Chapter VII Cosimplicial spaces

One of the most commonly used tools in homotopy theory is the homotopy spectral sequence of a cosimplicial space. This first appeared in the work of Bousfield and Kan [14] and has been further analyzed by Bousfield [10]. Two of the standard examples include the Bousfield-Kan spectral sequence — an unstable Adams spectral sequence that arose before the general example [7] — and the spectral sequence for computing the homotopy groups of the homotopy inverse limit of a diagram of pointed spaces. One of the main purposes of this chapter is to define and discuss this kind of spectral sequence. We will do much more however; for example, we will give a detailed analysis of the total tower of a cosimplicial space, which is the basic object from which the spectral sequence is derived.

We begin with a discussion of skeleta in a very general context. This is the purpose of Section 1, and the main results are Propositions 1.8 and 1.13, which discuss to what extent a simplicial object in a category $\mathcal{C}$ with enough colimits can be built by "attaching cells". One application is a characterization of cofibrations in the kind of model category considered in Section II.5. See Example 1.14.

If the category $\mathcal{C}$ happens to itself be a closed model category, then the category $s\mathcal{C}$ inherits a closed model category structure in which a morphism $X \to Y$ in $s\mathcal{C}$ is a weak equivalence in and only if $X_n \to Y_n$ is a weak equivalence for all $n \geq 0$. This is the Reedy model category structure on $s\mathcal{C}$ and one of the main results here is that there is a geometric realization functor

$$| \cdot | : s\mathcal{C} \longrightarrow \mathcal{C}$$

which preserves weak equivalences between Reedy cofibrant objects. See Proposition 3.6. The Reedy model category structure is discussed in detail in Section 2.

In Section 4, this theory is specialized to the case of cosimplicial spaces $c\mathbf{S}$; that is, to the case of the opposite category to the category of simplicial objects in $\mathbf{S}^{op}$. The resulting model category structure on $c\mathbf{S}$ — probably the original example of a Reedy model category structure — is the one discussed by Bousfield and Kan, and it has some unusual features. In particular, one can give a simple characterization of cofibrations. This is done in Proposition 4.16.

If $x \in c\mathbf{S}$, the total space of $\mathrm{Tot}(X)$ of X is the geometric realization of X; that is,

$$\mathrm{Tot}(\cdot) = | \cdot |^{op} : c\mathbf{S} = (s\mathbf{S})^{op} \to (\mathbf{S}^{op})^{op}.$$

As a result, if $X \in c\mathbf{S}$ is fibrant, $\mathrm{Tot}(X)$ is the inverse limit of a tower of fibrations. If X is pointed, the tower is pointed, and there is spectral sequence in homotopy groups. This is the spectral sequence of the cosimplicial space. There are several issues: defining the spectral sequence in low degrees in homotopy, where one might not have groups, or abelian groups; the question of whether the spectral sequence converges to $\pi_* \operatorname{Tot}(X)$ or not; and what the E_2 term is. This all occupies Sections 5, 6, and 7. The main examples are discussed at the end of Section 7.

Finally, there is a possibility that $\operatorname{Tot}(X)$ might be empty — it's an inverse limit, and inverse limits can be empty. To analyze this question, and to address the calculation of $\pi_0 \operatorname{Tot}(X)$ in general, Bousfield has developed an extensive obstruction theory in the reference cited above. We give a small, but very useful, example of this theory in Section 8. The main results are Propositions 8.3 and 8.4.

1. Decomposition of simplicial objects.

Let $\mathcal{C}$ be a category with all limits and colimits. The purpose of this section is to analyze how simplicial objects are constructed out of smaller components. We will use this inductive argument in later sections.

We begin with skeleta. The category $s\mathcal{C}$ is the functor category $\mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}}$. Let $i_n : \mathbf{\Delta}_n \subseteq \mathbf{\Delta}$ be the inclusion of the full subcategory with objects $\mathbf{k}$, $k \leq n$, and let $s_n\mathcal{C} = \mathcal{C}^{\mathbf{\Delta}_n^{\mathrm{op}}}$. There is a restriction function $i_{n*} : s\mathcal{C} \to s_n\mathcal{C}$ which simply forgets the k-simplices, $k > n$. This restriction functor has a left adjoint given by

$$(i_n^* X)_m = \varinjlim_{\mathbf{m}\to\mathbf{k}} X_k = \varinjlim_{\mathbf{m}\downarrow\mathbf{\Delta}_n} X_k \tag{1.1}$$

and the colimit is over morphisms $\mathbf{m} \to \mathbf{k}$ in $\mathbf{\Delta}$ with $k \leq n$. This is an example of a left Kan extension. Since every morphism $\mathbf{m} \to \mathbf{k}$ in $\mathbf{\Delta}$ can be factored uniquely as

$$\mathbf{m} \xrightarrow{\phi} \mathbf{k}' \xrightarrow{\psi} \mathbf{k}$$

where ϕ is a surjection and ψ is one-to-one, the surjections $\mathbf{m} \to \mathbf{k}$, $k \leq n$, are cofinal in the category of all morphisms used to define the colimit of (1.1) and we get

$$(i_n^* X)_m = \varinjlim_{\mathbf{m}\downarrow\mathbf{\Delta}_n^+} X_k \tag{1.2}$$

where $\mathbf{\Delta}_+ \subseteq \mathbf{\Delta}$ is the subcategory with the same objects but only surjections as morphisms and $\mathbf{\Delta}_n^+ = \mathbf{\Delta}_n \cap \mathbf{\Delta}_+$.

If $X \in s\mathcal{C}$ we define the n^{th} *skeleton* of X by the formula

$$\mathrm{sk}_n X = i_n^* i_{n*} X. \tag{1.3}$$

There are natural maps $\mathrm{sk}_m X \to \mathrm{sk}_n X$, $m \leq n$, and $\mathrm{sk}_n X \to X$. In light of (1.2), $(\mathrm{sk}_n X)_m = X_m$ for $m \leq n$ It is clear that there is a natural isomorphism

$$\varinjlim_n \mathrm{sk}_n X \xrightarrow{\cong} X.$$

Here is an example. Because $\mathcal{C}$ has limits and colimits, $s\mathcal{C}$ has a canonical structure as a simplicial category. (See Section II.2). In particular if $X \in s\mathcal{C}$ and $K \in \mathbf{S}$, then

$$(X \otimes K)_n = \bigsqcup_{K_n} X_n.$$

It is now a straightforward exercise to prove

Proposition 1.4. *If $X \in s\mathcal{C}$ is constant and $K \in \mathbf{S}$, then there is a natural isomorphism*

$$\mathrm{sk}_n X \otimes \mathrm{sk}_n K \cong \mathrm{sk}_n(X \otimes K).$$

To explain how $\mathrm{sk}_n X$ is built from $\mathrm{sk}_{n-1} X$ we define the n^{th} *latching object* $L_n X$ of X by the formula

$$\begin{aligned} L_n X &= (\mathrm{sk}_{n-1} X)_n \\ &\cong \varinjlim_{\mathbf{n}\downarrow\mathbf{\Delta}^+_{n-1}} X_k. \end{aligned} \tag{1.5}$$

If $Z \in \mathcal{C}$, we may regard Z as a constant object in $s\mathcal{C}$ and there is an adjoint isomorphism

$$\hom_{\mathcal{C}}(Z, X_n) \cong \hom_{s\mathcal{C}}(Z \otimes \Delta^n, X)$$

for all $n \geq 0$. This immediately supplies maps in $s\mathcal{C}$

$$L_n X \otimes \Delta^n \to \mathrm{sk}_{n-1} X$$

and

$$X_n \otimes \Delta^n \to \mathrm{sk}_n X.$$

Furthermore, by Proposition 1.4, $\mathrm{sk}_{n-1}(X_n \otimes \Delta^n) = X_n \otimes \mathrm{sk}_{n-1}\Delta^n = X_n \otimes \partial\Delta^n$ and we obtain a diagram

$$\begin{array}{ccc} L_nX \otimes \partial\Delta^n & \longrightarrow & L_nX \otimes \Delta^n \\ \downarrow & & \downarrow \\ X_n \otimes \partial\Delta^n & \longrightarrow & \mathrm{sk}_{n-1} X. \end{array} \tag{1.6}$$

PROPOSITION 1.7. *For all $X \in s\mathcal{C}$ there is a natural pushout diagram, $n \geq 0$,*

$$\begin{array}{ccc} X_n \otimes \partial\Delta^n \cup_{L_nX\otimes\partial\Delta^n} L_nX \otimes \Delta^n & \longrightarrow & \mathrm{sk}_{n-1} X \\ \downarrow & & \downarrow \\ X_n \otimes \Delta^n & \longrightarrow & \mathrm{sk}_n X \end{array}$$

PROOF: In light of Proposition 1.4, if we apply $i_n^* i_{n*} = \mathrm{sk}_n(\cdot)$ to this diagram we obtain an isomorphic diagram. Since $i_n^* : s_n\mathcal{C} \to s\mathcal{C}$ is a left adjoint, we need only show this is a pushout diagram in degrees less than or equal to n.

In degrees $m < n$, the map $(L_nX \otimes \partial\Delta^n)_m \to (L_nX \otimes \Delta^n)_m$ is an isomorphism, so

$$(X_n \otimes \partial\Delta^n \cup_{L_nX\otimes\partial\Delta^n} L_nX \otimes \Delta^n)_m \to (X_n \otimes \Delta^m)_m$$

is an isomorphism, and the assertion is that $(\mathrm{sk}_{n-1} X)_m \cong (\mathrm{sk}_n X)_m$, which is true: both are isomorphic to X_m. In degree n, the left vertical map is isomorphic to

$$\bigsqcup_{(\partial\Delta^n)_m} X_n \sqcup L_nX \to \bigsqcup_{(\partial\Delta^n)_m} X_n \sqcup X_n$$

so the assertion is that the right vertical map is isomorphic to the natural map $L_nX \to X_n$. This is true by the definition of L_nX. ∎

Morphisms also have skeletal fibrations. If $f : A \to X$ is a morphism in $s\mathcal{C}$, define $\mathrm{sk}_n^A X$ by setting $\mathrm{sk}_{-1}^A = A$ and, for $n \geq 0$, defining sk_n^A by the pushout diagram

$$\begin{array}{ccc} \mathrm{sk}_n A & \longrightarrow & A \\ {\scriptstyle \mathrm{sk}_n f}\downarrow & & \downarrow \\ \mathrm{sk}_n X & \longrightarrow & \mathrm{sk}_n^A X. \end{array}$$

The adjunct to Proposition 1.7 is the next result, which is proved in an identical manner. Let $L_n(f) = (\mathrm{sk}_{n-1}^A X)_n = A_n \cup_{L_nA} L_nX$.

PROPOSITION 1.8. *For all morphisms $A \to X$ in $s\mathcal{C}$ there is a pushout diagram*

$$\begin{tikzcd} X_n \otimes \partial\Delta^n \cup_{L_n(f)\otimes\partial\Delta^n} L_n(f) \otimes \Delta^n \arrow[r] \arrow[d] & \mathrm{sk}^A_{n-1} X \arrow[d] \\ X_n \otimes \Delta^n \arrow[r] & \mathrm{sk}^A_n X. \end{tikzcd}$$ ■

There is a situation under which the pushout diagrams of Proposition 1.7 and 1.8 simplify considerably. This we now explain.

If I is a small category, let I^δ be the discrete category with the same objects as I, but no non-identity morphism. The left adjoint r^* to the restriction functor $r_* : \mathcal{C}^I \to \mathcal{C}^{I^\delta}$ has a very simple form

$$(r^* Z)_j = \bigsqcup_{i \to j} Z_i.$$

We call such a diagram *I-free*. More generally, a morphism $f : A \to X$ in $\mathcal{C}^I$ is *I-free* if there is an I-free object $X' \in \mathcal{C}^I$ and an isomorphism under A of f with the inclusion of the summand $A \to A \sqcup X'$. This implies that there is an object $\{Z_i\} \in \mathcal{C}^{I^\delta}$ so that

$$X_j \cong A_j \sqcup (\bigsqcup_{i \to j} Z_i).$$

DEFINITION 1.9. An object $X \in s\mathcal{C}$ is *degeneracy free* if the underlying degeneracy diagram is free. That is, if $\mathbf{\Delta}_+ \subseteq \mathbf{\Delta}$ is the subcategory with same objects but only surjective morphisms, then X regarded as an object in $\mathcal{C}^{\mathbf{\Delta}_+^{\mathrm{op}}}$ is $\mathbf{\Delta}_+^{\mathrm{op}}$-free. A morphism $A \to X$ is degeneracy free if, when regarded as a morphism in $\mathcal{C}^{\mathbf{\Delta}_+^{\mathrm{op}}}$ it is $\mathbf{\Delta}_+^{\mathrm{op}}$ free.

