Chapter VIII Simplicial functors and homotopy coherence

Suppose that $\mathcal{A}$ is a simplicial category. The main objects of study in this chapter are the functors $X : \mathcal{A} \to \mathbf{S}$ taking values in simplicial sets, and which respect the simplicial structure of $\mathcal{A}$. In applications, the simplicial category $\mathcal{A}$ is typically a resolution of a category I, and the simplicial functor X describes a homotopy coherent diagram. The main result of this chapter (due to Dwyer and Kan, Theorem 2.13 below) is a generalization of the assertion that simplicial functors of the form $X : \mathcal{A} \to \mathbf{S}$ are equivalent to diagrams of the form $I \to \mathbf{S}$ in the case where $\mathcal{A}$ is a resolution of the category I. The proof of this theorem uses simplicial model structures for categories of simplicial functors, given in Section 1, and then the result itself is proved in Section 2.

The Dwyer-Kan theorem immediately leads to realization theorems for homotopy coherent diagrams in cases where the simplicial categories $\mathcal{A}$ model homotopy coherence phenomena. A realization of a homotopy coherent diagram $X : \mathcal{A} \to \mathbf{S}$ is a functor $Y : I \to \mathbf{S}$ which is weakly equivalent to X in a strong sense. Insofar as the information arising from X typically consists of simplicial set maps $X(\alpha) : X(i) \to X(j)$, one for each morphism α of I which only respect the composition laws of I up to some system of higher homotopies, a realization Y is a replacement of X, up to weak equivalence, by a collection of maps $Y(\alpha) : Y(i) \to Y(j)$ which define a functor on the nose. Diagrams of spaces which are not quite functorial are really very common: the machines which produce the algebraic K-theory spaces, for example, are not functors (on scheme categories in particular), but they have homotopy coherent output for categorical reasons [49].

Approaches to homotopy coherence for simplicial set diagrams arising from some specific resolution constructions are discussed in Section 3; traditional homotopy coherence (in the sense of [90]) is one of the examples.

But more generally, we take the point of view that a homotopy coherent diagram on a fixed index category I is a simplicial functor $X : \mathcal{A} \to \mathbf{S}$ defined on *any* simplicial resolution $\mathcal{A}$ of I. We can further ask for realization results concerning homotopy coherent diagrams $\mathcal{A} \to \mathcal{M}$ taking values in more general simplicial model categories $\mathcal{M}$. This is the subject of Section 4. We derive, in particular, realization theorems for homotopy coherent diagrams taking values in pointed simplicial sets, spectra and simplicial abelian groups (aka. chain complexes).

1. Simplicial functors.

We shall take the point of view throughout this chapter that a *simplicial category* $\mathcal{A}$ is a simplicial object in the category of categories having a discrete simplicial class of objects and satisfying the conditions of Definition 2.1 of Chapter II. This means, in particular that the objects $A \otimes K$ and $\mathbf{hom}(K, A)$ exist for every (finite) simplicial set K and every object A of $\mathcal{A}$, and that these objects satisfy the exponential laws. The full simplicial set category $\mathbf{S}$ with the function complexes $\mathbf{Hom}(X, Y)$ is the canonical example of a simplicial category. In all that follows, we shall assume that the simplicial category $\mathcal{A}$ has a small set of objects.

The simplicial set of morphisms from A to B in a simplicial category $\mathcal{A}$ is denoted by $\mathbf{Hom}(A, B)$; from this point of view, the corresponding set of n-simplices $\mathbf{Hom}(A, B)_n$ is the set of morphisms from A to B in the category at level n. Any morphism $\alpha : B \to C$ in $\mathbf{Hom}(B, C)_0$ induces a simplicial set map

$$\alpha_* : \mathbf{Hom}(A, B) \to \mathbf{Hom}(A, C),$$

which one understands to be composition with α, and which can be defined to be the composite simplicial set map

$$\mathbf{Hom}(A,B) \cong \mathbf{Hom}(A,B) \times \Delta^0 \xrightarrow{1 \times \iota_\alpha} \mathbf{Hom}(A,B) \times \mathbf{Hom}(B,C) \xrightarrow{\circ} \mathbf{Hom}(A,C).$$

Similarly, any morphism $\beta : A \to B$ induces a simplicial set map

$$\beta^* : \mathbf{Hom}(B, C) \to \mathbf{Hom}(A, C),$$

that one thinks of as precomposition. Composition and precomposition respect composition in α and β in the traditional sense.

A *simplicial functor* $f : \mathcal{A} \to \mathcal{B}$ is a morphism of simplicial categories. This means that f consists of a function $f : Ob(\mathcal{A}) \to Ob(\mathcal{B})$ and simplicial set maps $f : \mathbf{Hom}(A, B) \to \mathbf{Hom}(f(A), f(B))$ which respect identities and composition at all levels.

A *natural transformation* $\eta : f \to g$ of simplicial functors $f, g : \mathcal{A} \to \mathcal{B}$ consists of morphisms

$$\eta_A : f(A) \to g(A)$$

in $\hom(f(A), g(A)) = \mathbf{Hom}(f(A), g(A))_0$, one for each object A of $\mathcal{A}$, such that the following diagram of simplicial set maps commutes

$$\begin{tikzcd}
\mathbf{Hom}(A, B) \arrow[r, "f"] \arrow[d, "g"'] & \mathbf{Hom}(f(A), f(B)) \arrow[d, "\eta_{B*}"] \\
\mathbf{Hom}(g(A), g(B)) \arrow[r, "\eta_A^*"'] & \mathbf{Hom}(f(A), g(B))
\end{tikzcd}$$

for each pair of objects A, B of $\mathcal{A}$. Notice that this is just another way of saying that the various degeneracies of the morphisms η_A are natural transformations between the functors induced by f and g at all levels. The collection of all simplicial functors from $\mathcal{A}$ to $\mathcal{B}$ and all natural transformations between them form a category, which we shall denote by $\mathcal{B}^{\mathcal{A}}$. Write $\mathrm{Nat}(f, g)$ for the set of all natural transformations from f to g.

The category $\mathbf{S}^{\mathcal{A}}$ of simplicial functors taking values in the simplicial set category $\mathbf{S}$ are of particular interest, and can be given a much more explicit description. Suppose that $\mathcal{A}_n$ denotes the small category at level n within the simplicial category $\mathcal{A}$. Then a simplicial functor $X : \mathcal{A} \to \mathbf{S}$ consists of a function $X : Ob(\mathcal{A}) \to Ob(\mathbf{S})$ as before, together with a collection of simplicial set maps $X(\alpha) : X(A) \times \Delta^n \to X(B)$, one for each morphism $\alpha : A \to B$ in $\mathcal{A}_n$, $n \geq 0$, such that

(1) the simplicial set map $X(\beta\alpha)$ is the composite

$$X(A) \times \Delta^n \xrightarrow{1 \times \Delta^n} X(A) \times \Delta^n \times \Delta^n \xrightarrow{X(\alpha) \times 1} X(B) \times \Delta^n \xrightarrow{X(\beta)} X(C)$$

for each composable string of morphisms

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

in $\mathcal{A}_n$,

(2) the simplicial set map $X(1_A)$ associated to the identity on $A \in \mathcal{A}_n$ is the projection $X(A) \times \Delta^n \to X(A)$, and

(3) for each ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$ and each morphism $\alpha : A \to B$ of $\mathcal{A}_n$, the following diagram commutes:

$$\begin{tikzcd}
X(A) \times \Delta^m \arrow[d, "1 \times \theta_*"'] \arrow[dr, "X(\theta^*(\alpha))"] & \\
X(A) \times \Delta^n \arrow[r, "X(\alpha)"'] & X(B).
\end{tikzcd}$$

Then, from this point of view, a natural transformation $\eta : X \to Y$ of simplicial functors taking values in simplicial sets consists of simplicial set maps

$\eta_A : X(A) \to Y(A)$, one for each $A \in Ob(\mathcal{A})$, such that the diagram

$$\begin{tikzcd} X(A) \times \Delta^n \arrow[r, "X(\alpha)"] \arrow[d, "\eta_A \times 1"'] & X(B) \arrow[d, "\eta_B"] \\ Y(A) \times \Delta^n \arrow[r, "Y(\alpha)"'] & Y(B) \end{tikzcd}$$

commutes for each morphism $\alpha : A \to B$ of $\mathcal{A}_n$, and for all $n \geq 0$.

Now it's easy to see that the category $\mathbf{S}^{\mathcal{A}}$ is the category at level 0 of a simplicial category. Given a simplicial functor $X : \mathcal{A} \to \mathbf{S}$ and a simplicial set K, there is a simplicial functor $K \times X$ which assigns to each $A \in Ob(\mathcal{A})$ the simplicial set $K \times X(A)$ and to each morphism $\alpha : A \to B$ in $\mathcal{A}_n$ the simplicial set map

$$K \times X(A) \times \Delta^n \xrightarrow{1 \times X(\alpha)} K \times X(B).$$

Dually, there is a simplicial functor $\mathbf{hom}(K, X)$ which associates to each $A \in Ob(\mathcal{A})$ the simplicial set $\mathbf{hom}(K, X(A))$, and to each morphism $\alpha : A \to B$ of $\mathcal{A}_n$ the simplicial set map $\mathbf{hom}(K, X(A)) \times \Delta^n \to \mathbf{hom}(K, X(B))$ which is defined to be the adjoint of the composite

$$K \times \mathbf{hom}(K, X(A)) \times \Delta^n \xrightarrow{ev \times 1} X(A) \times \Delta^n \xrightarrow{X(\alpha)} X(B),$$

where $ev : K \times \mathbf{hom}(K, X(A)) \to X(A)$ is the standard evaluation map. It follows immediately that the collection of all such evaluation maps induces a natural bijection

$$\mathrm{Nat}(K \times Y, X) \cong \mathrm{Nat}(Y, \mathbf{hom}(K, X)).$$

There is also, plainly, a simplicial set $\mathbf{hom}(Y, X)$ whose set of n-simplices is the set $\mathrm{Nat}(\Delta^n \times Y, X)$. Finally, since the simplicial set K is a colimit of its simplices, there is a natural bijection (exponential law)

$$\hom(K, \mathbf{Hom}(Y, X)) \cong \mathrm{Nat}(K \times Y, X) \tag{1.1}$$

relating morphisms in the simplicial set category to natural transformations of simplicial functors.

