this is enough to give the desired simplicial model structure.

Suppose given maps of spectra

$$B \xleftarrow{j} A \xrightarrow{g} C$$

where j is a cofibration and g is a stable equivalence. To finish verifying **A3**, we have to show that the induced map $g_* : B \to B \cup_A C$ is a stable equivalence. Construct a diagram

$$\begin{tikzcd} & & \tilde{C} \arrow[d, "\pi_C"] \\ B & A \arrow[l, "j"'] \arrow[ur, "\tilde{g}"] \arrow[r, "g"'] & C \end{tikzcd}$$

where $\tilde{g}$ is a cofibration, and the map π_C is a strict fibrations and a strict weak equivalence. The glueing lemma for simplicial sets implies that the induced map $\pi_* : B \cup_A \tilde{C} \to B \cup_A C$ is a strict weak equivalence. It is therefore enough to show that the induced map $\tilde{g}_* : B \to B \cup_A \tilde{C}$ is a stable equivalence. But $\tilde{g}$ is a stable equivalence as well as a cofibration, and such maps are closed under pushout. ■

REMARKS 6.16.

1) In view of the coincidence of the homotopy groups of a space with the simplicial homotopy groups of its associated singular complex (Proposition I.11.1), the definition of QX implies that the homotopy groups $\pi_k QX^n$ coincide up to isomorphism with the stable homotopy groups $\pi_{k-n}|X|$ of the (pre)spectrum $|X|$, so that Q-equivalence is stable equivalence: a map $f : X \to Y$ of spectra is a Q-equivalence if and only if it induces an isomorphism in all stable homotopy groups. For this reason, as is standard, we say that Q-equivalences are *stable equivalences* and that Q-fibrations are *stable fibrations*. The stable homotopy

groups can, of course, be defined completely in terms of the simplicial homotopy groups arising from the strictly fibrant model $Ex^{\infty}X$.

2) Corollary 6.12 implies that a spectrum X is stably fibrant if and only if X is strictly fibrant and all associated adjoints $\sigma_* : X^n \to \mathbf{hom}_*(S^1, X^{n+1})$ of the bonding maps $\sigma : S^1 \wedge X^n \to X^{n+1}$ are weak equivalences. In other words, a stably fibrant spectrum X is a type of Ω-*spectrum* , and one commonly lapses into describing it that way.

Write $\mathbf{Spt}^Q$ for the closed model structure on the category of spectra of Theorem 6.15. The associated homotopy category $\mathrm{Ho}(\mathbf{Spt}^Q)$ is a model for the stable category, in view of the above remarks. This is far from being the only (or even best) construction of the stable category, but it has the advantage of giving descriptions of stable homotopy theory in a variety of non-standard situations [52]. There is even a collection of ways to describe the closed model structure $\mathbf{Spt}^Q$: one could, for example, localize $\mathbf{Spt}$ at the family of stably trivial cofibrations, as is done in [38].

Alternatively, the results of the last section can be used to show that $\mathbf{Spt}^Q$ arises from the strict category by an f-localization, for a suitable choice of map (or set of maps) f. The family of maps f is easy to describe: let S be the (simplicial) sphere spectrum

$$S^0 = \partial\Delta^1,\ S^1,\ S^1 \wedge S^1, \dots,$$

and consider the collection of canonical maps of the form $f : \Sigma S[n] \to S[n+1]$, $n \in \mathbb{Z}$. These maps are not cofibrations in the strict closed model category, but they can be replaced by such. Further, the objects $\Sigma S[n]$ and $S[n]$ are cofibrant, so a spectrum X is f-local for this family of maps f if and only if it is strictly fibrant and all induced maps

$$\mathbf{Hom}(S[-n], X) \xrightarrow{f^*} \mathbf{Hom}(\Sigma S[-n-1], X)$$

are weak equivalences of simplicial sets. An adjointness argument shows that this last requirement is equivalent to the assertion that all of the maps

$$X^n \xrightarrow{\sigma_*} \mathbf{hom}_*(S^1, X^{n+1})$$

are weak equivalences. In particular, a spectrum X is f-local if and only if it is stably fibrant. Further, a map $A \to B$ of cofibrant spectra is a stable equivalence if and only if the induced map $\mathbf{Hom}(B, X) \to \mathbf{Hom}(A, X)$ is a weak equivalence for all stably fibrant spectra X, on account of Lemma II.4.2. It therefore follows from the definition of f-local equivalence (Definition 4.13) that a map of spectra is a stable equivalence if and only if it is an f-local equivalence. The weak equivalences for the stable and f-local theories coincide, as do the cofibrations, so we recover the stable closed model structure $\mathbf{Spt}^Q$ from Proposition 4.22.