If X is degeneracy free, then there is a sequence $\{Z_n\}_{n \geq 0}$ of objects in $\mathcal{C}$ so that

$$X_n \cong \bigsqcup_{\phi : \mathbf{n} \to \mathbf{m}} Z_m$$

where ϕ runs over the epimorphisms in $\mathbf{\Delta}$. A degeneracy free map $A \to X$ yields a similar decomposition:

$$X_n \cong A_n \sqcup \bigsqcup_{\phi : \mathbf{n} \to \mathbf{m}} Z_m. \tag{1.10}$$

We say $A \to X$ is degeneracy free on $\{Z_m\}$. Although we have not made this explicit before, notice an object X is I-free if and only if the morphism $\phi \to X$ from the initial object in I-free.

The following result says that degeneracy free maps are closed under a variety of operations.

LEMMA 1.11.
1) Let $f_\alpha : A_\alpha \to X_\alpha$ be a set of maps so that f_α is degeneracy free on $\{Z_n\}$. Then $\bigsqcup_\alpha f_\alpha$ is free on $\{\bigsqcup_\alpha Z_n^\alpha\}$.
2) Suppose $f : A \to X$ is degeneracy free on $\{Z_n\}$ and

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & B \arrow[d, "g"] \\ X \arrow[r] & Y \end{tikzcd}$$

is a pushout diagram. Then g is degeneracy free on $\{Z_n\}$.
3) Let $A_0 \to A_1 \to A_2 \to \cdots$ be a sequence of morphisms so that $A_{j-1} \to A_j$ is degeneracy free on $\{Z_n^j\}$. Then $A_0 \to \varinjlim A_j$ is degeneracy free on $\{\bigsqcup_j Z_n^j\}$.
4) Let $F : \mathcal{C} \to \mathcal{D}$ be a functor that preserves coproducts. If $f : A \to X$ in $s\mathcal{C}$ is degeneracy free on $\{Z_n\}$, then $Ff : FA \to FX$ is degeneracy free on $\{FZ_n\}$.

LEMMA 1.12. *A morphism $f : A \to X$ is degeneracy free if and only if there are objects Z_n and maps $Z_n \to X_n$ so that the induced map*

$$(A_n \cup_{L_n A} L_n X) \sqcup Z_n \to X_n$$

is an isomorphism.

PROOF: If f is degeneracy free on $\{Z_n\}$, the decomposition follows from the formulas (1.5) and (1.10). To prove the converse, fix the given isomorphisms. Then Proposition 1.8 implies there is a pushout diagram in $\mathcal{C}^{\mathbf{\Delta}_+^{\mathrm{op}}}$

$$\begin{tikzcd} Z_n \otimes \partial\Delta^n \arrow[r] \arrow[d] & \mathrm{sk}_{n-1}^A X \arrow[d] \\ Z_n \otimes \Delta^n \arrow[r] & \mathrm{sk}_n^A X. \end{tikzcd}$$

However, the morphism of simplicial $\partial\Delta^n \to \Delta^n$ is degeneracy free on the canonical n-simplex in $(\Delta^n)_n$; hence, $\mathrm{sk}_{n-1}^A X \to \mathrm{sk}_n^A X$ is degeneracy free on Z_n in degree n. The result now follows from Lemma RefVII.1.11.3; indeed $A \to X$ is degeneracy free on $\{Z_n\}$. ∎

PROPOSITION 1.13. *Suppose $A \to X$ is degeneracy-free on $\{Z_n\}$ in $s\mathcal{C}$. Then for all $n \geq 0$ there is a pushout diagram,*

$$\begin{tikzcd} Z_n \otimes \partial\Delta^n \arrow[r] \arrow[d] & \mathrm{sk}_{n-1}^A X \arrow[d] \\ Z_n \otimes \Delta^n \arrow[r] & \mathrm{sk}_n^A X. \end{tikzcd}$$

PROOF: Combine Proposition 1.8 and Lemma 1.12. ∎

Notice that one can interpret this result as saying $\mathrm{sk}_n^A X$ is obtained from $\mathrm{sk}_{n-1}^A X$ by attaching n-cells.

EXAMPLE 1.14. Lemmas 1.11 and 1.12 provide any number of examples of degeneracy free morphisms. For example, a cofibration in simplicial sets is degeneracy free, by Lemma 1.12. Also, consider a category $\mathcal{C}$ equipped with a functor $G : \mathcal{C} \to \mathrm{Sets}$ with a left adjoint F and satisfying the hypotheses of Theorem II.5.1. Define a morphism $f : A \to X$ in $s\mathcal{C}$ to be *free* (this terminology is from Quillen) if there is a sequence of sets $\{Z_n\}$ so that f is degeneracy free on $\{FZ_n\}$. Then a morphism in $s\mathcal{C}$ is a cofibration if and only if it is a retract of a free map. To see this, note that the small object argument of Lemma II.5.2, coupled with Lemma 1.11 factors any morphism $A \to B$ as

$$A \xrightarrow{j} X \xrightarrow{q} Y$$

where j is a free map and a cofibration and q is a trivial fibration. Thus any cofibration is a retract of a free map. Conversely, Proposition 1.13 and Proposition II.3.4 imply any free map is a cofibration. Similar remarks apply to the model categories supplied in Theorems II.6.8 and II.7.8. In the latter case one must generalize Lemma 1.11.3 to longer colimits.

Dual to the notion of skeleta is the notion of coskeleta. The theory is analogous, and we give only an outline. The restriction functor $i_{n*} : s\mathcal{C} \to s_n\mathcal{C}$ has right adjoint $i_n^!$ with

$$(i_n^! X)_m = \varprojlim_{\mathbf{k}\to\mathbf{m}} X_k,$$

with the limit over all morphism $\mathbf{k} \to \mathbf{m}$ in $\mathbf{\Delta}$ with $k \leq n$. Equally, one can take the limit over morphisms $\mathbf{k} \to \mathbf{m}$ which are injections. The composite gives the n^{th} coskeleton functor:

$$\mathrm{cosk}_n X = i_n^! i_{n*} X. \tag{1.15}$$

More generally, if $f : X \to B$ is a morphism in $s\mathcal{C}$, let $\operatorname{cosk}^B_{-1} X = B$ and let $\operatorname{cosk}^B_n X$ be defined by the pullback

$$\begin{tikzcd} \operatorname{cosk}^B_n X \arrow[r] \arrow[d] & \operatorname{cosk}_n X \arrow[d] \\ B \arrow[r] & \operatorname{cosk}_n B. \end{tikzcd} \tag{1.16}$$

Then there are maps $\operatorname{cosk}^B_n X \to \operatorname{cosk}^B_{n-1} X$ and

$$X \cong \varprojlim_n \operatorname{cosk}^B_n X.$$

Note that if $B = *$, the terminal object, then $\operatorname{cosk}^B_n X = \operatorname{cosk}_n X$.

For $X \in s\mathcal{C}$, define the n^{th} *matching object* $H_n X$ by the formula

$$M_n X = (\operatorname{cosk}_{n-1} X)_n = \varprojlim_{\mathbf{k}\to\mathbf{n}} X_k \tag{1.17}$$

where $\mathbf{k} \to \mathbf{n}$ runs over all morphisms (or all monomorphisms) in $\mathbf{\Delta}$ with $k < n$.

To fit the map $\operatorname{cosk}^B_n X \to \operatorname{cosk}^B_{n-1} X$ into a pullback diagram, we make some definitions. Let $\rho_n : s\mathcal{C} \to \mathcal{C}$ be the functor $\rho_n X = X_n$. This is a restriction functor between diagram categories and has a right adjoint $\rho^!_n$. The functor $L_n : s\mathcal{C} \to \mathcal{C}$ assigning each simplicial object the latching object $L_n X$ also has a right adjoint, which we will call $\mu^!_n$. To see this, see Lemma 1.23 below, or note that we may write

$$L_n X = (\operatorname{sk}_{n-1} X)_n = \rho_n i^*_{n-1} i_{(n-1)*} X$$

where $i_{(n-1)*} : s\mathcal{C} \to s_{n-1}\mathcal{C}$ is the restriction functor. Hence

$$\mu^!_n Z = i^!_{n-1} i_{(n-1)*} \rho^!_n Z = \operatorname{cosk}_{n-1}(\rho^!_n Z).$$

The natural map $s : L_n X \to X$ gives a natural transformation $\rho^!_n \to \mu^!_n$. The reader is invited to prove the following result.

PROPOSITION 1.18. *Let $f : X \to B$ be a morphism in $s\mathcal{C}$. Then for all $n \geq 0$ there is a pullback diagram*

$$\begin{tikzcd} \operatorname{cosk}^B_n X \arrow[r] \arrow[d] & \rho^!_n X_n \arrow[d] \\ \operatorname{cosk}^B_{n-1} X \arrow[r] & \rho^!_n M_n(f) \times_{\mu^!_n M_n(f)} \mu^!_n X_n \end{tikzcd}$$

where $M_n(f) = B_n \times_{M_n B} M_n X$.

We have never encountered the analog of degeneracy-free morphisms ("face free"?) and don't include an exposition here.

The latching and matching object functors L_n and M_n are examples of a much more general sort of functor, which we now introduce and analyze. We begin with generalized matching objects.

Proposition 1.19. *Let $K \in \mathbf{S}$ be a simplicial set. Then the functor*

$$- \otimes K : \mathcal{C} \to s\mathcal{C}$$

has right adjoint M_K. For fixed $X \in s\mathcal{C}$, the assignment $K \mapsto M_K X$ induces a functor $\mathbf{S}^{\mathrm{op}} \to \mathcal{C}$ which has a left adjoint.