The *representable simplicial functor* $\mathbf{Hom}(A, \ )$ is the simplicial functor which associates to each $B \in Ob(\mathcal{A})$ the simplicial set $\mathbf{Hom}(A, B)$, and associates to each morphism $\beta : B \to C$ of $\mathcal{A}_n$ the composite simplicial set map

$$\mathbf{Hom}(A, B) \times \Delta^n \xrightarrow{1 \times \iota_\beta} \mathbf{Hom}(A, B) \times \mathbf{Hom}(B, C) \xrightarrow{\circ} \mathbf{Hom}(A, C).$$

It is central to observe that the representable simplicial functors in $\mathbf{S}^{\mathcal{A}}$ satisfy the Yoneda lemma:

Lemma 1.2. *Suppose that A is an object of a simplicial category $\mathcal{A}$, and that $X : \mathcal{A} \to \mathbf{S}$ is a simplicial functor. Then there is a natural isomorphism of simplicial sets*

$$\mathbf{Hom}(\mathbf{Hom}(A,\ ), X) \cong X(A).$$

Proof: The natural transformations $\eta : \mathbf{Hom}(A,\ ) \to X$ are in one to one correspondence with the vertices of $X(A)$. In effect, given a morphism $\alpha : A \to B$ of $\mathcal{A}_m$, there is a commutative diagram

$$\begin{tikzcd}
\mathbf{Hom}(A,A) \times \Delta^m \arrow[r, "\eta_A \times 1"] \arrow[d, "\alpha_*"'] & X(A) \times \Delta^m \arrow[d, "X(\alpha)"] \\
\mathbf{Hom}(A,B) \arrow[r, "\eta_B"'] & X(B),
\end{tikzcd}$$

so that

$$\eta_B(\alpha) = X(\alpha)(\eta_A(1_A) \times \iota_m),$$

where $\iota_m = 1_{\mathbf{m}} \in \Delta^m_m$ is the classifying simplex. But $1_A = s(1_{A,0})$ where $1_{A,0}$ denotes the identity on A in $\mathcal{A}_0$. Thus, η is completely determined by the vertex $\eta_A(1_{A,0})$ of $X(A)$.

It follows that there are natural bijections

$$\begin{aligned}
\mathrm{Nat}(\Delta^n \times \mathbf{Hom}(A,\ ), X) &\cong \mathrm{Nat}(\mathbf{Hom}(A,\ ), \mathbf{hom}(\Delta^n, X)) \\
&\cong \mathbf{hom}(\Delta^n, X(A))_0 \\
&\cong X(A)_n
\end{aligned}$$

and that these bijections respect the simplicial structure. ∎

A natural transformation $\eta : \Delta^n \times \mathbf{Hom}(A,\ ) \to X$ is completely determined by the n-simplex $\eta(\iota_n, 1_A)$ of the simplicial set $X(A)$: this is a corollary of the proof of Lemma 1.2.

The functor $\mathbf{S} \to \mathbf{S}$ defined by $K \mapsto K \times \Delta^n$ preserves all small colimits of simplicial sets and takes limits to limits fibred over Δ^n. It follows immediately that the category $\mathbf{S}^{\mathcal{A}}$ of simplicial functors from $\mathcal{A}$ to $\mathbf{S}$ is complete and co-complete, and that all limits and colimits are formed pointwise.

There is an analogue of the simplex category for each simplicial functor $X \in \mathbf{S}^{\mathcal{A}}$. Write $\mathcal{A} \downarrow X$ for the category whose objects are the transformations

$$\Delta^n \times \mathbf{Hom}(A,\ ) \to X,$$

and whose objects are all commutative triangles of transformations of the form

$$\begin{tikzcd}
\Delta^n \times \mathbf{Hom}(A,\ ) \arrow[d] \arrow[dr] & \\
& X \\
\Delta^m \times \mathbf{Hom}(B,\ ) \arrow[ur] &
\end{tikzcd}$$

The objects of this category are simplices of sections of X, and the morphisms form its structural data. It follows that X is a colimit of its simplices in $\mathbf{S}^{\mathcal{A}}$ in the sense that there is a natural isomorphism

$$X \cong \varinjlim_{\Delta^n \times \mathbf{Hom}(A,\ ) \to X} \Delta^n \times \mathbf{Hom}(A,\ ). \tag{1.3}$$

Say that a map $f : X \to Y$ of $\mathbf{S}^{\mathcal{A}}$ is a *fibration* (respectively *weak equivalence*) if the component maps $f : X(A) \to Y(A)$ are fibrations (respectively weak equivalences) of simplicial sets, for all objects A of $\mathcal{A}$. Such maps will be called *pointwise fibrations* and *pointwise weak equivalences*, respectively. A *cofibration* of $\mathbf{S}^{\mathcal{A}}$ is a map which has the left lifting property with respect to all trivial fibrations.

PROPOSITION 1.4. *With these definitions, the category $\mathbf{S}^{\mathcal{A}}$ of simplicial set-valued simplicial functors defined on a small simplicial category $\mathcal{A}$ satisfies the axioms for a simplicial model category.*

PROOF: **CM1** is a consequence of the completeness and cocompleteness of $\mathbf{S}^{\mathcal{A}}$, as described above. The weak equivalence axiom **CM2** and the retract axiom **CM3** are both trivial consequences of the corresponding axioms for simplicial sets.

Note that a map $f : X \to Y$ of $\mathbf{S}^{\mathcal{A}}$ is a fibration if and only if f has the right lifting property with respect to all maps

$$\Lambda^n_k \times \mathbf{Hom}(A,\ ) \xrightarrow{i \times 1} \Delta^n \times \mathbf{Hom}(A,\ ),$$

and f is a trivial fibration if and only if it has the right lifting property with respect to all maps

$$\partial\Delta^n \times \mathbf{Hom}(A,\ ) \xrightarrow{i' \times 1} \Delta^n \times \mathbf{Hom}(A,\ ),$$

where $i : \Lambda^n_k \hookrightarrow \Delta^n$ and $i' : \partial\Delta^n \hookrightarrow \Delta^n$ denote the respective canonical inclusions. Both statements follow from the exponential law (1.1) and Lemma 1.2. The map $i \times 1$ is a weak equivalence.

We may therefore apply standard small object arguments to prove the factorization axiom **CM5** for $\mathbf{S}^{\mathcal{A}}$. In particular, any map $f : X \to Y$ has a factorization

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & Z \arrow[d, "p"] \\ & Y, \end{tikzcd}$$

where p is a fibration and j is a filtered colimit of maps of the form $j_n : X_n \to X_{n+1}$, each of which is defined by a pushout diagram

$$\begin{tikzcd} \bigsqcup_r \Lambda^{n_r}_{k_r} \times \mathbf{Hom}(A_r, \ ) \arrow[r] \arrow[d, "\bigsqcup_r (i \times 1)"'] & X_n \arrow[d, "j_n"] \\ \bigsqcup_r \Delta^{n_r} \times \mathbf{Hom}(A_r, \ ) \arrow[r] & X_{n+1}. \end{tikzcd}$$

It follows that each of the maps j_n is a pointwise weak equivalence, and has the left lifting property with respect to all fibrations, and so the map j has the same properties, giving one of the factorizations required by **CM5** in particular. The other factorization has a similar construction.

The axiom **CM4** is a standard consequence of the method of proof of the factorization axiom **CM5**, in that any map $f : X \to Y$ which is a cofibration and a pointwise weak equivalence has a factorization $f = p \cdot j$, where p is a fibration and where j is a pointwise weak equivalence and has the right lifting property with respect to all fibrations. In particular, p is a pointwise weak equivalence, and so the indicated lifting exists in the diagram

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "f"'] & Z \arrow[d, "p"] \\ Y \arrow[r, "1_G"'] \arrow[ur, dashed] & Y. \end{tikzcd}$$

The map f is therefore a retract of j, and therefore has the left lifting property with respect to all fibrations.

The axiom **SM7** is a consequence of the corresponding statement for simplicial sets. ∎

Remark 1.5. Proposition 1.4 is a special case of a result of Dwyer and Kan [24]. The proof given here is much simpler.

Remark 1.6. The Bousfield-Kan closed model structure for I-diagrams of simplicial sets given in Example II.7.11 and Section IV.3.1 is a special case of Proposition 1.4. Every category I having all coproducts and products is a simplicial category, for which $\mathbf{Hom}(X,Y)$ is the discrete simplicial set on the set of morphisms $\hom(X,Y)$ from X to Y in I. The required isomorphisms

$$\hom(X\otimes K,Y)\cong \hom_{\mathbf{S}}(K,\mathbf{Hom}(X,Y))\cong \hom(X,\mathbf{hom}(K,Y))$$

respectively force $X\otimes K$ to be a copy of the coproduct $\bigsqcup_{\pi_0 K} X$, and dictate that $\mathbf{hom}(K,Y)$ is the product $\prod_{\pi_0 K} Y$. In the language of Section IV.3.1, the simplicial set $K\times \mathbf{Hom}(A,\ )$ is a copy of the Kan extension $F_A K$.

2. The Dwyer-Kan theorem.

A simplicial functor $X:\mathcal{A}\to\mathbf{S}$ can alternatively be described as a "rule" which associates to each object $\mathcal{A}$ a simplicial set $X(A)$, and to each morphism $\alpha:A\to B$ of $\mathcal{A}_n$ a function $\alpha_*:X(A)_n\to X(B)_n$ on the n-simplex level such that

(1) the assignment $\alpha\mapsto\alpha_*$ is functorial in morphisms α of $\mathcal{A}_n$, for all $n\geq 0$, and

(2) for each morphism $\alpha:A\to B$ of $\mathcal{A}_n$ and each ordinal number map $\theta:\mathbf{m}\to\mathbf{n}$, the following diagram commutes:

$$\begin{tikzcd} X(A)_n \arrow[r, "\alpha_*"] \arrow[d, "\theta^*"] & X(B)_n \arrow[d, "\theta^*"] \\ X(A)_m \arrow[r, "(\theta^*\alpha)_*"'] & X(B)_m. \end{tikzcd}$$

Given a simplicial functor X, with induced maps $\alpha_*:X(A)\times\Delta^n\to X(B)$ as defined above for $\alpha:A\to B$, the corresponding morphism $\alpha_*:X(A)_n\to X(B)_n$ is the assignment $x\mapsto\alpha_*(x,\iota_n)$. Conversely, given a system of maps $\alpha_*:X(A)_n\to X(B)_n$ having the properties described in (1) and (2) above, one defines a map $\alpha_*:X(A)\times\Delta^n\to X(B)$ on m-simplices (x,θ) by the assignment $(x,\theta)\mapsto(\theta^*\alpha)_*(x)$. These two assignments are inverse to each other.

EXAMPLE 2.1. Suppose that C is an object of a simplicial category $\mathcal{A}$. Then, according to this new description, the representable simplicial functor $\mathbf{Hom}(C,\ ):\mathcal{A}\to\mathbf{S}$ associates to each object A the simplicial set $\mathbf{Hom}(C,A)$ and to each morphism $\alpha:A\to B$ of A_n the function

$$\mathbf{Hom}(C,A)_n \xrightarrow{\alpha_*} \mathbf{Hom}(C,B)_n$$

defined by composition with α in A_n. It's easier to make sense of this if one recalls that $\mathbf{Hom}(C,B)_n$ is precisely the set of morphisms from C to B in the category $\mathcal{A}_n$.

From this new point of view, a natural transformation of simplicial functors $f:X\to Y$ is a collection of simplicial set maps $f:X(A)\to Y(A)$ such that, for each morphism $\alpha:A\to B$ of $\mathcal{A}_n$ the diagram of functions

$$\begin{tikzcd} X(A)_n \arrow[r, "\alpha_*"] \arrow[d, "f"'] & X(B)_n \arrow[d, "f"] \\ Y(A)_n \arrow[r, "\alpha_*"'] & Y(B)_n \end{tikzcd}$$

commutes.

Suppose that $f:\mathcal{A}\to\mathcal{B}$ and $X:\mathcal{A}\to\mathbf{S}$ are simplicial functors and that B is an object of the simplicial category $\mathcal{B}$, then the bisimplicial object $\tilde{f}^*X(B)$ defined by

$$\tilde{f}^*X(B)_{n,m}$$
$$=\bigsqcup X(A)_{0,m}\times\mathbf{Hom}(A_0,A_1)_m\times\cdots\times\mathbf{Hom}(A_{n-1},A_n)_m\times\mathbf{Hom}(fA_n,B)_m$$

The horizontal simplicial set $\tilde{f}^*X(B)_{*,m}$ is the nerve of the translation category associated to the composite functor

$$f\downarrow B\xrightarrow{Q}\mathcal{A}_m\xrightarrow{X_m}\mathbf{Sets},$$

and each morphism $\omega:B\to C$ of $\mathcal{B}_m$ determines a simplicial set map

$$\omega_*:\tilde{f}^*X(B)_{*,m}\to\tilde{f}^*X(C)_{*,m}.$$

On the other hand, if $\tau : \mathbf{k} \to \mathbf{m}$ is an ordinal number map, then there is a commutative diagram of simplicial set maps

$$\begin{tikzcd} \tilde{f}^* X(B)_{*,m} \arrow[r, "\omega_*"] \arrow[d, "\tau^*"'] & \tilde{f}^* X(C)_{*,m} \arrow[d, "\tau^*"] \\ \tilde{f}^* X(B)_{*,k} \arrow[r, "(\tau^*\omega)_*"'] & \tilde{f}^* X(C)_{*,k}. \end{tikzcd} \tag{2.2}$$

It follows that applying the diagonal simplicial set functor gives a simplicial functor $\mathcal{B} \to \mathbf{S}$ defined by $(\mathbf{n}, B) \mapsto \tilde{f}^* X(B)_{n,n}$, which will also be denoted by $\tilde{f}^* X$. The simplicial functor $\tilde{f}^* X$ is the *homotopy left Kan extension* of X along the simplicial functor f.

We've already noted that the simplicial set $\tilde{f}^* X(B)_{*,m}$ is the nerve of a translation category. Furthermore, the simplicial set diagram (2.2) is induced by a commutative diagram of functors. Suppose, for example, that the simplicial functor X is the representable functor $\mathbf{Hom}(A,\ )$. Then the corresponding simplicial set $\tilde{f}^*\mathbf{Hom}(A,\ )(B)_{*,m}$ can be identified with the nerve of the translation category associated to the composite contravariant functor

$$A \downarrow \mathcal{A}_m \xrightarrow{Q} \mathcal{A}_m \xrightarrow{f} \mathcal{B}_m \xrightarrow{\mathbf{Hom}(\ ,B)_m} \mathbf{Sets}.$$

The identity morphism on A is initial in $A \downarrow A_m$, so the *category*

$$\tilde{f}^*\mathbf{Hom}(A,\ )(B)_{*,m}$$

contracts canonically onto the discrete subcategory on the set of objects

$$\mathbf{Hom}(f(A), B)_m.$$

In this way, we obtain maps

$$\begin{cases} \mathbf{Hom}(f(A), B)_m \xrightarrow{s} \tilde{f}^*\mathbf{Hom}(A,\ )(B)_{*,m}, \\ \tilde{f}^*\mathbf{Hom}(A,\ )(B)_{*,m} \xrightarrow{r} \mathbf{Hom}(f(A), B)_m \end{cases} \tag{2.3}$$

such that $rs = 1$, as well as a natural transformation

$$\tilde{f}^*\mathbf{Hom}(A,\ )(B)_{*,m} \times \mathbf{1} \xrightarrow{H} \tilde{f}^*\mathbf{Hom}(A,\ )(B)_{*,m}$$

from the composite functor sr to the identity. All of this data is natural in B, and simplicial, and so we have proved

Lemma 2.4. *Suppose that $f : \mathcal{A} \to \mathcal{B}$ is a simplicial functor, and let A be an object of $\mathcal{A}$. Then the simplicial functor $\mathbf{Hom}(f(A),\ )$ is a strong deformation retract of $\tilde{f}^*\mathbf{Hom}(A,\ )$.*

We shall also need

Lemma 2.5. *The simplicial functors $\mathbf{Hom}(f(A),\ )$ and $\tilde{f}^*\mathbf{Hom}(A,\ )$ are cofibrant.*

Proof: The assertion about $\mathbf{Hom}(f(A),\ )$ is a consequence of the Yoneda lemma and the observation that any trivial fibration $p : X \to Y$ of $\mathbf{S}^{\mathcal{B}}$ consists of maps $p : X(B) \to Y(B)$ which are trivial fibrations of simplicial sets and are therefore surjective on vertices.

The object $\tilde{f}^*\mathbf{Hom}(A,\ )$ may also be interpreted as a simplicial object in the category $\mathbf{S}^{\mathcal{B}}$ of simplicial functors defined on $\mathcal{B}$.

Suppose, for the moment, that Z is an arbitrary simplicial object in $\mathbf{S}^{\mathcal{B}}$. Then Z consists of simplicial functors Z_n, $n \geq 0$, and simplicial structure maps $\theta^* : Z_n \to Z_m$ in $\mathbf{S}^{\mathcal{B}}$ corresponding to ordinal number maps $\theta : \mathbf{m} \to \mathbf{n}$. The object Z is, in other words, a type of diagram taking values in bisimplicial sets. The standard constructions on bisimplicial sets apply to Z. There is, in particular, an associated diagonal simplicial functor $d(Z)$ which is defined by $d(Z)_n = Z_{n,n}$, and $d(Z)$ is a co-end in the simplicial functor category for the diagrams of simplicial functor maps

$$\begin{tikzcd} Z_n \times \Delta^m \arrow[r, "1 \times \theta_*"] \arrow[d, "\theta^* \times 1"'] & Z_n \times \Delta^n \\ Z_m \times \Delta^n & \end{tikzcd}$$

arising from ordinal number maps $\theta : \mathbf{m} \to \mathbf{n}$. This means that there are simplicial functor maps $\gamma_n : Z_n \times \Delta^n \to d(Z)$ defined "in sections" by $(x, \tau) \mapsto \tau^*(x)$, and these maps assemble to give the universal arrow in a coequalizer diagram

$$\bigsqcup_{\theta:\mathbf{m}\to\mathbf{n}} Z_n \times \Delta^m \rightrightarrows \bigsqcup_{n} Z_n \to d(Z).$$

Finally, $d(Z)$ has a filtration $d(Z)^{(p)} \subset d(Z)^{(p+1)} \subset \dots$, where $d(Z)^{(p)}$ is generated by the images of the maps γ_r, $0 \leq r \leq p$. We are interested in proving that the simplicial functor $d(Z)$ is cofibrant, so the key, for us, is the existence of the pushout diagram

$$\begin{tikzcd}
(s_{[p]}Z_p \times \Delta^{p+1}) \cup (Z_{p+1} \times \partial\Delta^{p+1}) \arrow[r] \arrow[d, hook] & d(Z)^{(p+1)} \arrow[d, hook] \\
Z_{p+1} \times \Delta^{p+1} \arrow[r] & d(Z)^{(p+1)}.
\end{tikzcd}$$

in the category $\mathbf{S}^{\mathcal{B}}$. Here, the inclusion on the left is induced by the inclusion $s_{[p]}Z_p \hookrightarrow Z_{p+1}$ of the "horizontally degenerate part" in Z_{p+1}. The central observation is that the simplicial model structure on $\mathbf{S}^{\mathcal{B}}$ implies that $d(Z)$ is cofibrant if each of the inclusions $s_{[p]}Z_p \hookrightarrow Z_{p+1}$ is a cofibration.

The object at level n for the simplicial object $\tilde{f}^*\mathbf{Hom}(A,\ )$ is the disjoint union

$$\bigsqcup \mathbf{Hom}(A, A_0) \times \dots \mathbf{Hom}(A_{n-1}, A_n) \times \mathbf{Hom}(f(A_n),\ ),$$

which may in turn be written as a disjoint union

$$\bigsqcup_{C \in Ob(\mathcal{B})} B(A \downarrow \mathcal{A})_n^C \times \mathbf{Hom}(f(C),\ ),$$

where $B(A \downarrow \mathcal{A})_n^C$ indicates the simplicial set of strings

$$A \to A_0 \to \dots \to A_{n-1} \to A_n = C,$$

ending at C. The horizontal degeneracies preserve this decomposition, and so the inclusion of the degenerate part in

$$\bigsqcup_{C \in Ob(\mathcal{B})} B(A \downarrow \mathcal{A})_{n+1}^C \times \mathbf{Hom}(f(C),\ ),$$

can be identified with the map

$$\bigsqcup_{C \in Ob(\mathcal{B})} DB(A \downarrow \mathcal{A})_{n+1}^C \times \mathbf{Hom}(f(C),\ ) \hookrightarrow \bigsqcup_{C \in Ob(\mathcal{B})} B(A \downarrow \mathcal{A})_{n+1}^C \times \mathbf{Hom}(f(C),\ ), \tag{2.6}$$

which is induced by the simplicial set inclusions

$$DB(A \downarrow \mathcal{A})_{n+1}^C \subset B(A \downarrow \mathcal{A})_{n+1}^C.$$

Each of the simplicial functors $\mathbf{Hom}(f(C),\ )$ is cofibrant, so the map (2.6) is a cofibration of simplicial functors, and the simplicial functor $\tilde{f}^*\mathbf{Hom}(A,\ )$ is cofibrant, as claimed. ■

Suppose now that $Y : \mathcal{B} \to \mathbf{S}$ is a simplicial functor. Then a simplicial functor $\tilde{f}_*Y : \mathcal{A} \to \mathbf{S}$ is specified at an object A of $\mathcal{A}$ by

$$\tilde{f}_*Y(A) = \mathbf{Hom}(\tilde{f}^*\mathbf{Hom}(A,\ ), Y).$$

In other words, an n-simplex of $\tilde{f}_*(Y)(A)$, or rather a morphism

$$\Delta^n \times \mathbf{Hom}(A,\ ) \to \tilde{f}_*Y$$

is defined to be a morphism

$$\Delta^n \times \tilde{f}^*\mathbf{Hom}(A,\ ) \to Y.$$

There is an isomorphism

$$\Delta^n \times \tilde{f}^*\mathbf{Hom}(A,\ ) \cong \tilde{f}^*(\Delta^n \times \mathbf{Hom}(A,\ )),$$

by formal nonsense. The resulting isomorphisms

$$\begin{aligned} \mathrm{Nat}(\Delta^n \times \mathbf{Hom}(A,\ ), \tilde{f}_*Y) &\cong \mathrm{Nat}(\Delta^n \times \tilde{f}^*\mathbf{Hom}(A,\ ), Y) \\ &\cong \mathrm{Nat}(\tilde{f}^*(\Delta^n \times \mathbf{Hom}(A,\ )), Y) \end{aligned}$$

are natural with respect to maps

$$\Delta^m \times \mathbf{Hom}(B,\ ) \to \Delta^n \times \mathbf{Hom}(A,\ ).$$

Furthermore, every simplicial functor $X : \mathcal{A} \to \mathbf{S}$ is a colimit of its simplices, as in (1.3), and the functor $\tilde{f}^*$ preserves colimits. It follows that there is an adjunction isomorphism

$$\mathrm{Nat}(\tilde{f}^*X, Y) \cong \mathrm{Nat}(X, \tilde{f}_*Y). \tag{2.7}$$

The simplicial functor $f : \mathcal{A} \to \mathcal{B}$ also induces a functor $f_* : \mathbf{S}^{\mathcal{B}} \to \mathbf{S}^{\mathcal{A}}$, which is defined by $f_*Y(A) = Y(f(A))$. The notation is supposed to remind one of a direct image functor.