Proof: If $Z \in s\mathcal{C}$ is constant, $\hom_{s\mathcal{C}}(Z, Y) = \hom_{\mathcal{C}}(Z, Y_0)$. Hence we may set $M_K X = (X^K)_0$, since

$$\hom_{s\mathcal{C}}(Z \otimes K, X) \cong \hom_{s\mathcal{C}}(Z, X^K).$$

Furthermore

$$\hom_{\mathcal{C}}(Z, M_K X) \cong \hom_{s\mathcal{C}}(Z \otimes K, X) \cong \hom_{\mathbf{S}}(K, \mathbf{Hom}_{s\mathcal{C}}(Z, X)). \qquad \blacksquare$$

Since $\hom_{s\mathcal{C}}(Z \otimes \Delta^n, X) \cong \hom_{\mathcal{C}}(Z, X_n)$, $M_{\Delta^n} X \cong X_n$. More generally, we can give a decomposition of $M_K X$ as follows. Let $\mathbf{\Delta} \downarrow K$ be the simplex category of K. Thus an object of $\mathbf{\Delta} \downarrow K$ is a morphism $\sigma : \Delta^n \to K$ in $\mathbf{S}$ and a morphism in $\mathbf{\Delta} \downarrow K$ is a triangle in $\mathbf{S}$

$$\begin{tikzcd} \Delta^n \arrow[r, "\sigma"] \arrow[d, "\phi"'] & K \\ \Delta^m . \arrow[ur, "\tau"'] & \end{tikzcd}$$

If we define a functor $F_K : \mathbf{\Delta} \downarrow K \to \mathbf{S}$ by

$$F_K(\sigma : \Delta^n \to K) = \Delta^n$$

then

$$\varinjlim_{\mathbf{\Delta}\downarrow K} F_K \cong K.$$

Since $M_K X$ sends colimits in the K variable to limits we have

$$M_K X \cong \varprojlim_{\mathbf{\Delta}\downarrow K^{\mathrm{op}}} M_{F_K} X \cong \varprojlim_{\mathbf{\Delta}\downarrow K^{\mathrm{op}}} X \tag{1.20}$$

where we write $X : \mathbf{\Delta} \downarrow K^{\mathrm{op}} \to \mathcal{C}$ to be

$$X(\sigma : \Delta^n \to K) = X_n.$$

EXAMPLE 1.21. Let $\phi : \Delta^k \to \Delta^n$ be any morphism in $\mathbf{S}$. Then ϕ factors uniquely as a composition

$$\Delta^k \xrightarrow{\phi'} \Delta^m \xrightarrow{\psi} \Delta^n$$

where ϕ' is a surjection and ψ is an injection. Thus if $K \subseteq \Delta^n$ is any subcomplex, the full subcategory $\mathbf{\Delta} \downarrow K_0 \subseteq \mathbf{\Delta} \downarrow K$ with objects

$$\sigma : \Delta^m \to K$$

with σ an injection is cofinal. Equivalently $\mathbf{\Delta} \downarrow K_0^{\mathrm{op}} \subseteq \mathbf{\Delta} \downarrow K^{\mathrm{op}}$ is final; hence

$$M_K X \cong \varprojlim_{\mathbf{\Delta}\downarrow K_0^{\mathrm{op}}} X.$$

In particular $M_{\partial\Delta^n} X \cong \varprojlim_{\phi:\mathbf{k}\to\mathbf{n}} X_k$ where $\phi \in \mathbf{\Delta}$ is an injection and $k < n$. Thus $M_{\partial\Delta^n} X \cong M_n X$. Note that the inclusion map $\partial\Delta^n \to \Delta^n$ induces the projection

$$X_n \to M_n X.$$

To generalize the latching objects we appropriate the formulation of matching objects presented in (1.20). Let J be a small category and $F : J \to \mathbf{\Delta}^{\mathrm{op}}$. For $X \in s\mathcal{C} = \mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}}$, define the generalized latching object to be

$$\begin{aligned} L_J X &= \varinjlim_J X \circ F \\ &= \varinjlim_j X_{F(j)}. \end{aligned} \tag{1.22}$$

We are primarily interested in sub-categories J of $\mathbf{\Delta}^{\mathrm{op}}$. We write $X\big|_J$ for $X \circ F$.

LEMMA 1.23. *For fixed $F : J \to \mathbf{\Delta}^{\mathrm{op}}$, the functor $L_J : s\mathcal{C} \to \mathcal{C}$ has a right adjoint and, hence, preserves colimits.*

Proof: For $\hom_{\mathcal{C}}(L_J X, Z) \cong \hom_{\mathcal{C}^J}(X \circ F, Z)$ where Z is regarded as a constant diagram. But

$$\hom_{\mathcal{C}^J}(X\big|_J, Z) = \hom_{s\mathcal{C}}(X, F^! Z)$$

where $F^!$ is the right Kan extension functor. ∎

This restriction to sub-categories of $\mathbf{\Delta}^{\mathrm{op}}$ is practical, but perhaps gratuitous.

An example of a generalized latching object is the following. Let $\mathcal{O}_n$ be the category with objects the morphisms $\phi : \mathbf{n} \to \mathbf{m}$ with ϕ surjective and $m < n$. The morphisms in $\mathcal{O}_n$ are commutative triangles in $\mathbf{\Delta}$ under $\mathbf{n}$. Define $F : \mathcal{O}_n^{\mathrm{op}} \to \mathbf{\Delta}^{\mathrm{op}}$ by $F(\phi : \mathbf{n} \to \mathbf{m}) = \mathbf{m}$. Then there is a natural isomorphism

$$L_n X \cong L_{\mathcal{O}_n^{\mathrm{op}}} X.$$

In the next section we will need a decomposition of $L_n X$. To accomplish this define sub-categories $\mathcal{M}_{n,k} \subseteq \mathcal{O}_n$, $0 \leq k \leq n$, to be the full sub-category of surjections $\phi : \mathbf{n} \to \mathbf{m}$, $m < n$, with $\phi(k) < k$. Define, for $X \in s\mathcal{C}$

$$L_{n,k} X = L_{\mathcal{M}_{n,k}^{\mathrm{op}}} X.$$

Then $L_{n,0} X = \phi$ is the initial object in $\mathcal{C}$ (since $\mathcal{M}_{n,0}$ is empty) and $L_{n,n} X = L_n X$, since $\mathcal{M}_{n,n} = \mathcal{O}_n$.

Also define $\mathcal{M}(k) \subseteq \mathcal{O}_n$ to be the full subcategory of surjections $\phi : \mathbf{n} \to \mathbf{m}$, $m < n$, with $\phi(k) = \phi(k+1)$. Notice that $s^k : \mathbf{n} \to \mathbf{n-1}$ is the initial object of $\mathcal{M}(k)$. Hence for $X \in s\mathcal{C}$

$$L_{\mathcal{M}(k)^{\mathrm{op}}} X \cong X_{n-1}.$$

The reader can verify the following statements about these subcategories.

Lemma 1.24.
1) $\mathcal{M}_{n,k}$ and $\mathcal{M}(k)$ are subcategories of $\mathcal{M}_{n,k_1}$, and every object in $\mathcal{M}_{n,k+1}$ is in $\mathcal{M}_{n,k}$ or $\mathcal{M}(k)$ (or both).

2) There is an isomorphism of categories

$$- \circ s^k : \mathcal{M}_{n-1,k} \to \mathcal{M}(k) \cap \mathcal{M}_{n,k}$$

sending ϕ to $\phi \circ s^k$.

3) If ϕ is an object of $\mathcal{M}_{n,k}$ (or $\mathcal{M}(k)$) and $\phi \to \psi$ is a morphism in $\mathcal{M}_{n,k+1}$, then $\phi \to \psi$ is a morphism in $\mathcal{M}_{n,k}$ (or $\mathcal{M}(k)$).

The following example, illustrating the case $n = 3$, $k = 2$, might be helpful.

In the following diagram the symbol 012 near a dot ($\bullet$) indicates the object $s^0 s^1 s^2$ in the appropriate category. The unlabeled arrows indicated composition with s^i for some i; for example $0\bullet \to \bullet 02$ means $s^0 \mapsto s^1 s^0 = s^0 s^2$.

$$\begin{tikzcd}
\overset{0}{\bullet} \arrow[d] & \overset{1}{\bullet} \arrow[dl] \\
\underset{01}{\bullet} &
\end{tikzcd}$$

$\mathcal{M}_{2,2} = \mathcal{O}_2$

$$\begin{tikzcd}
\overset{0}{\bullet} \arrow[d] \arrow[dr] & \overset{1}{\bullet} \arrow[dl] \arrow[dr] & \\
01\ \bullet \arrow[dr] & \bullet\ 02 \arrow[d] & \bullet\ 12 \\
& \underset{012}{\bullet} &
\end{tikzcd}$$

$\mathcal{M}_{3,2}$

$$\begin{tikzcd}
& \overset{2}{\bullet} \arrow[dl] \arrow[d] \\
02\ \bullet \arrow[d] & \bullet\ 12 \arrow[dl] \\
\underset{012}{\bullet} &
\end{tikzcd}$$

$\mathcal{M}(2)$

$$\begin{tikzcd}
\overset{0}{\bullet} \arrow[d] \arrow[dr] & \overset{1}{\bullet} \arrow[dl] \arrow[dr] & \overset{2}{\bullet} \arrow[dl] \arrow[d] \\
01\ \bullet \arrow[dr] & \underset{02}{\bullet} \arrow[d] & \bullet\ 12 \arrow[dl] \\
& \underset{012}{\bullet} &
\end{tikzcd}$$

$\mathcal{M}_{3,3} = \mathcal{O}_3$

PROPOSITION 1.25. *Let $X \in s\mathcal{C}$. Then there is a pushout diagram in $\mathcal{C}$*

$$\begin{tikzcd}
L_{n-1,k}X \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\
L_{n,k}X \arrow[r] & L_{n,k+1}X.
\end{tikzcd}$$

Proof: Lemma 1.24 implies that

$$
\begin{tikzcd}
\mathcal{M}_{n-1,k}X \arrow[r] \arrow[d] & \mathcal{M}(k) \arrow[d] \\
\mathcal{M}_{n,k}X \arrow[r] & \mathcal{M}_{n,k+1}
\end{tikzcd}
$$

is a pushout in the category of small categories. The same is then true of the opposite categories. The result follows. ■

2. Reedy model category structures.

Let $\mathcal{C}$ be a closed model category. Using the considerations of the previous section, this structure can be promoted to a closed model category structure on the category $s\mathcal{C}$ of simplicial objects in $\mathcal{C}$ that is particularly useful for dealing with geometric realization. The results of this section are a recapitulation of the highly influential, but unpublished paper of C.L. Reedy [80].

In the following definition, let L_0X and M_0X denote the initial and final object of $\mathcal{D}$ respectively.

Definition 2.1. A morphism $f : X \to Y$ in $s\mathcal{C}$ is a

1) Reedy weak equivalence if $f : X_n \to Y_n$ is a weak equivalence for all $n \geq 0$;
2) a Reedy fibration if
$$X_n \to Y_n \times_{M_nY} M_nX$$
is a fibration for all $n \geq 0$;
3) a Reedy cofibration if
$$X_n \cup_{L_nX} L_nY \to Y_n$$
is a cofibration for all $n \geq 0$.

The main result is that this defines a model category structure on $s\mathcal{C}$. Before proving this we give the following lemma.

LEMMA 2.2. *A morphism $f : X \to Y$ in $s\mathcal{C}$ is a*

1) *Reedy trivial fibration if and only if*

$$X_n \to Y_n \times_{M_n Y} M_n X$$

is a trivial fibration for $n \geq 0$;

2) *a Reedy trivial cofibration if and only if*

$$X_n \cup_{L_n X} L_n Y \to Y_n$$

is a trivial cofibration for all $n \geq 0$.

PROOF: Let $\Delta^{n,k} = d^0\Delta^{n-1} \cup \cdots \cup d^k\Delta^{n-1} \subseteq \partial\Delta^n$, $-1 \leq k \leq n$. Then $\Delta^{n,-1} = \phi$ and $\Delta^{n,n} = \partial\Delta^n$. There are pushout diagrams $-1 \leq k \leq n-1$

$$\begin{tikzcd} \Delta^{n-1,k} \arrow[r] \arrow[d] & \Delta^{n-1} \arrow[d] \\ \Delta^{n,k} \arrow[r] & \Delta^{n,k+1}. \end{tikzcd}$$

Taking matching objects yields a natural pullback square

$$\begin{tikzcd} M_{n,k+1}X \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\ M_{n,k}X \arrow[r] & M_{n-1,k}X \end{tikzcd}$$

where we have written $M_{n,k}X$ for $M_{\Delta^{n,k}}X$. This yields a pullback square

$$\begin{tikzcd} Y_n \times_{M_{n,k+1}Y} M_{n,k+1}X \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\ Y_n \times_{M_{n,k}Y} M_{n,k}X \arrow[r] & Y_{n-1} \times_{M_{n-1,k}Y} M_{n-1,k}X. \end{tikzcd} \tag{2.3}$$

Assume $f : X \to Y$ is a Reedy fibration. Then $X_0 \to Y_0$ is a fibration and this begins an induction where the induction hypothesis is that

$$X_{n-1} \to Y_{n-1} \times_{M_{n-1,k}Y} M_{n-1,k}X \tag{2.4}$$

is a fibration. To complete the inductive step, one uses (2.3) to show

$$Y_n \times_{M_{n,k+1}Y} M_{n,k+1}X \to Y_n \times_{M_{n,k}Y} M_{n,k}X \tag{2.5}$$

is a fibration for all k, $-1 \leq k \leq n-1$. Since composites of fibrations are fibrations and $X_n \to Y_n \times_{M_nY} M_nX$ is a fibration, we close the loop.