Note in particular, that the adjoint of a n-simplex

$$\Delta^n \times \mathbf{Hom}(A,\ ) \xrightarrow{x} \tilde{f}_*Y$$

is the composite

$$\tilde{f}^*(\Delta^n \times \mathbf{Hom}(A,\ )) \cong \Delta^n \times \tilde{f}^*\mathbf{Hom}(A,\ ) \xrightarrow{x} Y$$

It follows that the canonical map

$$\eta : \Delta^n \times \mathbf{Hom}(A,\ ) \to \tilde{f}_*\tilde{f}^*(\Delta^n \times \mathbf{Hom}(A,\ ))$$

is the map which sends the classifying simplex $(\iota_n, 1_A)$ to the canonical isomorphism

$$\Delta^n \times \tilde{f}^*\mathbf{Hom}(A,\ ) \xrightarrow{\cong} \tilde{f}^*(\Delta^n \times \mathbf{Hom}(A,\ )),$$

since η must be adjoint to the identity on $\tilde{f}^*(\Delta^n \times \mathbf{Hom}(A,\ ))$. Thus, if x is an n-simplex of $X(A)$, then $\eta(x) \in \tilde{f}_*\tilde{f}^*X(A)$ is the simplex defined by the composite

$$\Delta^n \times \tilde{f}^*\mathbf{Hom}(A,\ ) \xrightarrow{\cong} \tilde{f}^*(\Delta^n \times \mathbf{Hom}(A,\ )) \xrightarrow{\tilde{f}^*(x)} \tilde{f}^*X. \tag{2.8}$$

The adjunction (2.7) is easily promoted to a natural isomorphism of simplicial sets

$$\mathbf{Hom}(\tilde{f}^*X, Y) \cong \mathbf{Hom}(X, \tilde{f}_*Y). \tag{2.9}$$

We have therefore proved the first part of the following result:

Proposition 2.10. *Suppose that $f : \mathcal{A} \to \mathcal{B}$ is a functor between small simplicial categories. Then*

(1) *The functors*

$$\tilde{f}^* : \mathbf{S}^{\mathcal{A}} \leftrightarrow \mathbf{S}^{\mathcal{B}} : \tilde{f}_*$$

are simplicially adjoint in the sense that there is a natural isomorphism of simplicial sets

$$\mathbf{Hom}(\tilde{f}^*X, Y) \cong \mathbf{Hom}(X, \tilde{f}_*Y).$$

(2) *the functor $\tilde{f}_*$ preserves weak equivalences and fibrations.*

(3) *the functor $\tilde{f}^*$ preserves weak equivalences and cofibrations.*

PROOF: To prove Part (2), observe that $\tilde{f}^*\mathbf{Hom}(A,\ \ )$ is cofibrant for every object $A \in \mathcal{A}$, by Lemma 2.5, so that every map

$$\Lambda^n_k \times \tilde{f}^*\mathbf{Hom}(A,\ \ ) \to \Delta^n \times \tilde{f}^*\mathbf{Hom}(A,\ \ )$$

is a trivial cofibration on account of the simplicial model structure of the category $\mathbf{S}^{\mathcal{B}}$. It follows that the functor $\tilde{f}_*$ preserves fibrations.

The retraction map

$$r : \tilde{f}^*\mathbf{Hom}(A,\ \ ) \to \mathbf{Hom}(f(A),\ \ )$$

is natural in A, and therefore induces a map

$$r^* : f_*Y \to \tilde{f}_*Y$$

which is natural in $Y \in \mathbf{S}^{\mathcal{B}}$. It follows that $\tilde{f}_*$ preserves weak equivalences, if it can be shown that the map r^* is a natural weak equivalence.

Recall that the map r has a section

$$s : \mathbf{Hom}(f(A),\ \ ) \to \tilde{f}^*\mathbf{Hom}(A,\ \ ),$$

and that there is a homotopy

$$H : \tilde{f}^*\mathbf{Hom}(A,\ \ ) \times \Delta^1 \to \tilde{f}^*\mathbf{Hom}(A,\ \ )$$

from the composite sr to the identity. The induced map

$$H^* : \mathbf{Hom}(\tilde{f}^*\mathbf{Hom}(A,\ \ ), Y) \to \mathbf{Hom}(\tilde{f}^*\mathbf{Hom}(A,\ \ ) \times \Delta^1, Y)$$

can be composed with the canonical map

$$\mathbf{Hom}(\tilde{f}^*\mathbf{Hom}(A,\ \ ) \times \Delta^1, Y) \to \mathbf{hom}(\Delta^1, \mathbf{Hom}(\tilde{f}^*\mathbf{Hom}(A,\ \ ), Y))$$

to give a map which is adjoint to a simplicial homotopy

$$\mathbf{Hom}(\tilde{f}^*\mathbf{Hom}(A,\ \ ), Y) \times \Delta^1 \to \mathbf{Hom}(\tilde{f}^*\mathbf{Hom}(A,\ \ ), Y)$$

from the identity map to s^*r^*. This implies that r^* is a pointwise weak equivalence, as required.

For Part (3), the functor $\tilde{f}^*$ preserves weak equivalences as a consequence of its definition as a type of homotopy colimit. Also, $\tilde{f}^*$ is left adjoint to the functor $\tilde{f}_*$, and it is a consequence of Part (2) that the latter preserves trivial fibrations. Standard closed model category tricks therefore imply that the functor $\tilde{f}^*$ preserves cofibrations. ∎

COROLLARY 2.11. *The functor $f_* : \mathbf{S}^{\mathcal{B}} \to \mathbf{S}^{\mathcal{A}}$ induces an equivalence*

$$f_* : \mathrm{Ho}(\mathbf{S}^{\mathcal{B}}) \to \mathrm{Ho}(\mathbf{S}^{\mathcal{A}})$$

of homotopy categories if and only if the adjunction maps

$$\eta : X \to \tilde{f}_* \tilde{f}^* X \qquad \text{and} \qquad \epsilon : \tilde{f}^* \tilde{f}_* Y \to Y$$

are weak equivalences.

PROOF:
The functors $\tilde{f}^*$ and $\tilde{f}_*$ induce an adjoint pair of functors

$$\tilde{f}^* : \mathrm{Ho}(\mathbf{S}^{\mathcal{A}}) \leftrightarrow \mathrm{Ho}(\mathbf{S}^{\mathcal{B}}) : \tilde{f}_*$$

between the associated homotopy categories, by Proposition 2.10. If the induced functor $\tilde{f}_*$ is an equivalence of categories, then it is part of an adjoint equivalence (see [66, p.91]). The adjoint to $\tilde{f}_*$ of that adjoint equivalence is therefore naturally isomorphic to $\tilde{f}^*$ in $\mathrm{Ho}(\mathbf{S}^{\mathcal{B}})$, and so the adjunction maps η and ϵ induce isomorphisms on the homotopy category level. It follows that η and ϵ are weak equivalences.

The converse is clear. ■

There is a simplicial map $i : X(A) \to \tilde{f}^* X(f(A))$ that is defined on the bisimplicial set level to be the map

$$X(A) \to \bigsqcup X(A_0) \times \mathbf{Hom}(A_0, A_1) \times \cdots \times \mathbf{Hom}(A_{n-1}, A_n) \times \mathbf{Hom}(f(A_n), f(A))$$

by sending an n-simplex x to the element $(x, 1_A, \dots, 1_A, 1_{f(A)})$ in the summand

$$X(A) \times \mathbf{Hom}(A, A) \times \cdots \times \mathbf{Hom}(A, A) \times \mathbf{Hom}(f(A), f(A))$$

corresponding to $A_i = A$ for $0 \leq i \leq n$. It follows from the description of the adjunction map η given in (2.8) that i is the composite of the map

$$X(A) \xrightarrow{\eta} \tilde{f}_* \tilde{f}^* X(A) = \mathbf{Hom}(\tilde{f}^* \mathbf{Hom}(A, \ ), \tilde{f}^* X)$$

with the map

$$\mathbf{Hom}(\tilde{f}^* \mathbf{Hom}(A, \ ), \tilde{f}^* X) \xrightarrow{s^*} \mathbf{Hom}(\mathbf{Hom}(f(A), \ ), \tilde{f}^* X) \cong \tilde{f}^* X(f(A))$$

induced by precomposition with the section $s : \mathbf{Hom}(f(A), \ ) \to \tilde{f}^* \mathbf{Hom}(A, \ )$ given in (2.3). Neither s nor i is natural in A.

COROLLARY 2.12. *The functor $f_* : \mathbf{S}^{\mathcal{B}} \to \mathbf{S}^{\mathcal{A}}$ induces an equivalence*

$$f_* : \mathrm{Ho}(\mathbf{S}^{\mathcal{B}}) \to \mathrm{Ho}(\mathbf{S}^{\mathcal{A}})$$

of homotopy categories if and only if the following conditions are satisfied:

(1) *for every simplicial functor $X \in \mathbf{S}^{\mathcal{A}}$ and every object $A \in \mathcal{A}$ the map*

$$i : X(A) \to \tilde{f}^* X(f(A)) = f_* \tilde{f}^* X(A)$$

is a weak equivalence,

(2) *a map $\alpha : Y_1 \to Y_2$ of $\mathbf{S}^{\mathcal{B}}$ is a weak equivalence if and only if the induced map $f_*\alpha : f_*Y_1 \to f_*Y_2$ is a weak equivalence of $\mathbf{S}^{\mathcal{A}}$.*

PROOF: If f_* is an equivalence of homotopy categories, then (2) holds, by Lemma II4.1. Furthermore, the canonical map map η is a weak equivalence by Corollary 2.11. It follows from the proof of Proposition 2.10 that s^* is a weak equivalence in general, and so the map i is a weak equivalence as claimed.

Conversely, if conditions (1) and (2) are satisfied, then η is a weak equivalence, and then $\tilde{f}_*(\epsilon)$ is a weak equivalence on account of a triangle identity. The conditions for Corollary 2.11 are therefore met, since $f_*(\epsilon)$ must also be a weak equivalence. ■

Dwyer and Kan say that a functor $f : \mathcal{A} \to \mathcal{B}$ between small simplicial categories is a *weak r-equivalence* if two conditions hold:

(a) for every pair of objects A_1, A_2 of $\mathcal{A}$, the functor f induces a weak equivalence of simplicial sets

$$\mathbf{Hom}(A_1, A_2) \to \mathbf{Hom}(f(A_1), f(A_2)),$$

(b) every object in the category of components $\pi_0\mathcal{B}$ is a retract of an object in the image of $\pi_0 f$.

Every functor $f : \mathcal{A} \to \mathcal{B}$ which satisfies condition (a) and is surjective on objects is a weak r-equivalence. Most examples of weak r-equivalences that are encountered in nature have this form.