Now suppose $f : X \to Y$ is a Reedy trivial fibration. Then, inductively, each of the maps of (2.4) is a trivial fibration. For the inductive step, use (2.3) to show each of the maps (2.5) is a trivial fibration. Then the axiom **CM2** of closed model categories and the fact that $X_n \to Y_n$ a trivial fibration finishes the argument. In particular $X_n \to Y_n \times_{M_nY} M_nX$ is a trivial fibration for all n.

Conversely, suppose $X_n \to X_n \times_{M_nY} M_nX$ is a trivial fibration. Then one again runs the same sort of argument to conclude $X_n \to Y_n$ is a trivial fibration.

The argument for part 2) is similar, using the pushout diagram

$$\begin{tikzcd}
L_{n-1,k}X \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\
L_{n,k}X \arrow[r] & L_{n,k+1}X
\end{tikzcd}$$

of Proposition 1.25. ■

We break the proof of the verification of the Reedy model category structure into several steps.

LEMMA 2.6. *The Reedy structure on $s\mathcal{C}$ satisfies the lifting axiom* **CM4**.

PROOF: Suppose we are given a lifting problem in $s\mathcal{C}$

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "j"'] & X \arrow[d, "q"] \\
B \arrow[r] & Y
\end{tikzcd}$$

where j is cofibration, q is a fibration and either j or q is a weak equivalence. We inductively solve the lifting problems

$$\begin{tikzcd} \operatorname{sk}^A_{n-1} B \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{sk}^A_n B \arrow[r] & Y \end{tikzcd}$$

for $n \geq 0$. Because of the pushout diagram of Proposition 1.8 it is sufficient to solve the lifting problems, with $L_n(j) = A_n \cup_{L_n A} L_n B$

$$\begin{tikzcd} L_n(j) \otimes \Delta^n \cup_{L_n(j) \otimes \partial\Delta^n} B_n \otimes \partial\Delta^n \arrow[r] \arrow[d] & X \arrow[d] \\ B_n \otimes \Delta^n \arrow[r] & Y. \end{tikzcd}$$

By an adjunction argument this is equivalent to the lifting problem

$$\begin{tikzcd} A_n \cup_{L_n A} L_n B \arrow[r] \arrow[d] & X_n \arrow[d] \\ B_n \arrow[r] & Y_n \times_{M_n Y} M_n X. \end{tikzcd}$$

This is solvable by Lemma 2.2. ∎

For the proof of the factorization axiom we need to know how much data we need to build a simplicial object.

Lemma 2.7. *Let $X \in s\mathcal{C}$. Then $\operatorname{sk}_n X$ is determined up to natural isomorphism by the following natural data: $\operatorname{sk}_{n-1} X$, X_n, and maps $L_n X \to X_n \to M_n X$ so that the composite $L_n X \to M_n X$ is the canonical map.*

Proof: The given map $X_n \to M_n X$ is adjoint to a map $X_n \otimes \partial\Delta^n \to X$ which factors uniquely through $\operatorname{sk}_{n-1} X$. The data listed thus yields the pushout square

$$\begin{tikzcd} L_n X \otimes \Delta^n \cup_{L_n X \otimes \partial\Delta^n} X_n \otimes \partial\Delta^n \arrow[r] \arrow[d] & \operatorname{sk}_{n-1} X \arrow[d] \\ X_n \otimes \Delta^n \arrow[r] & \operatorname{sk}_n X \end{tikzcd}$$

of Proposition 1.7. ∎

This can be restricted: let $s_n\mathcal{C}$ be the functor category $\mathcal{C}^{\mathbf{\Delta}_n^{\mathrm{op}}}$ and $X \in s_n\mathcal{C}$. Let $r_*X \in s_{n-1}\mathcal{C}$ be the restriction and $r^* : s_{n-1}\mathcal{C} \to s_n\mathcal{C}$ the left Kan extension functor. Hence $(r^*r_*X)_n = L_nX$.

Lemma 2.7 immediately implies

LEMMA 2.8. *Let $X \in s_n\mathcal{C}$. Then X is determined up to natural isomorphism by the following natural data: r_*X, X_n, and maps $L_nX \to X_n \to M_nX$ so that the composite $L_nX \to M_nX$ is the canonical map.*

PROOF: Let $i_{n*} : s\mathcal{C} \to s_n\mathcal{C}$ be the restriction and $i_n^* : s_n\mathcal{C} \to s\mathcal{C}$ the left adjoint. Then $i_{n*}i_n^* \cong 1$ and $i_n^*i_{n*} = \mathrm{sk}_n$. Thus

$$i_n^*X \cong \mathrm{sk}_n\, i_n^*X$$

is determined up to natural isomorphism by $\mathrm{sk}_{n-1}\, i_n^*X \cong i_{n-1}^*r_*X$ (where $i_{n-1}^* : s_{n-1}\mathcal{C} \to s\mathcal{C}$ is left adjoint to restriction), X_n and maps $L_ni_n^*X \to X_n \to M_ni_n^*X$ that compose to the canonical map $L_ni_n^*X \to M_ni_n^*X$. But $L_nX \cong L_ni_n^*X$ and $M_nX \cong M_ni_n^*X$. ∎

LEMMA 2.9. *The Reedy structure on $s\mathcal{C}$ satisfies the factorization axiom* **CM5**.

PROOF: Let us do the trivial cofibration-fibration factorization (compare the proof of Lemma IV.3.5).

Let $X \to Y$ be a morphism in $s\mathcal{C}$ and let $i_{n*}X \to i_{n*}Y$ be the induced morphism in $s_n\mathcal{C}$. For each $n \geq 0$ we construct a factorization

$$i_{n*}X \to Z(n) \to i_{n*}Y$$

in $s_n\mathcal{C}$ with the property that restricted to $s_{n-1}\mathcal{C}$ we get the factorization $i_{(n-1)*}X \to Z(n-1) \to i_{(n-1)*}Y$.

For $n = 0$, simply factor $X_0 \to Y_0$ and $X_0 \xrightarrow{j} Z(0) \xrightarrow{q} Y_0$ where j is a trivial cofibration in $\mathcal{C}$ and q is a fibration in $\mathcal{C}$. Suppose the factorization in $s_{n-1}\mathcal{C}$ has been constructed. Let $r_* : s_n\mathcal{C} \to s_{n-1}\mathcal{C}$ be the restriction functor and $r^*, r^! : s_{n-1}\mathcal{C} \to s_n\mathcal{C}$ be respectively, the left and right Kan extension functors. Thus

$$r^*i_{(n-1)*}X = i_{(n-1)*}\,\mathrm{sk}_{n-1}\,X \text{ and } r^!i_{(n-1)*}X = i_{n*}\,\mathrm{cosk}_{n-1}\,X.$$

Furthermore, the morphism $r_* r^* Z(n-1) \to Z(n-1)$ yields a map $r^* Z(n-1) \to r^! Z(n-1)$ so that the following diagram commutes

$$\begin{tikzcd}
r^* i_{(n-1)*} X \arrow[r] \arrow[d] & r^* Z(n-1) \arrow[r] & r^! Z(n-1) \arrow[r] & r^! i_{(n-1)*} Y \\
i_{n*} X \arrow[rrr] & & & i_{n*} Y. \arrow[u]
\end{tikzcd}$$

At level n this yields a map (write $L_n Z = L_n Z(n-1)$, etc.)

$$X_n \cup_{L_n X} L_n Z \to Y_n \times_{M_n Y} M_n Z.$$

Factor this map as

$$X_n \cup_{L_n X} L_n Z \xrightarrow{j} Z_n \xrightarrow{q} Y_n \times_{M_n Y} M_n Z \tag{2.10}$$

where j is a trivial cofibration and q is a fibration. The morphisms j and q yield diagrams

$$\begin{tikzcd}
L_n X \arrow[r] \arrow[d] & X_n \arrow[r] \arrow[d] & M_n X \arrow[d] \\
L_n Z \arrow[r] \arrow[d] & Z_n \arrow[r] \arrow[d] & M_n Z \arrow[d] \\
L_n Y \arrow[r] & Y_n \arrow[r] & M_n Y
\end{tikzcd}$$

such that all the horizontal maps are induced from the canonical natural transformation $L_n \to M_n$. Hence Lemma 2.8 produces

$$i_{n*} X \to Z(n) \to i_{n*} Y.$$

Finally, define $Z \in s\mathcal{C}$ by $Z_k = Z(n)_k$, $k \leq n$. There is a factoring

$$X \xrightarrow{j} Z \xrightarrow{q} Y$$

and using (2.10) j is a trivial cofibration by Lemma 2.2 and q is a fibration by definition.

The other factorization is similar. ■

We now state

Theorem 2.11. *With the definitions of Reedy weak equivalence, cofibration, and fibration given in 2.1, $s\mathcal{C}$ is a closed model category.*

Proof: The axioms **CM1–CM3** are easy and Lemmas 2.6 and 2.9 prove **CM4** and **CM5** respectively. ∎

Next suppose $\mathcal{C}$ is in fact a simplicial model category. For $K \in \mathbf{S}$ and $Y, Z \in \mathcal{C}$ write $Z \square K$, Z^K, and $\mathbf{Hom}_{\mathcal{C}}(Y, Z)$ for the tensor object, the exponential object, and the mapping space. We use this notation to distinguish it from $Z \otimes K$ defined by the simplicial structure.

The category $s\mathcal{C}$ inherits a simplicial structure. If $X \in s\mathcal{C}$ and $K \in \mathbf{S}$, then $X \square K$ and X^K are defined level-wise

$$(X \square K)_n = X_n \square K \text{ and } (X^K)_n = X_n^K.$$

The mapping space is defined by the usual formula

$$\mathbf{Hom}_{s\mathcal{C}}(X, Y)_n \cong \hom_{s\mathcal{C}}(X \square \Delta^n, Y).$$

We call this the internal structure on $s\mathcal{C}$.

Corollary 2.12. *With this internal simplicial structure on $s\mathcal{C}$, the Reedy model category structure is a simplicial model category.*

Proof: We claim $M_n(Y^K) \cong M_n(Y)^K$. For there is a sequence of natural isomorphisms, $Z \in \mathcal{C}$,

$$\begin{aligned}
\hom_{\mathcal{C}}(Z, M_n(Y^K)) &\cong \hom_{s\mathcal{C}}(Z \otimes \partial\Delta^n, Y^K)\\
&\cong \hom_{s\mathcal{C}}((Z \otimes \partial\Delta^n) \square K, Y)\\
&\cong \hom_{s\mathcal{C}}((Z \square K) \otimes \partial\Delta^n, Y)\\
&\cong \hom_{s\mathcal{C}}(X \square K, M_n Y) \cong \hom_{s\mathcal{C}}(Z(M_n Y)^K).
\end{aligned}$$

The isomorphism $(Z \otimes \partial\Delta^n) \square K \cong (Z \square K) \otimes \partial\Delta^n$ follows by a level-wise calculation. The result follows from the claim using Proposition II.3.13. ∎

One can ask if $s\mathcal{C}$ in the Reedy model category is a simplicial model category in the standard simplicial structure obtained by Quillen's method (as in the previous section). The answer is no; for if $Z \in \mathcal{C}$ is cofibrant in $\mathcal{C}$, then

$$1 \otimes d^0 : Z \otimes \Delta^0 \to Z \otimes \Delta^1$$

is a Reedy cofibration, but not, in general, a Reedy weak equivalence. However we do have

PROPOSITION 2.13.
1) Let $f : X \to Y$ be a Reedy cofibration in $s\mathcal{C}$ and $j : K \to L$ a cofibration in $\mathbf{S}$. *Then*

$$X \otimes L \cup_{X \otimes K} Y \otimes K \to Y \otimes L$$

is a Reedy cofibration which is a Reedy weak equivalence if f is a Reedy weak equivalence.
2) Suppose $f : X \to Y$ is a Reedy fibration in $s\mathcal{C}$ and $j : K \to L$ is a cofibration in $\mathbf{S}$. *Then*

$$X^L \to X^K \times_{Y^K} Y^L$$

is a Reedy fibration which is a Reedy weak equivalence if f is a Reedy weak equivalence.

PROOF: By an adjunction argument these two statements are equivalent. Let $K \in \mathbf{S}$. Define $L_n K \subseteq K_n$ to be the degenerate n-simplices; that is

$$L_n K = \varinjlim_{\phi : \mathbf{n} \to \mathbf{k}} K_k$$

where ϕ runs over the surjective morphisms. Then if $X \in s\mathcal{C}$

$$L_n(X \otimes K) \cong \bigsqcup_{x \in L_n K} L_n X.$$

We leave the rest of the proof as an exercise. ■

3. Geometric realization.

Suppose $\mathcal{C}$ is a simplicial category and $X \in s\mathcal{C}$. Then the geometric realization $|X| \in \mathcal{C}$ is defined by the coequalizer diagram

$$\bigsqcup_{\phi : \mathbf{n} \to \mathbf{m}} X_m \square \Delta^n \underset{d_1}{\overset{d_0}{\rightrightarrows}} \bigsqcup_{n \geq 0} X_n \square \Delta^n \to |X| \tag{3.1}$$

where ϕ runs over the morphisms of $\mathbf{\Delta}$ and d_0 and d_1 or the factor associated to $\phi : \mathbf{n} \to \mathbf{m}$ are respectively

$$X_m \square \Delta^n \xrightarrow{\phi^* \square 1} X_n \square \Delta^n \longrightarrow \bigsqcup_{n \geq 0} X_n \square \Delta^n$$

$$X_m \square \Delta^n \xrightarrow{1 \square \phi} X_m \square \Delta^m \longrightarrow \bigsqcup_{n \geq 0} X_n \square \Delta^n.$$

This is the obvious generalization of the geometric realization of Chapters I and III. Note that $|X|$ is a coend:

$$|X| = \int^{\mathbf{\Delta}} X \square \Delta,$$

where Δ denotes the covariant functor $\mathbf{n} \mapsto \Delta^n$ on $\mathbf{\Delta}$. We discuss the homotopical properties of $|X|$.

First note that $|\cdot| : s\mathcal{C} \to \mathcal{C}$ has a right adjoint

$$Y \mapsto Y^{\Delta} = \{Y^{\Delta^n}\}. \tag{3.2}$$

If we give $s\mathcal{C}$ the internal (or level-wise) simplicial structure induced from $\mathcal{C}$, it follows immediately that if $X \in s\mathcal{C}$ and $K \in \mathbf{S}$, then

$$|X \square K| \cong |X| \square K. \tag{3.3}$$

Indeed, $\hom_{s\mathcal{C}}(X \square K, Y^{\Delta}) \cong \hom_{s\mathcal{C}}(X, (Y^K)^{\Delta})$.

Now assume $\mathcal{C}$ is a simplicial model category. Endow $s\mathcal{C}$ with the Reedy model category structure. By Corollary 2.12, this is a simplicial model category in the internal simplicial structure.

LEMMA 3.4. *The functor $(\cdot)^{\Delta} : \mathcal{C} \to s\mathcal{C}$ preserves fibrations and trivial fibrations.*

PROOF: The first point to be proved is this: if $K \in \mathbf{S}$ and $Y \in \mathcal{C}$, then

$$M_K(Y^{\Delta}) \cong Y^K. \tag{3.5}$$

Let $\mathbf{\Delta} \downarrow K$ be the simplex category of K (cf. Example 1.21). Then

$$\begin{aligned} M_K(Y^{\Delta}) &\cong \varprojlim_{\mathbf{\Delta}\downarrow K^{\mathrm{op}}} (Y^{\Delta})_n \\ &\cong \varprojlim_{\mathbf{\Delta}\downarrow K^{\mathrm{op}}} Y^{\Delta^n} \cong Y^{\varinjlim_{\mathbf{\Delta}\downarrow K} \Delta^n} \cong Y^K. \end{aligned}$$

Now let $X \to Y$ be a fibration in $\mathcal{C}$. Then

$$(X^{\Delta})_n \to (Y^{\Delta})_n \times_{M_n(Y^{\Delta})} M_n(X^{\Delta})$$

is isomorphic to

$$X^{\Delta^n} \to Y^{\Delta^n} \times_{Y^{\partial\Delta^n}} X^{\partial\Delta^n}$$

and the result follows from Lemma 2.2 and **SM7** for $\mathcal{C}$. ∎

PROPOSITION 3.6. *The geometric realization functor $|\cdot| : s\mathcal{C} \to \mathcal{C}$ preserves cofibrations, trivial cofibrations and weak equivalences between Reedy cofibrant objects.*

PROOF: Use Lemma 3.4 and Lemma II.8.9. ∎

Implicit in the proof of Lemma 3.4, is the assertion that if $Z \in s\mathcal{C}$ is *constant* and $K \in \mathbf{S}$ then there is a natural isomorphism

$$|Z \otimes K| \cong Z \square K. \tag{3.7}$$

Indeed, using the isomorphism (3.5), we have

$$\hom_{\mathcal{C}}(|Z \otimes K|, Y) \cong \hom_{\mathcal{C}}(Z, Y^K),$$

and the assertion follows. Therefore, for $X \in s\mathcal{C}$ Reedy cofibrant, the realization comes with a natural skeletal filtration. Define

$$\mathrm{sk}_n |X| = |\,\mathrm{sk}_n X|.$$

Then Proposition 1.7 and the natural isomorphism of (3.7) yield that there are natural pushout squares

$$\begin{array}{ccc} X_n \square \partial\Delta^n \cup_{L_n X \square \partial\Delta^n} L_n \square \Delta^n & \longrightarrow & \mathrm{sk}_{n-1} |X| \\ \downarrow & & \downarrow \\ X_n \square \Delta^n & \longrightarrow & \mathrm{sk}_n |X|. \end{array} \tag{3.8}$$

This is because $|\cdot|$, as a left adjoint commutes with all colimits. If X is cofibrant, Proposition 3.6 implies that each of the maps $\mathrm{sk}_{n-1} |X| \to \mathrm{sk}_n |X|$ is a cofibration. Furthermore, again since $|\cdot|$ commutes with colimits

$$\varinjlim_n \mathrm{sk}_n |X| \cong |X|. \tag{3.9}$$

Finally if X happened to s-free on same set $\{Z_n\}$ of objects in $\mathcal{C}$, then (3.8) can be refined (as in Proposition 1.13) to a pushout diagram

$$\begin{array}{ccc} Z_n \square \partial\Delta^n & \longrightarrow & \mathrm{sk}_{n-1} |X| \\ \downarrow & & \downarrow \\ Z_n \square \Delta^n & \longrightarrow & \mathrm{sk}_n |X|. \end{array} \tag{3.10}$$

The object $\mathrm{sk}_n |X|$ can also be described as a coend. Let $\mathrm{sk}_n \Delta$ be the functor from $\mathbf{\Delta}$ to $\mathbf{S}$ with

$$\mathbf{m} \mapsto \mathrm{sk}_n \Delta^m.$$

Proposition 3.11. *Let $X \in s\mathcal{C}$. Then there is a natural isomorphism*

$$\mathrm{sk}_n |X| \cong \int^{\mathbf{\Delta}} X \square \, \mathrm{sk}_n \Delta$$

and this isomorphism is compatible with the skeletal filtrations of source and target.

Proof: There is a sequence of natural isomorphisms, where

$$i_{n*} : s\mathcal{C} \to s_n\mathcal{C}$$

is the restriction functor

$$\begin{aligned} \hom_{\mathcal{C}}(\mathrm{sk}_n |X|, Y) &\cong \hom_{s\mathcal{C}}(\mathrm{sk}_n X, Y^{\Delta}) \\ &\cong \hom_{s\mathcal{C}}(i_n^* i_{n*} X, Y^{\Delta}) \\ &\cong \hom_{s\mathcal{C}}(X, i_n^! i_{n*} Y^{\Delta}) \end{aligned}$$

Now $i_n^! i_{n*} Y^{\Delta} \cong \mathrm{cosk}_n(Y^{\Delta}) \cong Y^{\mathrm{sk}_n \Delta}$. Hence

$$\begin{aligned} \hom_{\mathcal{C}}(\mathrm{sk}_n |X|, Y) &\cong \hom_{s\mathcal{C}}(X, Y^{\mathrm{sk}_n \Delta}) \\ &\cong \hom_{\mathcal{C}}(\int^{\Delta} X \square \, \mathrm{sk}_n \Delta, Y). \end{aligned}$$ ∎

4. Cosimplicial spaces.

The language and technology of the previous three sections can be used to give a discussion of the homotopy theory of cosimplicial spaces; that is, of the category $c\mathbf{S} = \mathbf{S}^{\mathbf{\Delta}}$ of functors from the ordinal number category to simplicial sets. We go through some of the details and give some examples. It turns out that cofibrations in $c\mathbf{S}$ have a very simple characterization; we close the section with a proof of this fact.

We begin with two important examples.

EXAMPLE 4.1. Let $R = \mathbb{F}_p$, the prime field with $p > 0$, or let R be a subring of the rationals. The forgetful functor from R-modules to sets has left adjoint $X \mapsto RX$, where RX is the free R-module on X. These functors prolong to an adjoint pair between simplicial R-modules and simplicial sets. By abuse of notation we write

$$R : \mathbf{S} \to \mathbf{S}$$

for the composite of these two functors. Then R is the functor underlying a cotriple on $\mathbf{S}$ and, if $X \in \mathbf{S}$,

$$\pi_* RX \cong H_*(X; R).$$

Let $T : \mathbf{S} \to \mathbf{S}$ be any cotriple on $\mathbf{S}$ with natural structure maps $\eta : X \to TX$ and $\epsilon : T^2X \to TX$. If $X \in \mathbf{S}$ is any object, there is a natural augmented cosimplicial space

$$X \to T^\bullet X$$

with $(T^\bullet X)^n = T^{n+1}X$ and

$$\begin{aligned} d^i &= T^i \eta T : (T^\bullet X)^n \to (T^\bullet X)^{n+1} \\ s^i &= T^i \epsilon T^{n-i} : (T^\bullet X)^{n+1} \to (T^\bullet X)^n. \end{aligned}$$

The augmentation is given by $\eta : X \to TX = (T^\bullet X)^0$; note that

$$d^0\eta = d^1\eta : X \to (T^\bullet X)^1.$$

In particular, if we let $T = R : \mathbf{S} \to \mathbf{S}$ we get an augmented cosimplicial space

$$X \to R^\bullet X$$

with the property that d^i, $i \geq 1$, and s^i, $i \geq 0$, are all morphisms of simplicial R-modules. Furthermore, if we apply R one more time, the augmented cosimplicial R-module

$$RX \to R(R^\bullet X)$$

has a cosimplicial contraction; hence

$$\pi^p H_*(R^\bullet X; R) \cong \begin{cases} H_*(X; R), & p = 0 \\ 0, & p > 0. \end{cases}$$

The object $X \to R^\bullet X$ is a variation on the Bousfield-Kan R-resolution of X.

EXAMPLE 4.2. Let J be a small category and $\mathbf{S}^J$ the category of J-diagrams in $\mathbf{S}$. Let J^δ be the category with some objects as J but no non-identity morphisms—J^δ is J made discrete. There is an inclusion functor $J^\delta \to J$, hence a restriction functor

$$r_* : \mathbf{S}^J \to \mathbf{S}^{J^\delta}.$$

The functor r_* has a right adjoint $r^!$ given by right Kan extension; in formulas

$$(r^! X)_j = \prod_{j \to i} X_i$$

where the product is over morphisms in J with source j. Let

$$T = r^! r_* : \mathbf{S}^J \to \mathbf{S}^J.$$

Then T is the functor of a cotriple on $\mathbf{S}^J$, and if $Y \in \mathbf{S}^J$, there is a natural cosimplicial object in $\mathbf{S}^J$

$$Y \to T^\bullet Y. \tag{4.3}$$

This cosimplicial object has the property that the underlying J^δ diagram has a cosimplicial contraction. Put another way, for each $j \in J$, the augmented cosimplicial space

$$Y_j \to (T^\bullet Y)_j$$

has a cosimplicial contraction. We can apply the functor $\varprojlim_J(\cdot)$ to (4.3) to obtain an augmented cosimplicial space

$$\varprojlim_J Y \to \varprojlim_J T^\bullet Y. \tag{4.4}$$

Note that $\varprojlim_J (T^\bullet Y)^n$ can be easily computed because

$$\varprojlim_J r^! X \cong \prod_j X_j$$

where j runs over the objects on J. This last follows from the isomorphisms

$$\begin{aligned} \hom_{\mathbf{S}}(Z, \varprojlim_J r_! X) &\cong \hom_{\mathbf{S}^{J^\delta}}(Z, r_! X) \\ &\cong \hom_{\mathbf{S}^{J^\delta}}(Z, X) \cong \prod_j \hom_{\mathbf{S}}(Z, X_j) \end{aligned}$$

where $Z \in \mathbf{S}$ is regarded as a constant diagram in $\mathbf{S}^J$ or $\mathbf{S}^{J^s}$.