The following is the main result of this chapter:

THEOREM 2.13 (DWYER-KAN). *Suppose that $f : \mathcal{A} \to \mathcal{B}$ is a functor between small simplicial categories. Then the induced functor*

$$f_* : \mathrm{Ho}(\mathbf{S}^{\mathcal{B}}) \to \mathrm{Ho}(\mathbf{S}^{\mathcal{A}})$$

of homotopy categories is an equivalence of categories if and only if the functor f is a weak r-equivalence.

PROOF: Suppose that the functor f is a weak r-equivalence. We shall verify the conditions of Corollary 2.12.

The instance of i corresponding to the identity functor $1_{\mathcal{A}}$ on $\mathcal{A}$ has the form

$$i : X(A) \to \tilde{1}_{\mathcal{A}}^{*} X(A).$$

This map is a weak equivalence since $1_{\mathcal{A}}$ induces an equivalence $\operatorname{Ho}(\mathbf{S}^{\mathcal{A}}) \to \operatorname{Ho}(\mathbf{S}^{\mathcal{A}})$, and it fits into a commutative diagram

$$\begin{tikzcd} & \tilde{1}_{\mathcal{A}}^{*} X(A) \arrow[dd, "f_*"] \\ X(A) \arrow[ur, "i"] \arrow[dr, "i"'] & \\ & \tilde{f}^{*} X(f(A)), \end{tikzcd}$$

where the indicated map f_* is induced by a bisimplicial set map given on summands by the simplicial set maps

$$X(A_0) \times \cdots \times \mathbf{Hom}(A_n, A) \to X(A_0) \times \cdots \times \mathbf{Hom}(f(A_n), f(A))$$

defined by $1 \times \cdots \times 1 \times f$. This map f_* is a weak equivalence, by assumption, and so the map $i : X(A) \to \tilde{f}^{*} X(f(A))$ is a weak equivalence as well.

Suppose that $\alpha : Y_1 \to Y_2$ is a map of $\mathbf{S}^{\mathcal{B}}$ such that $f_*\alpha : f_*Y_1 \to f_*Y_2$ is a weak equivalence of $\mathbf{S}^{\mathcal{A}}$. Then the induced simplicial set maps $\alpha : Y_1(f(A)) \to Y_2(f(A))$ are weak equivalences for all objects $A \in \mathcal{A}$. The assumption that f is a weak r-equivalence means, in part, that for each $B \in \mathcal{B}$ there is an object $A \in \mathcal{A}$ and maps $j : B \to f(A)$ and $q : f(A) \to B$ in $\mathcal{B}$ such that qj maps to the identity in $\pi_0\mathcal{B}$. It follows that, for each $Y \in \mathbf{S}^{\mathcal{B}}$, there is a path of simplicial homotopies from the composite simplicial set map

$$Y(B) \xrightarrow{Y(j)} Y(f(A)) \xrightarrow{Y(q)} Y(B)$$

to the identity on $Y(B)$. It follows that this composite is a weak equivalence for any Y. In particular, in the diagram

$$\begin{tikzcd} Y_1(B) \arrow[r, "Y_1(j)"] \arrow[d, "\alpha"'] & Y_1(f(A)) \arrow[r, "Y_1(q)"] \arrow[d, "\alpha"] & Y_1(B) \arrow[d, "\alpha"] \\ Y_2(B) \arrow[r, "Y_2(j)"'] & Y_2(f(B)) \arrow[r, "Y_2(q)"'] & Y_2(B) \end{tikzcd}$$

both horizontal composites are weak equivalences, so that $\alpha : Y_1(B) \to Y_2(B)$ is a weak equivalence as well.

Conversely, suppose that $f : \mathcal{A} \to \mathcal{B}$ induces the indicated equivalence of homotopy categories. The following diagram commutes:

$$\begin{tikzcd} \mathbf{Hom}(B,A) \arrow[r, "i"] \arrow[dr, "f"'] & \tilde{f}^*\mathbf{Hom}(B,\ )(A) \arrow[d, "r"] \\ & \mathbf{Hom}(f(B), f(A)) \end{tikzcd}$$

The map i is a weak equivalence by assumption, and r is the weak equivalence of Lemma 2.4, and so the condition (a) for f to be weak r-equivalence is verified.

The simplicial functor $\tilde{f}^*\tilde{f}_*\mathbf{Hom}(B,\ )$ is the colimit

$$\varinjlim_{x:\tilde{f}^*\mathbf{Hom}(A,\ )\times\Delta^n \to \mathbf{Hom}(B,\ )} \tilde{f}^*\mathbf{Hom}(A,\ ),$$

and the canonical map

$$\epsilon : \tilde{f}^*\tilde{f}_*\mathbf{Hom}(B,\ ) \to \mathbf{Hom}(B,\ )$$

is the unique map having components given by the morphisms

$$x : \tilde{f}^*\mathbf{Hom}(A,\ ) \times \Delta^n \to \mathbf{Hom}(B,\ )$$

in the definition of the colimit. The natural map ϵ is a weak equivalence by assumption, and the path component functor commutes with colimits, so there is a map

$$\tilde{f}^*\mathbf{Hom}(A,\ ) \times \Delta^n \to \mathbf{Hom}(B,\ )$$

such that $1_B \in \mathbf{Hom}(B,B)_0$ lifts to a vertex of $\tilde{f}^*\mathbf{Hom}(A,B) \times \Delta^n$ up to (iterated) homotopy. The map

$$\mathbf{Hom}(f(A),B) \times \Delta^n \xrightarrow{s\times 1} \tilde{f}^*\mathbf{Hom}(A,B) \times \Delta^n$$

is a weak equivalence, so that 1_B is in the image of the composite

$$\mathbf{Hom}(f(A),B) \times \Delta^n \xrightarrow{s\times 1} \tilde{f}^*\mathbf{Hom}(A,B) \times \Delta^n \to \mathbf{Hom}(B,B)$$

up to homotopy. This composite classifies an n-simplex α of $\mathbf{Hom}(B, f(A))$, and one verifies that this maps sends a 0-simplex (γ, v) of $\mathbf{Hom}(f(A),B) \times \Delta^n$ maps to the composite $\gamma \circ v^*(\alpha)$. ∎

3. Homotopy coherence.

Homotopy coherence theory is more a collection of phenomena than theorems. The whole point of the theory is to determine when a diagram which commutes up to a system of higher homotopies can be replaced by a diagram which commutes on the nose. To this end, the game is either to recognize when systems of higher homotopies can be suitably defined to serve as input for the Dwyer-Kan theorem (Theorem 2.13), or to avoid that result altogether, as one does (preferably) with lax functors. Various examples of these phenomena will be described here. This material pertains to homotopy coherent diagrams of simplicial sets, or spaces. A discussion of homotopy coherence phenomena for other selected simplicial model categories appears in Section 4.

3.1. Classical homotopy coherence.

Suppose that I is a small category, and write UI for the underlying directed graph of I. The graph UI is *pointed* in the sense that there is a distinguished element, namely 1_a, in the set $I(a,a)$ of arrows from a to itself, for all objects $a \in I$. There is a free category FX associated to each pointed directed graph X, which has the same objects as X, and all finite composable strings of non-identity arrows in X as morphisms. Composition in FX is given by concatenation. The free category functor F is left adjoint to the underlying graph functor U, with canonical maps $\eta : 1 \to UF$ and $\epsilon : FU \to 1$. The canonical maps can be used, along with various iterations of the composite functor FU to define a simplicial category F_*I. The category F_nI of n-simplices of F_*I has the form $F_nI = (FU)^{n+1}I$, and the faces and degeneracies of F_*I are defined by

$$\begin{cases} d_i = (FU)^{i-1}\epsilon(FU)^{n-i}, & \text{and} \\ s_j = (FU)^j F\eta U(FU)^{n-j}. \end{cases}$$

One can appeal to the dual of results of [66, p.134,171] to see that this definition works, or check the simplicial identities directly. The diagram

$$FUFU(I) \underset{d_1}{\overset{d_0}{\rightrightarrows}} FU(I) \xrightarrow{\epsilon} I$$

is a coequalizer in the category of small categories. It follows that, for any pair of objects a, b of I, the diagram of functions between morphism sets

$$FUFU(I)(a,b) \underset{d_1}{\overset{d_0}{\rightrightarrows}} FU(I)(a,b) \xrightarrow{\epsilon} I(a,b)$$

is a coequalizer. It can be shown directly that the underlying simplicial graph UF_*I has an extra degeneracy (see Section III.5), given by the functors

$$s_{-1} = \eta U(FU)^n(I) \to U(FU)^{n+1}(I).$$

Now, $UF_*(I)(a,b) = F_*(I)(a,b)$ as a simplicial set, so that $F_*(I)(a,b)$ has an extra degeneracy. The simplicial set map

$$F_*I(a,b) \to K(I(a,b),0)$$

induced by ϵ is therefore a weak equivalence by Lemma III.5.1, so we have constructed a weak r-equivalence

$$\epsilon : F_*I \to K(I,0)$$

of simplicial categories called *simplicial free resolution* of the category I. It is a special case of a construction that is quite ubiquitous.

The simplicial free resolution F_*I of a category I is the traditional basis of the definition of homotopy coherence. Explicitly, a *homotopy coherent* I-diagram is classically defined to be a simplicial functor $X : F_*I \to \mathbf{S}$.

Homotopy coherent diagrams are notoriously difficult to interpret, much less construct. Intuitively, $X : F_*I \to \mathbf{S}$ associates a simplicial map $\alpha_* : X(a) \to X(b)$ to each non-identity morphism $\alpha : a \to b$ of I (α_* is the image of the string $(\alpha) \in FU(I)$). Given $\beta : b \to c$ in I, it's *not* the case that $(\beta\alpha)_* = \beta_*\alpha_*$, but rather that the two maps are homotopic in a precise way, via the homotopy $X((\beta,\alpha)) : X(a) \times \Delta^1 \to X(c)$ which is the image of the arrow $((\beta,\alpha)) \in FUFU(I)$. Given yet another I-morphism $\gamma : c \to d$, one sees that the associativity relationship $\gamma(\beta\alpha) = \gamma\beta\alpha$ in I in the sense that the morphism $(((\gamma),(\beta,\alpha)))$ of $FUFUFU(I)$ determines a map $X(a) \times \Delta^2 \to X(d)$, which is a higher homotopy that has the homotopies

$$\begin{tikzcd}
(\gamma\beta\alpha)_* \arrow[r, "X((\gamma,\beta\alpha))"] \arrow[dr, "X((\gamma,\beta,\alpha))"'] & \gamma_*(\beta\alpha)_* \arrow[d, "X((\gamma),(\beta,\alpha))"] \\
& \gamma_*\beta_*\alpha_*
\end{tikzcd}$$

appearing in its 1-skeleton. The higher, or iterated, associativity relation

$$\omega(\gamma(\alpha\beta)) = \omega\gamma\alpha\beta$$

corresponds to the 3-simplex $((((\omega)),((\gamma),(\beta,\alpha))))$ of F_*I. The homotopy coherent diagram X is determined by all higher associativities or iterated bracketings of strings $\alpha_n\alpha_{n-1}\cdots\alpha_1$ of morphisms in I.

Homotopy coherent diagrams are very rarely constructed from scratch, although there are obstruction theoretic techniques for doing so [27]. They nevertheless appear quite naturally, usually as the output of large categorical machines — more will be said about this below.