The cosimplicial object (4.4) is used in analyzing the homotopy inverse limit of $Y \in \mathbf{S}^J$ — see Section IX.5.

With these examples in hand, we begin to analyze the homotopy theory of $c\mathbf{S}$. Since $\mathbf{S}$ is a simplicial model category, so is the opposite category $\mathbf{S}^{\mathrm{op}}$. Thus the category $(s(\mathbf{S}^{\mathrm{op}}))^{\mathrm{op}} = c\mathbf{S}$ acquires a Reedy model category structure as in section 2 of this chapter. We take some care with the definitions as the opposite category device can be confusing.

To begin with, $c\mathbf{S}$ is a simplicial category: if $K \in \mathbf{S}$ and $X \in c\mathbf{S}$ define $X \square K$ and X^K in $c\mathbf{S}$ by

$$(X \square K)^n = X^n \times K \tag{4.5}$$

and

$$(X^K)^n = (X^n)^K. \tag{4.6}$$

One often writes $X \otimes K$ for $X \square K$; however, we are—in this chapter—reserving the tensor product notation for the external operation on $s\mathcal{C}$. The mapping spaces functor is then

$$\mathbf{Hom}_{c\mathbf{S}}(X, Y)_n \cong \hom_{c\mathbf{S}}(X \square \Delta^n, Y). \tag{4.7}$$

There are also latching and matching objects. At this point the literature goes to pieces. The matching objects in $c\mathbf{S}$ defined in [BK] are the latching objects in $s(\mathbf{S}^{\mathrm{op}})$ as defined in Section 1. Since we would hope the reader will turn to the work of Bousfield and Kan as needed, we will be explicit.

Let $X \in c\mathbf{S}$. The n^{th} *matching space* $M^n X \in \mathbf{S}$ is the $(n+1)^{\text{st}}$ *latching* object $L_{n+1}X$ of $X \in s(\mathbf{S}^{\mathrm{op}})$. Specifically,

$$M^n X \cong \varprojlim_{\phi : \mathbf{n+1} \to \mathbf{k}} X^k \tag{4.8}$$

where $\phi : \mathbf{n} \to \mathbf{k}$ runs over the surjections in $\mathbf{\Delta}$ with $k \leq n$. Thus $n \geq -1$. There are natural maps $s : X^{n+1} \to M^n X$ and $f : X \to Y$ in $c\mathbf{S}$ is a *fibration* if and only if

$$X^{n+1} \to Y^{n+1} \times_{M^n Y} M^n X \tag{4.9}$$

is a fibration for $n \geq -1$.

The reader should be aware of 1) the shift in indices $M^n X = L_{n+1}X$ and 2) the superscript versus the subscript: $M^n X \neq M_n X$.

Similarly, there are latching objects. If $X \in c\mathbf{S}$, $L^n X = M_{n+1}X$, the matching spaces of $X \in s(\mathbf{S}^{\mathrm{op}})$; thus

$$L_n X = \varinjlim_{\phi:\mathbf{k}\to\mathbf{n+1}} X^k$$

where ϕ runs over the injections in $\mathbf{\Delta}$. A morphism $X \to Y$ in $c\mathbf{S}$ is a *cofibration* if and only if

$$X^{n+1} \cup_{L_n X} L_n Y \to Y^{n+1} \tag{4.10}$$

is a cofibration (that is, inclusion) of simplicial sets.

Finally, we define a morphism $X \to Y$ in $c\mathbf{S}$ to be a weak equivalence if $X^n \to Y^n$ is a weak equivalence for all $n \geq 1$. Theorem 2.11 and Corollary 2.12 now imply the following result.

PROPOSITION 4.11. *With the definitions above, the category $c\mathbf{S}$ is a simplicial model category.*

We can give a simple characterization of cofibrations in $c\mathbf{S}$, and, along the way, show that $c\mathbf{S}$ is cofibrantly generated. First, let us define a set of specific cofibrations.

The functors from $c\mathbf{S} \to \mathbf{S}$

$$\rho_n : X \mapsto X^n, n \geq 0$$

and

$$\mu_n : X \mapsto M^n X, n \geq -1$$

all have left adjoints, given by variations on left Kan extension. Indeed, the adjoint to ρ_n is given by the formula

$$(\rho_n^* Y)^k = \bigsqcup_{\phi:\mathbf{n}\to\mathbf{k}} Y$$

where ϕ runs over all morphisms in $\mathbf{\Delta}$ with source $\mathbf{n}$. This is a left Kan extension. The adjoint to μ_n is slightly more complicated: if J is the category with objects surjections $\mathbf{n+1} \to \mathbf{k}$ in $\mathbf{\Delta}$, and $r : J \to \mathbf{\Delta}$ the functor sending $\mathbf{n+1} \to \mathbf{k}$ to $\mathbf{k}$, then the left adjoint μ_n^* to μ_n is characterized by

$$\begin{aligned}\hom_{c\mathbf{S}}(\mu_n^* Z, X) &\cong \hom_{S^J}(Z, r_* X)\\ &\cong \hom_{\mathbf{S}}(Z, M_n X)\end{aligned}$$

where $Z \in \mathbf{S}^J$ is the constant diagram. Thus $\mu_n^* Z$ is a left Kan extension of a constant diagram. Note that the natural transformation

$$X_n \to M_{n-1}X$$

gives a natural transformation

$$\mu_{n-1}^* Z \to \rho_n^* Z.$$

Define morphisms in $c\mathbf{S}$ as follows:

$$\rho_n^* \partial\Delta^m \cup_{\mu_{n-1}^* \partial\Delta^m} \mu_{n-1}^* \Delta^m = \partial\Delta \left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right] \xrightarrow{i_n^m} \Delta \left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right] = \rho_n^* \Delta^m, \tag{4.12}$$

for $n \geq 0$, $m \geq 0$, and

$$\Delta \begin{bmatrix} m\ k \\ n \end{bmatrix} = \rho_n^* \Delta_k^m \cup_{\mu_{n-1}^* \Delta_k^m} \mu_{n-1}^* \Delta^m \xrightarrow{j_n^{m,k}} \Delta \begin{bmatrix} m \\ n \end{bmatrix} = \rho_n^* \Delta^m, \tag{4.13}$$

for $n \geq 0$, $0 \leq k \leq m$, $m \geq 1$.

LEMMA 4.14. *A morphism $f : X \to Y$ in $c\mathbf{S}$ is a fibration if and only if it has the right lifting property with respect to the morphisms $j_n^{m,k}$ of (4.13). A morphism in $c\mathbf{S}$ is a trivial fibration if and only if it has the right lifting property with respect to the morphisms i_n^m of (4.12).*

PROOF: We prove the trivial fibration case; the other is similar. A lifting problem

$$\begin{CD} \partial\Delta \begin{bmatrix} m \\ n \end{bmatrix} @>>> X \\ @VVV @VVV \\ \Delta \begin{bmatrix} m \\ n \end{bmatrix} @>>> Y \end{CD}$$

is equivalent, by adjointness to a lifting problem

$$\begin{CD} \partial\Delta^m @>>> X_n \\ @VVV @VV{s}V \\ \Delta^m @>>> Y_n \times_{M^{n-1}Y} M^{n-1}X. \end{CD}$$

Lemma 2.2.2 (remember $M^{n-1}X = L_n X$, where $X \in s(\mathbf{S}^{\mathrm{op}})$) implies that $f : X \to Y$ is a trivial fibration if and only if $s : X_n \to Y_n \times_{M^{n-1}Y} M^{n-1}X$ is a trivial fibration for all n. The result follows. ∎

PROPOSITION 4.15. *The simplicial model category structure on $c\mathbf{S}$ is cofibrantly generated: the morphisms i_n^m of (4.12) generate the cofibrations and the morphisms $j_n^{m,k}$ of (4.13) generate the trivial cofibrations.*

PROOF: In light of Lemma 4.14, the small object argument now applies. ∎

We can use this result to characterize cofibrations. If $X \in c\mathcal{C}$ is a cosimplicial object in any category, define the maximal augmentation aX by the equalizer diagram in $\mathcal{C}$

$$aX \to X^0 \underset{d^1}{\overset{d^0}{\rightrightarrows}} X^1.$$

Let $d^0 : aX \to X^0$ be the natural map.

PROPOSITION 4.16. *A morphism $f : X \to Y$ in $c\mathbf{S}$ is a cofibration if and only if $X^n \to Y^n$ is a cofibration in $\mathbf{S}$ for all $n \geq 0$ and the induced map $aX \to aY$ of maximal augmentations is an isomorphism.*

We give the proof below, after some technical preliminaries.

Let $\mathbf{\Delta}_{-1}$ be the augmented ordinal number category; this has objects

$$\mathbf{n} = \{0, 1, \dots, n-1\}, n \geq -1,$$

$(-\mathbf{1} = \phi)$ and ordering preserving maps. An augmented cosimplicial object in $\mathcal{C}$ is a functor $X : \mathbf{\Delta}_{-1} \to \mathcal{C}$. We write $d^0 : X^{-1} \to X^0$ for the unique map. Note that the maximal augmentation extends any cosimplicial object to an augmented cosimplicial object.

LEMMA 4.17. *Let X be an augmented cosimplicial set, and*

$$Z^n = X^n - \bigcup_{i=0}^{n} d^i X^{n-1}$$

If $d^0 : X^{-1} \to X^0$ isomorphic to the inclusion of the maximal augmentation, then

$$\bigsqcup_{\phi} Z^k \to X^n$$

with $\phi : \mathbf{k} \to \mathbf{n}$ running over all injections, $-1 \leq k \leq n$, is an isomorphism.

PROOF: Out of any cosimplicial set X we may construct a simplicial set Y "without d^0" as follows: $Y_n = X^n$ and

$$d_i = s^{n-i} : Y_n \to Y_{n-1}, 1 \leq i \leq n$$
$$s_i = d^{n-i} : Y_n \to Y_{n+1}, 0 \leq i \leq n.$$

Notice that this construction does not use $d^n : X^{n-1} \to X^n$. Let $Z_n \subseteq Y_n$ be the non-degenerate simplices:

$$Z_n = Y_n - \bigcup_i s_i Y_{n-1}.$$

The standard argument for simplicial sets (see Example 1.14) shows that

$$Y_n \cong \coprod_{\psi:\mathbf{n}\to\mathbf{k}} Z_k$$

where ψ runs over the surjections in $\mathbf{\Delta}$. Unraveling the definitions shows that our claim will follow if we can show that $d^n : X^{n-1} \to X^n$ induces an isomorphism $Z^n \sqcup Z^{n-1} \xrightarrow{\cong} Z_n$ or, equivalently, an isomorphism

$$Z^{n-1} \xrightarrow{\cong} Z_n \cap d^n X_{n-1}.$$

First note that d^n does induce an injection

$$d^n : Z^{n-1} \to Z_n \cap d^n X_{n-1}. \tag{4.18}$$

To show this it is sufficient to show that if $y \in Z^{n-1}$, then $d^n y \in Z_n$; that is, if $y \notin \bigcup_{i=0}^{n-1} d^i X_{n-2}$, then $d^n y \notin \bigcup_{i=0}^{n-1} d^i X_{n-1}$. The contrapositive of this statement reads: if $d^n y \in \bigcup_{i=0}^{n-1} d^i X_{n-1}$, then $y \in \bigcup_{i=0}^{n-1} d^i X_{n-2}$. So assume $d^n y = d^i z$ with $i < n$. If $i < n-1$, then

$$z = s^i d^n y = d^{n-1} s^i y$$

so $d^n y = d^i z = d^i d^{n-1} s^i y = d^n d^i s^i y$ and $y = d^i s^i y$. If $i = n-1$ and $n > 1$, then

$$y = s^{n-1} d^n y = s^{n-1} d^{n-1} z = z$$

so $d^n y = d^{n-1} y$; hence

$$y = s^{n-2} d^{n-1} y = s^{n-2} d^n y = d^{n-1} s^{n-2} y.$$

If $n = 1$ and $i = n-1$, we have $d^1 y = d^0 z$, hence $y = z$; since $d^0 : X^{-1} \to X^0$ is the inclusion of the maximal augmentation, $y = d^0 w$ for some w This is where the hypothesis is used.