Most of the point of having a homotopy coherent diagram in hand is that one can immediately replace it up to pointwise weak equivalence by a diagram that commutes on the nose. This is a consequence of the following realization theorem:

THEOREM 3.1 (REALIZATION). *Suppose that $X : F_*I \to \mathbf{S}$ is a homotopy coherent diagram in the category of simplicial sets. Then X is naturally pointwise weakly equivalent to a diagram ϵ_*Y, for some ordinary I-diagram $Y : I \to \mathbf{S}$.*

PROOF: This result follows easily from the Dwyer-Kan theorem 2.13 and its proof: the simplicial functor $\epsilon : F_*I \to K(I,0)$ is a weak r-equivalence, so there are natural pointwise weak equivalences

$$X \xrightarrow{i} \tilde{\epsilon}_*\tilde{\epsilon}^*X \xleftarrow{r^*} \epsilon_*\tilde{\epsilon}^*X.$$

Take Y to be $\tilde{\epsilon}^*X$. ■

3.2. Homotopy coherence: an expanded version.

The classical definition of homotopy coherence is interesting and complicated, but not flexible enough for all applications. It does not, for example, take into account diagrams where images of identity maps could wiggle away from actual identities up to controlled homotopy The realization result, Theorem 3.1, is also obviously just a special case of a much broader statement.

Define a *resolution of a category I* to be a simplicial functor $\pi : \mathcal{A} \to K(I,0)$ such the π is a weak r-equivalence. A *homotopy coherent diagram* on the category I shall henceforth be defined to be a simplicial functor $X : \mathcal{A} \to \mathbf{S}$, where $\pi : \mathcal{A} \to K(I,0)$ is a resolution of I. We now have an expanded form of the realization theorem:

THEOREM 3.2. *Suppose that $X : \mathcal{A} \to \mathbf{S}$ is a homotopy coherent diagram which is defined with respect to some realization $\pi : \mathcal{A} \to K(I,0)$ of I. Then X is naturally pointwise equivalent to a diagram of the form π_*Y, for some I-diagram $Y : I \to \mathbf{S}$.*

PROOF: The proof is a copy of the proof of Theorem 3.1. ■

The rest of this section will be taken up with a description of a natural resolution $BI_s \to K(I,0)$ of the category I which is different from the simplicial free resolution of the previous section. Homotopy coherent diagrams $X : BI_s \to \mathbf{S}$ provide for variance of morphisms induced by the identities of I up to homotopy.

Let a and b be objects of the small category I. There is a category, denoted $I_s(a,b)$, whose objects are the functors of the form $\theta : \mathbf{n} \to I$, with $\theta(0) = a$ and $\theta(n) = b$. The morphisms of $I_s(a,b)$ are commutative diagrams

$$\begin{tikzcd} \mathbf{n}_0 \arrow[dr, "\theta_0"] \arrow[dd, "\gamma"'] & \\ & I \\ \mathbf{n}_1 \arrow[ur, "\theta_1"'] & \end{tikzcd}$$

where θ_0 and θ_1 are objects of $I_s(a,b)$ and $\gamma : \mathbf{n}_0 \to \mathbf{n}_1$ is an ordinal number map which is *end point preserving* in the sense that $\gamma(0) = 0$ and $\gamma(n_0) = n_1$.

Suppose that $\theta : \mathbf{n} \to I$ is an object of $I_s(a,b)$ and $\omega : \mathbf{m} \to I$ is an object of $I_s(b,c)$. The poset join $\mathbf{n} * \mathbf{m}$ may be identified up to isomorphism with the ordinal number $\mathbf{n} + \mathbf{m}$, in such a way that the inclusion $\mathbf{n} \hookrightarrow \mathbf{n} * \mathbf{m}$ is identified with the ordinal number map $\mathbf{n} \to \mathbf{n} + \mathbf{m}$ defined by $j \mapsto j + n$ for $0 \leq j \leq m$. The objects θ and γ together determine a functor $\theta * \gamma : \mathbf{n} + \mathbf{m} \to I$, defined by

$$\theta * \gamma(j) = \begin{cases} \theta(j), & \text{if } 0 \leq j \leq n, \\ \gamma(j-n), & \text{if } n \leq j \leq m+n. \end{cases}$$

This is plainly the object level description of a functor

$$I_s(a,b) \times I_s(b,c) \xrightarrow{*} I_s(a,c),$$

which is called the *join functor*. This operation is associative, and has a two-sided identity in each $I_s(a,a)$ given by the object $a : \mathbf{0} \to I$.

We have constructed a category I_s which has the same objects as I, and is enriched in the category **cat** of small categories. Applying the nerve construction to each of the categories $I_s(a,b)$ gives simplicial sets $BI_s(a,b)$, which then form the morphism objects for a simplicial category BI_s, which has the same objects as I. These constructions are obviously natural: a functor $f : I \to J$ induces a functor $f : I_s \to J_s$ of categories enriched in **cat**, and hence determines a functor $f : BI_s \to BJ_s$ of of simplicial categories.

The category $I_s(a,b)$ has the form

$$I_s(a,b) = \bigsqcup_{\substack{f:a\to b \\ \text{in } I}} I_s(a,b)_f,$$

where $I_s(a,b)_f$ is the subcategory of those strings $\alpha : \mathbf{n} \to I_s$ whose composite is $f : a \to b$. The category $I_s(a,b)_f$ has an initial object, namely the functor $f : \mathbf{1} \to I_s$ which is determined by f, so that, on the simplicial set level, we have a decomposition

$$BI_s(a,b) = \bigsqcup_{\substack{f:a\to b \\ \text{in } I}} BI_s(a,b)_f,$$

of $BI_s(a,b)$ into connected components, each of which is contractible. It follows that the path component functor determines a resolution $BI_s \to K(I,0)$ of the category I.

A homotopy coherent diagram $X : BI_s \to \mathbf{S}$ exhibits the standard homotopy coherence phenomena. Suppose that

$$a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} a_2$$

is a composable pair of morphisms of I, and let these morphisms canonically determine a functor $\alpha : \mathbf{2} \to I$, which in turn is a 0-simplex of $BI_s(a_0,a_2)$. Then X associates simplicial set morphisms $(\alpha_1)_* : X(a_0) \to X(a_1)$ and $(\alpha_2)_* : X(a_1) \to X(a_2)$ to the morphisms (aka. 0-simplices) α_1 and α_2 respectively, and associates the composite simplicial set map

$$(\alpha_2)_*(\alpha_1)_* : X(a_0) \to X(a_2)$$

to the simplex α, since X takes joins to composites. There is a 1-simplex σ of $BI_s(a_0,a_2)$, defined by the picture

$$\begin{tikzcd} \mathbf{2} \arrow[dr, "\alpha"] & \\ & I_s, \\ \mathbf{1} \arrow[uu, "d^1"] \arrow[ur, "\alpha_2\alpha_1"'] & \end{tikzcd}$$

where $\mathbf{1} \to \mathbf{2}$ is the unique endpoint preserving ordinal number map, and this 1-simplex is mapped to a homotopy $X(\sigma) : X(a_0) \times \Delta^1 \to X(a_2)$ from $(\alpha_2\alpha_1)_*$ to the composite $(\alpha_2)_*(\alpha_1)_*$. In the same way, the associativity relation $\alpha_3(\alpha_2\alpha_1) = \alpha_3\alpha_2\alpha_1$ gives rise to a 2-simplex

$$\begin{tikzcd} \mathbf{1} \arrow[r, "d^1"] \arrow[dr, "\alpha_3\alpha_2\alpha_1"'] & \mathbf{2} \arrow[r, "d^2"] \arrow[d, "(\alpha_3\alpha_2,\alpha_1)"'] & \mathbf{3} \arrow[dl, "(\alpha_3,\alpha_2,\alpha_1)"] \\ & I & \end{tikzcd}$$

and hence to a higher homotopy $X(a_0) \times \Delta^2 \to X(a_3)$.

If a is an object of I, then the identity map $1_a : a \to a$ defines a 0-simplex $1_a : \mathbf{1} \to I$, and hence gets mapped to the simplicial set map $(1_a)_* : X(a) \to X(a)$, whereas the 0-simplex $a : \mathbf{0} \to I$ is an identity for BI_s, and is therefore sent to the identity $1_{X(a)}$ on $X(a)$. There is a 1-simplex η of $BI_s(a,a)$, of the form

$$\begin{tikzcd} \mathbf{0} \arrow[dr, "a"] & \\ & I \\ \mathbf{1} \arrow[uu] \arrow[ur, "1_a"'] & \end{tikzcd}$$

which is then mapped to a homotopy $X(\eta) : X(a) \times \Delta^1 \to X(\alpha)$ from $(1_a)_*$ to $1_{X(a)}$.

3.3. Lax functors.

A *lax functor* $F : I \rightsquigarrow \mathbf{cat}$ associates a category $F(a)$ to each object a of I, and associates a functor $\alpha_* : F(a) \to F(b)$ to each morphism $\alpha : a \to b$, in such a way that there are natural transformations

$$\theta(\beta,\alpha) : (\beta\alpha)_* \to \beta_*\alpha_* \quad \text{and} \quad \eta_a : (1_a)_* \to 1_{F(a)},$$

which together satisfy the cocycle conditions

$$\begin{tikzcd} (\beta\alpha\gamma)_* \arrow[r, "\theta(\beta\alpha,\gamma)"] \arrow[d, "\theta(\beta,\alpha\gamma)"'] & (\beta\alpha)_*\gamma_* \arrow[d, "\theta(\beta,\alpha)\gamma_*"] \\ \beta_*(\alpha\gamma)_* \arrow[r, "\beta_*\theta(\alpha,\gamma)"'] & \beta_*\alpha_*\gamma_* \end{tikzcd} \tag{3.3}$$

$$(3.4)\qquad \begin{tikzcd} \alpha_* 1_{F(a)} & \alpha_*(1_a)_* \arrow[l, "\alpha_*\eta_a"'] \\ \alpha_* \arrow[u, equal] \arrow[r, equal] & (\alpha 1_a)_* \arrow[u, "\theta(\alpha, 1_a)"'] \end{tikzcd} \qquad \begin{tikzcd} (1_b)_*\alpha_* \arrow[r, "\eta_b\alpha_*"] & 1_{F(b)}\alpha_* \\ (1_b\alpha)_* \arrow[u, "\theta(1_b, \alpha)"] \arrow[r, equal] & \alpha_*. \arrow[u, equal] \end{tikzcd}$$

The category **cat** of categories is enriched in categories. Its morphism categories $\mathbf{Hom}(A, B)$ have the functors from A to B as objects with natural transformations as morphisms. The composition functor

$$\mathbf{Hom}(A, B) \times \mathbf{Hom}(B, C) \to \mathbf{Hom}(A, C)$$

is defined on morphisms as follows: given natural transformations α_1 and α_2

$$A \quad \begin{matrix} \xrightarrow{\quad F_1 \quad} \\ \downarrow \alpha_1 \\ \xrightarrow[\quad G_1 \quad]{} \end{matrix} \quad B \quad \begin{matrix} \xrightarrow{\quad F_2 \quad} \\ \downarrow \alpha_2 \\ \xrightarrow[\quad G_2 \quad]{} \end{matrix} \quad C$$

one defines the composition $\alpha_2\alpha_1$ of α_1 and α_2 to be the diagonal transformation appearing in the commutative diagram

$$\begin{tikzcd} F_2F_1 \arrow[r, "F_2\alpha_1"] \arrow[d, "\alpha_2 F_1"'] \arrow[dr] & F_2G_1 \arrow[d, "\alpha_2 G_1"] \\ G_2F_1 \arrow[r, "G_2\alpha_1"'] & G_2G_1. \end{tikzcd}$$

There is a one to one correspondence between functors

$$I_s \to \mathbf{cat}$$

and lax functors on I. In particular, given a lax functor F as above, associate to each string $\mathbf{n} \to I$ in I of the form

$$a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} a_n$$

the composite functor

$$(\alpha_n)_* \dots (\alpha_1)_* : F(a_0) \to F(a_n).$$

The requisite functors

$$I_s(a_0, a_n) \to \mathbf{Hom}(F(a_0), F(a_n))$$

are defined on morphisms of $I_s(a_0, a_n)$ by first making a definition on cofaces and codegeneracies, which are completely determined up to join by the transformations θ and η, and then by showing that the relevant cosimplicial identities are satisfied. The non-trivial cosimplicial identities amount to the cocycle conditions which appear in the definition of the lax functor F.

The object **cat** has associated to it a simplicial category $B\mathbf{cat}$ in the obvious way: one uses the nerves $B\mathbf{Hom}(A, C)$ of the categories of morphisms $\mathbf{Hom}(A, C)$ appearing in **cat**. Note that $B\mathbf{Hom}(A, C)$ is canonically isomorphic to the function complex $\mathbf{Hom}(BA, BC)$. The collection of such canonical isomorphisms respects the composition laws of $B\mathbf{cat}$ and $\mathbf{S}$ and hence determine a functor $B\mathbf{cat} \to \mathbf{S}$. It follows in particular that any lax functor $F : I_s \to \mathbf{cat}$ determines a homotopy coherent diagram

$$BI_s \xrightarrow{BF} B\mathbf{cat} \to \mathbf{S}$$

on I.

3.4. The Grothendieck construction.

Homotopy coherent diagrams arising from lax functors are most often realized (or *rectified*) by using a category theoretic method that is known as the Grothendieck construction in place of a result like Theorem 3.2. Suppose that $F : I \rightsquigarrow \mathbf{cat}$ is a lax functor. The *Grothendieck construction* associated to F is a category LF whose set of objects consists of all pairs (a, x), where a is an object of I and x is an object of the category $F(a)$. A morphism $(\alpha, f) : (a, x) \to (b, y)$ is a pair consisting of a morphism $\alpha : a \to b$ of the base category I and a morphism $\alpha_*(x) \to y$ of the category $F(b)$. The composite of (α, x) with the morphism $(\beta, g) : (b, y) \to (c, z)$ of LF is defined to be the map $(\beta\alpha, g * f)$, where $g * f$ is the composite

$$(\beta\alpha)_*(x) \xrightarrow{\theta(\beta,\alpha)} \beta_*\alpha_*(x) \xrightarrow{\beta_*(f)} \beta_*(y) \xrightarrow{g} z$$

of $F(c)$. The identity morphism on an element (a, x) of LF is the morphism $(1_a, \eta_a)$. The associativity of the composition operation in LF and the fact that the morphisms $(1_a, x)$ are two-sided identities are, respectively, consequences of the cocycle conditions (3.3) and (3.4).

Projection onto the first variable, for both objects and morphisms, defines a canonical functor $\pi_F : LF \to I$. The comma category $\pi_F \downarrow c$ associated to an object c of I has all morphisms $\alpha : \pi_F(a,x) \to c$ for objects. There is a functor $f_c : \pi_F \downarrow c \to F(c)$ which associates the object $\alpha_*(c) \in F(I)$ to the object $\alpha : \pi_F(a,x) \to c$. Also, there is a functor $g_c : F(c) \to \pi_F \downarrow c$ which associates the object $1_c : \pi_F(c,x) \to c$ to $x \in F(c)$. The functor f_c is left adjoint to g_c, as can be seen by observing that the commutative diagram

$$\begin{tikzcd} \pi_F(a,x) \arrow[d, "{(\alpha,h)}"'] \arrow[dr, "\alpha"] & \\ \pi_F(c,y) \arrow[r, "1_c"'] & c \end{tikzcd}$$

is uniquely determined by the map $h : \alpha_*(x) \to y$ of $F(c)$.

Any morphism $\beta : c \to d$ induces a functor $\beta_* : \pi_F \downarrow c \to \pi_F \downarrow d$ in an obvious way, and this assignment determines a functor $\pi_F \downarrow ? : I \to \mathbf{cat}$. The diagram of functors

$$\begin{tikzcd} F(c) \arrow[r, "g_c"] \arrow[d, "\beta_*"'] & \pi_F \downarrow c \arrow[d, "\beta_*"] \\ F(c) \arrow[r, "g_d"'] & \pi_F \downarrow d \end{tikzcd}$$

commutes up to canonically determined natural transformation.

Generally, any functor $f : D \to E$ that has a left or right adjoint induces a homotopy equivalence $f_* : BD \to BE$ of the associated nerves. This is a result of the fact that any natural transformation of functors gives rise to a homotopy of the respective induced maps of simplicial sets. The Grothendieck construction therefore gives rise to homotopy equivalences

$$g_{c*} : BF(c) \xrightarrow{\simeq} B(\pi_F \downarrow c)$$

such that the diagrams

$$\begin{tikzcd} BF(c) \arrow[r, "g_{c*}"] \arrow[d, "\beta_*"'] & B(\pi_F \downarrow c) \arrow[d, "\beta_*"] \\ BF(c) \arrow[r, "g_{d*}"'] & B(\pi_F \downarrow d) \end{tikzcd}$$

commute up to homotopy. In other words, the I-diagram $c \mapsto B(\pi_F \downarrow c)$ is a realization of the homotopy coherent diagram $c \mapsto BF(c)$.

4. Realization theorems.

Suppose that $\pi : \mathcal{A} \to K(I,0)$ is a resolution of a small category I, and let $X : \mathcal{A} \to \mathbf{S}$ be a homotopy coherent diagram on I. A *realization* of X is a simplicial set valued functor $Y : I \to \mathbf{S}$ such that there is a pointwise weak equivalence $X \simeq \pi_* Y$ in the simplicial functor category $\mathbf{S}^{\mathcal{A}}$. Theorem 3.2 says that any homotopy coherent diagram X has a realization, and that the weak equivalence in $\mathbf{S}^{\mathcal{A}}$ can be chosen canonically.

Given an arbitrary simplicial model category $\mathcal{M}$ and a resolution $\pi : \mathcal{A} \to K(I,0)$, it is certainly sensible to say that a homotopy coherent diagram on I is a simplicial functor $X : \mathcal{A} \to \mathcal{M}$. One analogously defines a realization of X to be a functor $Y : I \to \mathcal{M}$ such that there is a pointwise weak equivalence $X \simeq \pi_* Y$. The purpose of this section is to show that all homotopy coherent diagrams in $\mathcal{M}$ admit realizations if the simplicial model category $\mathcal{M}$ has an adequate notion of homotopy colimit. This is done by giving a proof of Theorem 3.2 which does not depend on the Yoneda lemma 1.2. The proof is achieved by thinking about the functor $\tilde{\pi}^*$ in a different way; we lose the functor $\tilde{\pi}_*$ and along with it any notion of an equivalence of homotopy categories associated to the categories $\mathbf{S}^{\mathcal{A}}$ and $\mathbf{S}^I$. We begin with a weakened version of the Dwyer-Kan theorem.

THEOREM 4.1. *Suppose that a simplicial functor $f : \mathcal{A} \to \mathcal{B}$ is homotopically full and faithful in the sense that all induced simplicial set maps*

$$f_* : \mathbf{Hom}(A,B) \to \mathbf{Hom}(f(A), f(B))$$

are weak equivalences, and let $X : \mathcal{A} \to \mathbf{S}$ be a simplicial functor. Then X is naturally pointwise weakly equivalent to the simplicial functor $f_* \tilde{f}^* X$.

Proof: The assumption on f implies that the map of simplicial objects in $\mathbf{S}^{\mathcal{B}}$ defined by

$$\begin{array}{c} \displaystyle\bigsqcup_{(A_0,\dots,A_n)} X(A_0)\times\mathbf{Hom}(A_0,A_1)\times\dots\times\mathbf{Hom}(A_n,A) \\ \Big\downarrow{\scriptstyle 1\times f_*} \\ \displaystyle\bigsqcup_{(A_0,\dots,A_n)} X(A_0)\times\mathbf{Hom}(A_0,A_1)\times\dots\times\mathbf{Hom}(f(A_n),f(A)) \end{array}$$

is a levelwise weak equivalence, which is natural in $A\in\mathcal{A}$. The simplicial object

$$\bigsqcup_{(A_0,\dots,A_n)} X(A_0)\times\mathbf{Hom}(A_0,A_1)\times\dots\times\mathbf{Hom}(A_n,A)$$

is, in n-simplices, the nerve of the translation category associated to the composite functor

$$\mathcal{A}_m\downarrow A\xrightarrow{Q}\mathcal{A}_m\xrightarrow{X_m}\mathbf{Sets},$$

and the identity element 1_A is terminal in $\mathcal{A}_m\downarrow A$. It follows (see the development around (2.3)) that there is a canonical weak equivalence induced by the bisimplicial set map

$$r:\bigsqcup_{(A_0,\dots,A_n)} X(A_0)\times\mathbf{Hom}(A_0,A_1)\times\dots\times\mathbf{Hom}(A_n,A)\to X(A)$$

which is induced on the translation category level by functors

$$(x,\alpha:A'\to A)\mapsto\alpha_*(x).$$

We therefore have natural pointwise weak equivalences

$$f_*\tilde{f}^*X\xleftarrow{\simeq}1_*\tilde{1}^*X\xrightarrow[\simeq]{r}X. \qquad \blacksquare$$

Suppose now that $f:\mathcal{A}\to\mathcal{B}$ is a simplicial functor, and that $Y:\mathcal{A}\to\mathcal{M}$ is a simplicial functor taking values in a simplicial model category $\mathcal{M}$. We are entitled to an analogue of the functor $\tilde{f}^*$; in particular, we define $\tilde{f}^*Y(B)$ to be the simplicial object having n-simplices

$$\bigsqcup_{(A_0,\dots,A_n)} Y(A_0) \otimes \mathbf{Hom}(A_0, A_1) \otimes \cdots \otimes \mathbf{Hom}(f(A_n), B).$$

In this way, we define a functor

$$\tilde{f}^* : \mathcal{M}^{\mathcal{A}} \to \mathbf{S}(\mathcal{M}^{\mathcal{B}}),$$

where $\mathbf{S}(\mathcal{M}^{\mathcal{B}})$ denotes the category of simplicial objects in $\mathcal{M}^{\mathcal{B}}$. We are also entitled to the following analogues of the maps appearing in the proof of Theorem 4.1 in the category $\mathbf{S}(\mathcal{M}^{\mathcal{B}})$:
(4.2)

$$\begin{array}{ccc} \displaystyle\bigsqcup_{(A_0,\dots,A_n)} Y(A_0) \otimes \mathbf{Hom}(A_0, A_1) \otimes \cdots \otimes \mathbf{Hom}(A_n, A) & \xrightarrow{\quad r \quad} & Y(A) \\ \downarrow{\scriptstyle f_*} & & \\ \displaystyle\bigsqcup_{(A_0,\dots,A_n)} Y(A_0) \otimes \mathbf{Hom}(A_0, A_1) \otimes \cdots \otimes \mathbf{Hom}(f(A_n), f(A)). & & \end{array}$$

The trick, either for a given simplicial model category or for a particular class of objects of the simplicial functor category $\mathcal{M}^{\mathcal{A}}$, is to find a realization functor $\mathbf{S}(\mathcal{M}) \to \mathcal{M}$ which takes the maps in this diagram to weak equivalences of $\mathcal{M}^{\mathcal{B}}$.