We now must show that d^n, as in (4.18) is onto. If $n = 0$, $Z_0 = X_0$ and the result is clear. If $n \geq 1$, we need to show that if $x = d^n y$ and $x \notin \bigcup_{i=0}^{n-1} d^i X_{n-1}$, then $y \notin \bigcup_{i=0}^{n-1} d^i X^{n-2}$. The contrapositive of this statement is: if $x = d^n y$ and $y = d^i w$, $i \leq n-1$, then $x = d^j z$ for $j \leq n-1$. But

$$x = d^n y = d^n d^i w = d^i d^{n-1} w. \qquad \blacksquare$$

We can now prove Proposition 4.16:

PROOF OF 4.16: For the purposes of this argument, we say a morphism $f : X \to Y$ in $c\mathbf{S}$ has *Property* $\mathbf{C}$ if $X^n \to Y^n$ is a cofibration in $\mathbf{S}$ for all $n \geq 0$ and $aX \cong aY$. We leave it as an exercise to show that the class of morphisms satisfying Property $\mathbf{C}$ is closed under isomorphisms, coproducts, retracts, cobase change, and colimits over ordinal numbers. Only the statement about cobase change is non-trivial. Furthermore, the generating cofibrations $\iota^m_n : \partial\Delta \left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right] \to \Delta \left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right]$ have Property $\mathbf{C}$. Hence Proposition 4.15 implies all cofibrations have Property $\mathbf{C}$.

For the converse, suppose $f : X \to Y$ has Property $\mathbf{C}$. Referring to Lemma 4.17, write $Z^n(X_m)$ for Z^n obtained from the cosimplicial set of m simplices X_k. Then, Lemma 4.17 implies

$$(L^0 X)_m \cong Z^0(X_m) \sqcup Z^{-1}(X_m) \sqcup Z^0(X_m) \sqcup Z^{-1}(X_m)$$

and if $n > 1$,

$$(L^{n-1} X)_m \cong \bigsqcup_{\phi:\mathbf{k}\to\mathbf{n}} Z^k(X_m)$$

with ϕ running over injections with $-1 \leq k < n$. Since $f : X^n_m \to Y^n_m$ is one-to-one, $Z^k(\cdot)$ is natural in f. Since $Z^{-1}(X_m) \cong Z^{-1}(Y_m)$,

$$[X^n \cup_{L^{n-1}X} L^{n-1} Y]_m \cong Z^n(X_m) \sqcup \bigsqcup_{\phi:\mathbf{k}\to\mathbf{n}} Z^k(Y_m)$$

for all $n \geq 0$. Again ϕ runs over injections k, $-1 \leq k < n$. The result follows. $\blacksquare$

Proposition 4.16 makes it very easy to decide when an object of $c\mathbf{S}$ is cofibrant; for example, a constant object on a non-empty simplicial set is not cofibrant, but the standard simplices Δ^n form a cosimplicial space $\{\Delta^n\}_n = \Delta$ which is cofibrant.

5. The total tower of a cosimplicial space.

The results on geometric realization from Section 3, interpreted in the context of cosimplicial spaces, leads to the study of the total space of a cosimplicial space. The purpose of this section is to work through these results.

If $X \in c\mathbf{S} \cong (s(\mathbf{S}^{\mathrm{op}}))^{\mathrm{op}}$, we can define the geometric realization $|X| \in (\mathbf{S}^{\mathrm{op}})^{\mathrm{op}} = \mathbf{S}$; we will write $\mathrm{Tot}(X)$ for this object. Using the coequalizer (3.1), suitably interpreted, we find that $\mathrm{Tot}(X)$ fits into an equalizer diagram

$$\mathrm{Tot}(X) \to \prod_{n \geq 0} X_n^{\Delta^n} \rightrightarrows \prod_{\phi:\mathbf{n}\to\mathbf{m}} X_m^{\Delta^n}. \tag{5.1}$$

If $\Delta = \{\Delta^n\} \in c\mathbf{S}$ is the cosimplicial space composed of the standard simplices, (5.1) immediately implies there is a natural isomorphism in $\mathbf{S}$

$$\mathrm{Tot}(X) \cong \mathbf{Hom}_{s\mathcal{C}}(\Delta, X). \tag{5.2}$$

Since $\Delta \in c\mathbf{S}$ is cofibrant, **SM7** for $c\mathbf{S}$ implies that $\mathrm{Tot}(\cdot)$ preserves weak equivalences between fibrant objects; alternatively, this fact is a consequence of Proposition 3.6.

The skeletal filtration on the geometric realization (as in (3.8) and (3.9) writes $\mathrm{Tot}(X)$ as the inverse limit of a tower. We define $\mathrm{Tot}_n(X) = \mathrm{sk}_n\,|X| \in (\mathbf{S}^{\mathrm{op}})^{\mathrm{op}}$; arguing as in (5.1) and (5.2) and using Proposition 3.11, we have

$$\mathrm{Tot}_n(X) \cong \mathbf{Hom}_{c\mathbf{S}}(\mathrm{sk}_n\,\Delta, X) \tag{5.3}$$

and the natural projection $\mathrm{Tot}_n(X) \to \mathrm{Tot}_{n-1}(X)$ is induced by the inclusion $\mathrm{sk}_{n-1}\,\Delta \to \mathrm{sk}_n\,\Delta$. Note

$$\mathrm{Tot}(X) \cong \varprojlim \mathrm{Tot}_n(X). \tag{5.4}$$

By Proposition 4.16, $\mathrm{sk}_{n-1}\,\Delta \to \mathrm{sk}_n\,\Delta$ is a cofibration in $c\mathbf{S}$, hence, if $X \in c\mathbf{S}$ is fibrant, **SM7** for $c\mathbf{S}$ implies

$$\{\mathrm{Tot}_n(X)\}_{n\geq 0} \tag{5.5}$$

is a tower of fibrations. This fact equally follows from Proposition 3.6 This is the *total tower* of X.

The diagrams (3.8) and (3.10) tell us how $\mathrm{Tot}_n(X)$ is built from $\mathrm{Tot}_{n-1}(X)$. Recalling that for $X \in c\mathbf{S}$, $M^{n-1}X \cong L_nX$, $X \in s(\mathbf{S}^{\mathrm{op}})$, (3.8) implies that for all $X \in c\mathbf{S}$ there is a pullback diagram

$$\begin{array}{ccc} \mathrm{Tot}_n(X) & \longrightarrow & X_n^{\Delta^n} \\ \downarrow & & \downarrow \\ \mathrm{Tot}_{n-1}(X) & \longrightarrow & X_n^{\partial\Delta^n} \times_{M_{n-1}^{\partial\Delta^n} X} (M_{n-1}X)^{\Delta^n}. \end{array} \tag{5.6}$$

Note $\mathrm{Tot}_0(X) \cong X^0$. If X is fibrant, the vertical maps in (5.6) are fibrations. Finally, if the cosimplicial object $X \in c\mathbf{S}$ is codegeneracy-free in the sense that the underlying codegeneracy diagram is the right Kan extension of a discrete diagram $\{Z^n\}$, then (3.10) implies that there is a pullback diagram in $\mathbf{S}$.

$$\begin{array}{ccc} \mathrm{Tot}_n X & \longrightarrow & (Z^n)^{\Delta^n} \\ \downarrow & & \downarrow \\ \mathrm{Tot}_{n-1} X & \longrightarrow & (Z^n)^{\partial\Delta^n}. \end{array} \tag{5.7}$$

The rest of the section is devoted to showing that our two examples 4.1 and 4.2 are codegeneracy free in this sense.

Example 5.8. The category of cosimplicial abelian groups is equivalent to the category $(s(\mathbf{Ab}^{\mathrm{op}}))^{\mathrm{op}}$. Since $\mathbf{Ab}$ is abelian, so is $\mathbf{Ab}^{\mathrm{op}}$; hence the normalization functor $N : s(\mathbf{Ab}^{\mathrm{op}}) \to C_*\mathbf{Ab}^{\mathrm{op}}$ defines an equivalence of categories. See Section III.2. In particular every object $A \in s(\mathbf{Ab}^{\mathrm{op}})$ is naturally degeneracy free on $\{NA_s\}$. An examination of the proof shows that d_0 plays no role in the decomposition of $A \in s(\mathbf{Ab}^{\mathrm{op}})$; specifically if $n\mathbf{Ab}^{\mathrm{op}}$ denotes the category with objects $\{X_n\}_{n\geq 0}$, $X_n \in \mathbf{Ab}^{\mathrm{op}}$, then normalization and its adjoint define an equivalence of categories

$$N : s_0(\mathbf{Ab})^{\mathrm{op}} \to n\mathbf{Ab}^{\mathrm{op}}.$$

Here $s_0\mathcal{C}$ denotes the category of functors $\mathbf{\Delta}_0^{\mathrm{op}} \to \mathcal{C}$ where $\mathbf{\Delta}_0 \subseteq \mathbf{\Delta}$ is the subcategory with some objects and morphisms restricted $\phi(0) = 0$. These are simplicial objects without d_0.

If $c^+\mathbf{Ab}$ denotes the cosimplicial abelian groups with without d^0, these remarks supply an equivalence categories

$$N^* : c^+\mathbf{Ab} \to n\mathbf{Ab}$$

sending A to N^*A with

$$(N^*A)^n = \bigcap_{s=0}^{n-1} \ker\{s^i : A^n \to A^{n-1}\}.$$

In particular any object $A \in c^+\mathbf{Ab}$ is naturally codegeneracy free on N^*A. Now let $X \in \mathbf{S}$ and $R^\bullet X$ the Bousfield-Kan R-resolution of Example 4.1. Then $R^\bullet X \in c\mathbf{S}$ is not a cosimplicial simplicial abelian group; however, when restricted an object $R^\bullet X \in c^+\mathbf{S}$ is has a natural structure as an object in $c^+(s\mathbf{Ab})$. Since products in $s\mathbf{Ab}$ are products in $\mathbf{S}$, we have that $R^\bullet X$ is naturally codegeneracy free on $N^*R^\bullet X$. Then (5.7) gives the existence of a pullback diagram

$$\begin{array}{ccc}
\mathrm{Tot}_n\, R^\bullet X & \longrightarrow & \mathbf{Hom}(\Delta^n, N^*R^nX) \\
\downarrow & & \downarrow \\
\mathrm{Tot}_{n-1}\, R^\bullet X & \longrightarrow & \mathbf{Hom}(\partial\Delta^n, N^*R^nX).
\end{array}$$

EXAMPLE 5.9. Consider the cosimplicial object $T^\bullet X \in c\mathbf{S}^J$ constructed in Example 4.2. Examining the definitions we have

$$(T^\bullet X)^n = T^{n+1}(X)_j = \prod_{j\to i_0\to\cdots\to i_n} X_{i_n}.$$

It follows that $T^\bullet X$ is naturally codegeneracy free as an object in $c\mathbf{S}^J$ on $Z^n(X)$ where

$$Z^n(X) = \prod_{j\to i_0\to i_1\to\cdots\to i_n} X_{i_n}$$

where $i_t \to i_{t+1}$ is not the identity for any t, $0 \le t \le n-1$. It follows that $\underleftarrow{\lim}_J T^\bullet X$ at $c\mathbf{S}$ is also codegeneracy free, since $\underleftarrow{\lim}$ preserves products.