The naive realization $d(Z)$ of a simplicial object Z in $\mathcal{M}$ is the coend, given by the coequalizer

$$\bigsqcup_{\mathbf{m}\to\mathbf{n}} Z_n \otimes \Delta^m \rightrightarrows \bigsqcup_{n\geq 0} Z_n \otimes \Delta^n \to d(Z).$$

The object $d(Z)$ has a filtration

$$d(Z)^{(n)} \subset d(Z)^{(n+1)} \subset \dots,$$

where $d(Z)^{(n)}$ is the epimorphic image of the restricted map

$$\bigsqcup_{p\leq n} Z_p \otimes \Delta^p \to d(Z).$$

There are pushout diagrams

$$\begin{tikzcd}
(s_{[n]}Zn \otimes \Delta^{n+1}) \cup (Z_{n+1} \otimes \partial\Delta^{n+1}) \arrow[r] \arrow[d, "j_*"'] & d(Z)^{(n)} \arrow[d] \\
Z_{n+1} \otimes \Delta^{n+1} \arrow[r] & d(Z)^{(n+1)},
\end{tikzcd}$$

in $\mathcal{M}$, where the map j_* is canonically induced by the monomorphism $j : s_{[n]}Z_n \hookrightarrow Z_{n+1}$ given by taking the epimorphic image of the map

$$s : \bigsqcup_{0 \leq i \leq n} Z_n \to Z_{n+1}$$

which applies the i^{th} degeneracy s_i on the i^{th} summand. The monomorphism $d(Z)^{(n)} \hookrightarrow d(Z)^{(n+1)}$ is a cofibration of $\mathcal{M}$ if $j : s_{[n]}Z_n \to Z_{n+1}$ is a cofibration.

Say that a simplicial object Z in $\mathcal{M}$ is *diagonally cofibrant* if

(1) Z_0 is a cofibrant object of $\mathcal{M}$, and
(2) each morphism $j : s_{[n]}Z_n \to Z_{n+1}$ is a cofibration.

If Z is a diagonally cofibrant simplicial object of $\mathcal{M}$ in this sense, then

$$d(Z)^{(0)} = Z_0$$

is cofibrant and all of the maps $d(Z)^{(n)} \hookrightarrow d(Z)^{(n+1)}$ are cofibrations, and so the realization $d(Z)$ is a cofibrant object of $\mathcal{M}$. It also follows that if $f : Z \to W$ is a map of diagonally cofibrant simplicial objects which is a levelwise weak equivalence in the sense that all of the maps $f : Z_n \to W_n$ are weak equivalences of $\mathcal{M}$, then the induced map $f_* : d(Z) \to d(W)$ is a weak equivalence as well.

THEOREM 4.3. *Suppose that $f : \mathcal{A} \to \mathcal{B}$ is a simplicial functor which is homotopically full and faithful. Suppose that $Y : \mathcal{A} \to \mathcal{M}$ is a simplicial functor taking values in simplicial model category $\mathcal{M}$, such that $Y(A)$ is a cofibrant object of $\mathcal{M}$ for all objects $A \in \mathcal{A}$. Then the morphisms f_* and r induce pointwise weak equivalences*

$$f_*(d\tilde{f}^*Y) \cong d(f_*\tilde{f}^*Y) \underset{\simeq}{\xleftarrow{f_*}} d(1_*\tilde{1}^*Y) \underset{\simeq}{\xrightarrow{r}} d(Y) = Y.$$

Proof: The assumptions imply that the simplicial object $\tilde{f}^*Y(B)$ is diagonally cofibrant, for all $B \in \mathcal{B}$. This is seen by observing that the object

$$\bigsqcup_{(A_0,\dots,A_n)} Y(A_0) \otimes \mathbf{Hom}(A_0, A_1) \otimes \cdots \otimes \mathbf{Hom}(f(A_n), B)$$

can be rewritten in the form

$$\bigsqcup_{A_0} Y(A_0) \otimes B(f \downarrow B)^{A_0}_n,$$

where

$$B(f \downarrow B)^{A_0}_n = \mathbf{Hom}(A_0, A_1) \times \cdots \times \mathbf{Hom}(f(A_n), B)$$

means strings of length n in the simplicial nerve $B(f \downarrow B)$ which begin at A_0. This decomposition is preserved by all degeneracies, and so we have

$$s_{[n]}\tilde{f}^*(B) = \bigsqcup_{A_0} X(A_0) \otimes DB(f \downarrow B)^{A_0}_{n+1},$$

where $DB(f \downarrow B)^{A_0}_{n+1}$ denotes the degenerate strings of length $n+1$ which begin at A_0. Each map

$$Y(A_0) \otimes DB(f \downarrow B)^{A_0}_{n+1} \to Y(A_0) \otimes B(f \downarrow B)^{A_0}_{n+1}$$

induced by the simplicial set inclusion $DB(f \downarrow B)^{A_0}_{n+1} \subset B(f \downarrow B)^{A_0}_{n+1}$ is a cofibration, since $Y(A_0)$ is cofibrant, and the claim is verified.

The maps

$$\begin{array}{c}
\displaystyle\bigsqcup_{(A_0,\dots,A_n)} Y(A_0) \otimes \mathbf{Hom}(A_0, A_1) \otimes \cdots \otimes \mathbf{Hom}(A_n, A) \\
\Big\downarrow {\scriptstyle 1 \otimes f_*} \\
\displaystyle\bigsqcup_{(A_0,\dots,A_n)} Y(A_0) \otimes \mathbf{Hom}(A_0, A_1) \otimes \cdots \otimes \mathbf{Hom}(f(A_n), f(A))
\end{array}$$

are weak equivalences since all $Y(A_0)$ are cofibrant and f_* is a weak equivalence by assumption. It follows that the map

$$f_* : d(1_*\tilde{1}^*Y(A)) \to d(f_*\tilde{f}^*(f(A)))$$

is a weak equivalence, since the simplicial objects at issue are diagonally cofibrant.

The retraction map $r : 1_*\tilde{1}^*Y(A) \to Y(A)$ has a section $s : Y(A) \to 1_*\tilde{1}^*Y(A)$, along with an associated homotopy

$$h : 1_*\tilde{1}^*Y(A) \otimes \Delta^1 \to 1_*\tilde{1}^*Y(A)$$

from sr to the identity on $1_*\tilde{1}^*Y(A)$. Applying the coend functor to h gives a homotopy

$$d(h) : d(1_*\tilde{1}^*Y(A)) \otimes \Delta^1 \to d(1_*\tilde{1}^*Y(A)),$$

from s_*r_* to the identity, again since $1_*\tilde{1}^*Y(A)$ is diagonally cofibrant. It follows from Lemma II.4.1 that r_* is a weak equivalence of $\mathcal{M}$. ■

All pointed simplicial sets are cofibrant, so Theorem 4.3 immediately implies

COROLLARY 4.4. *Suppose that a simplicial functor $f : \mathcal{A} \to \mathcal{B}$ is homotopically full and faithful, and let $X : \mathcal{A} \to \mathbf{S}_*$ be a simplicial functor taking values in the category $\mathbf{S}_*$ of pointed simplicial sets. Then X is naturally pointwise weakly equivalent to the simplicial functor $f_*d(\tilde{f}^*X)$.*

Spectra, and the Bousfield-Friedlander model for the stable category, are formally discussed in Section IX.6 below. Within that model, it is not true that all spectra are cofibrant, but homotopy colimits in the category of spectra are constructed levelwise within the category of pointed simplicial sets, giving

COROLLARY 4.5. *Suppose that a simplicial functor $f : \mathcal{A} \to \mathcal{B}$ is homotopically full and faithful, and let $X : \mathcal{A} \to \mathbf{Spt}$ be a simplicial functor taking values in the category $\mathbf{Spt}$ of spectra. Then X is naturally pointwise strictly weakly equivalent to the simplicial functor $f_*d(\tilde{f}^*X)$.*

A simplicial functor $X : \mathcal{A} \to \mathbf{Spt}$ taking values in the category of spectra can be identified with a spectrum object in the category $\mathbf{S}_*^{\mathcal{A}}$ of pointed simplicial set valued simplicial functors. The object X therefore consists of simplicial functors $X_n : \mathcal{A} \to \mathbf{S}_*$ and pointed transformations $S^1 \wedge X_n \to X_{n+1}$. A map $X \to Y$ in $\mathbf{Spt}^{\mathcal{A}}$ is a pointwise strict weak equivalence if all of the maps $X_n(A) \to Y_n(A)$ are weak equivalences of pointed simplicial sets.

Despite all the huffing and puffing in its proof, Theorem 4.3 is a formal result which may not always be the best tool. To illustrate, suppose that $Y : \mathcal{A} \to s\mathbf{Ab}$ is a simplicial functor taking values in the category of simplicial abelian groups, and consider the maps of bisimplicial abelian groups corresponding to the diagram (4.2). One uses spectral sequence arguments and the generalized Eilenberg-Zilber theorem (Theorem IV.2.5) to see that, if $f : \mathcal{A} \to \mathcal{B}$ is homotopically faithful, then the maps f_* and r induce weak equivalences of the associated diagonal simplicial abelian groups, proving

LEMMA 4.6. *Suppose that a simplicial functor $f : \mathcal{A} \to \mathcal{B}$ is homotopically full and faithful, and let $Y : \mathcal{A} \to s\mathbf{Ab}$ be a simplicial functor taking values in the category $s\mathbf{Ab}$ of simplicial abelian groups. Then Y is naturally pointwise weakly equivalent to the simplicial functor $f_* d(\tilde{f}^* Y)$.*

Here, d is the diagonal functor. Note that the simplicial abelian groups $Y(A)$ do not have to be cofibrant.

The Dold-Kan equivalence (Corollary III.2.3) relating the categories $s\mathbf{Ab}$ of simplicial abelian groups and the category $\mathbf{Ch}_+$ of ordinary chain complexes, gives a simplicial model structure to the chain complex category. It also immediately implies the following result:

COROLLARY 4.7. *Suppose that a simplicial functor $f : \mathcal{A} \to \mathcal{B}$ is homotopically full and faithful, and let $Y : \mathcal{A} \to \mathbf{Ch}_+$ be a simplicial functor taking values in the category $\mathbf{Ch}_+$ of ordinary chain complexes. Then Y is naturally pointwise weakly equivalent to the simplicial functor $f_* d(\tilde{f}^* Y)$.*

In this case, $d(\tilde{f}^* Y)$ is the normalized chain complex $Nd(\tilde{f}^*(\Gamma Y))$, where ΓY is the diagram of simplicial abelian groups associated to the diagram of chain complexes Y.