6. The homotopy spectral sequence of a tower of fibrations.

Given a tower of pointed fibrations, the exact sequences in homotopy give rise to a spectral sequence. There are a number of problems with this spectral sequence. First, there is the problem that π_0 and π_1 are not generally abelian groups; second, the spectral sequence need not converge to anything homotopical; third, the spectral sequence is "fringed" in the sense that E_∞ can contain extra elements; and, fourth, the E_2 term may not have a sensible description in terms of homological algebra. We analyze these difficulties in turn. The first two difficulties are addressed in this section, and the last in the next. The fringing effect will be dealt with in Section 8.

So let

$$\begin{tikzcd} \cdots \arrow[r] & X_n \arrow[r, "p_n"] & X_{n-1} \arrow[r] & \cdots \arrow[r] & X_1 \arrow[r, "p_1"] & X_0 \arrow[r] & * \\ & F_n \arrow[u, "i_n"] & F_{n-1} \arrow[u, "i_{n-1}"] & & F_1 \arrow[u, "i_1"] & F_0 \arrow[u, "="] & \end{tikzcd}$$

be a tower of pointed fibrations, with F_n the fiber of p_n. Applying homotopy to each of the fiber sequences, yields

$$\begin{tikzcd} \cdots \arrow[r] & \pi_* X_n \arrow[r] & \pi_* X_{n-1} \arrow[r] \arrow[dl, dashed] & \cdots \arrow[r] \arrow[dl, dashed] & \pi_* X_1 \arrow[r] & \pi_* X_0 \arrow[dl, dashed] \\ & \pi_* F_n \arrow[u] & \pi_* F_{n-1} \arrow[u] & & \pi_* F_1 \arrow[u] & \pi_* F_0 \arrow[u, equal] \end{tikzcd}$$

where each of the dotted maps is of degree -1. This gives an exact couple and hence a spectral sequence. We will be explicit because π_1 is not abelian and because

$$\pi_1 X_{n-1} \to \pi_0 F_n \to \pi_0 X_n \to \pi_0 X_{n-1}$$

is exact only in the sense of pointed sets, with the additional proviso that the action of $\pi_1 X_{n-1}$ on $\pi_0 F_n$ extends to an injection of sets

$$* \to \pi_0 F_n / \pi_1 X_{n-1} \to \pi_0 X_n.$$

Note that $\pi_0 X_n \to \pi_0 X_{n-1}$ need not be onto, although the pre-image of the basepoint is $\pi_0 F_n / \pi_1 X_{n-1}$. (Here and throughout, if a group G acts on a set Z, Z/G denotes the orbits.)

The spectral sequence is now defined as follows. Let

$$\pi_i X_n^{(r)} = \operatorname{Im}(\pi_i X_{n+r} \to \pi_i X_n) \subseteq \pi_i X_n, \tag{6.1}$$

and for $t - s \geq 0$, let

$$Z_r^{s,t} = \operatorname{Ker}(\pi_{t-s} F_s \to \pi_{t-s} X_s / \pi_{t-s} X_s^{(r-1)}). \tag{6.2}$$

Then let

$$B_r^{s,t} = \mathrm{Ker}(\pi_{t-s+1}X_{s-1} \to \pi_{t-s+1}X_{s-r-1}). \tag{6.3}$$

Then we set

$$E_r^{s,t} = Z_r^{s,t}/B_r^{s,t} \tag{6.4}$$

where this formula must be interpreted: if $t-s \geq 1$ and $j_* : \pi_{i+1}X_{n-1} \to \pi_i F_n$ is the connecting homomorphism, $j_* B_r^{s,,t} \subseteq Z_r^{s,t}$ is a normal subgroup and $E_r^{s,t}$ is the quotient group, and if $t-s=0$, $B_r^{s,t}$ acts on $Z_r^{s,t}$ and $E_r^{s,t}$ is the set of orbits. Define differentials for $t-s \geq 1$

$$d_r : E_r^{s,t} \to E_r^{s+r,t+r-1} \tag{6.5}$$

by the composite map

$$E_r^{s,t} \to \pi_{t-s}X_s^{(r-1)} \to E_r^{s+r,t+r-1} \tag{6.6}$$

where the first map is induced by noticing that if $x \in Z_r^{s,t} \subseteq \pi_{t-s}F_s$, then $(i_s)_* x \in \pi_{t-s}X_s^{(r-1)}$ and the second by noticing that if y is in $\pi_{t-s}X_s^{(r-1)}$ we may choose $z \in \pi_{s-1}X_{s+r-1}$ mapping to y and

$$j_* z \in E_r^{s+r,t+r-1}$$

is independent of that choice. Notice that no differential is defined on $E_r^{s,s}$ and, indeed, that $E_r^{s,t}$ is not defined for $t-s<0$.

The following result is left as an exercise:

LEMMA 6.7.
1) $E_r^{s,t}$ is a pointed set if $t-s \geq 0$, a group if $t-s \geq 1$, and an abelian group if $t-s \geq 2$;

2) The function $d_r : E_r^{s,t} \to E_s^{s+r,t+r-1}$ is a homomorphism if $t-s \geq 2$ and a map of pointed sets if $t-s \geq 1$.

3) The image of $d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$ is in the center of $E_r^{s+r,t+r-1}$ if $t-s \geq 2$ and if $t-s \geq 1$, $E_{r+1}^{s,t} = \mathrm{Ker}(d_r)/\mathrm{Im}(d_r)$.

4) The map $d_r : E_r^{s,s+1} \to E_r^{s+r,s+r}$ extends to an action of the source on the target and $E_{r+1}^{s,s} \subseteq E_r^{s,s}/E_r^{s-r,s-r+1}$.

PROOF: All these facts follow from the properties of the long exact sequence in homotopy of a fibration. ■

REMARK 6.8. The fact that $E_{r+1}^{s,s} \subseteq E_r^{s,s}/E_r^{s-r,s-r+1}$ and is not necessarily equal to what is meant by saying the spectral sequence is *fringed*. The failure of equality arises from the fact that $\pi_0 X_n \to \pi_0 X_{n-1}$ need not be onto.

The collection $\{E_r^{s,t}; d_r\}_{r\geq 1}$ is the homotopy spectral sequence of the tower of fibrations. It is natural in the tower and we may write $\{E_r^{s,t}X\}$. We now begin to address what it might converge to.

Notice that if $r > s$, $E_{r+1}^{s,t} \subset E_r^{s,t}$. Define

$$E_\infty^{s,t} = \varprojlim_{r>s} E_r^{s,t} = \bigcap_{r>s} E_r^{s,t}. \tag{6.9}$$

One hopes that $E_\infty^{s,t}$ may have something to do with the homotopy groups of $\varprojlim X_n$. For this, we must first decide how to compute $\pi_*(\varprojlim X_n)$ in terms of the tower of groups $\{\pi_* X_n\}$. This is done by a $\varprojlim^1$ exact sequence, once we have defined $\varprojlim^1$ for non-abelian groups.

First recall that $\varprojlim^1$ for a tower of abelian groups $A = \{A_n\}_{n\geq 0}$ is defined by the exact sequence

$$0 \to \varprojlim A_n \to \prod_n A_n \xrightarrow{\partial} \prod_n A_n \to \varprojlim{}^1 A_n \to 0$$

where $\partial(a_n) = (a_n - f(a_{n+1}))$, with $f : A_{n+1} \to A_n$ the maps in the tower. The following is an easy exercise in homological algebra.

LEMMA 6.10. *The functor* $\varprojlim^1 : \mathbf{tow}(\mathbf{Ab}) \to \mathbf{Ab}$ *is characterized up to natural isomorphism by:*

1) If $0 \to A \to B \to C \to 0$ *is a short exact of towers of abelian groups, then there is a six-term exact sequence*

$$0 \to \varprojlim A \to \varprojlim B \to \varprojlim C \to \varprojlim{}^1 A \to \varprojlim{}^1 B \to \varprojlim{}^1 C \to 0.$$

2) If A *is any tower so that* $A_{n+1} \to A_n$ *is surjective for all* n*, then* $\varprojlim^1 A = 0$.

We give a similar statement for non-abelian groups. Let $G = \{G_n\}$ be a tower of not necessarily abelian groups. Then the group $\prod_n G_n$ acts on the set $Z = \prod_n G_n$ by $(g_n) \circ (x_n) = (g_n x_n f(g_{n+1})^{-1})$. Then the stabilizer subgroup of $e \in Z$ is $\varprojlim G_n$ and we define $\varprojlim^1 G_n$ to be the pointed set of orbits. If G is a tower of abelian groups these definitions agree with those above.

LEMMA 6.11. *The functor* $\varprojlim^1 : \mathbf{tow}(\mathbf{Gr}) \to \text{pointed sets}$ *is characterized up to natural isomorphism by*
1) If $* \to G_1 \to G_2 \to G_2 \to *$ *is short exact sequence of towers of groups then there is a six term sequence*

$$* \to \varprojlim G_1 \to \varprojlim G_2 \to \varprojlim G_3 \to \varprojlim^1 G_1 \to \varprojlim^1 G_2 \to \varprojlim^1 G_3 \to *$$

which is exact as groups at the first three terms, as pointed sets at the last three terms, and $\varprojlim G_3$ *acts on* $\varprojlim^1 G_1$ *and there is an induced injection*

$$\varprojlim^1 G_1 / \varprojlim G_3 \to \varprojlim^1 G_2.$$

2) If G *is any tower so that* $G_{n+1} \to G_n$ *is surjective for all* n*, then* $\varprojlim^1 G_n = *$.

PROOF: The proof is the same as in the abelian case. ■

REMARK 6.12. It is good to have examples where $\varprojlim^1$ is non-zero. Let B be any group and let B also be the constant tower on that group. let $A_n \subseteq B$ be a descending ($A_{n+1} \subset A_n$) sequence of normal subgroups so that $\bigcap_n A_n = e$. Let $A = \{A_n\}$ be the resulting tower. Then one gets a short exact sequence

$$* \to B \to \varprojlim_n (B/A_n) \to \varprojlim^1 A \to * \tag{6.13}$$

so that $\varprojlim^1 A = *$ if and only if B is complete in the topology defined by the A_n. For example, if $B = \mathbb{Z}$ and $A_n = p^n\mathbb{Z}$ for some prime p, (6.13) becomes

$$0 \to \mathbb{Z} \to \mathbb{Z}_p \to \mathbb{Z}_p/\mathbb{Z} \to 0$$

where $\mathbb{Z}_p$ is the group of p-adic integers, which is uncountable. One could take $B = \bigoplus_{s\geq 1} \mathbb{Z}$ and $A_n = \bigoplus_{s \geq n} \mathbb{Z} \subseteq B$. Then (6.13) becomes

$$0 \to \bigoplus_{s\geq 1} \mathbb{Z} \to \prod_{s\geq 1} \mathbb{Z} \to \prod_{s\geq 1} \mathbb{Z} / \bigoplus_{s\geq 1} \mathbb{Z} \to 0.$$

Now let $X = \{X_n\}$ be a tower of pointed fibrations.

PROPOSITION 6.14. *For every* $t \geq 0$ *there is a sequence*

$$* \to \varprojlim^1 \pi_{t+1} X_n \to \pi_t(\varprojlim X_n) \to \varprojlim \pi_t X_n \to *$$

which is exact as groups if $t \geq 1$ *and as pointed sets if* $t = 0$.

Proof: Consider the description of the homotopy inverse limit given in Section IX.5. Then one has a homotopy fibration sequence

$$\prod_n \Omega X_n \to \underleftarrow{\operatorname{holim}} X_n \to \prod X_n.$$

A simple calculation with the homotopy exact sequence of this fibration gives the result except possibly for the claim that $\pi_0 \underleftarrow{\operatorname{holim}} X_n \to \underleftarrow{\lim} \pi_0 X_n$ is onto. But it is an easy exercise to show that for a tower of fibrations $\pi_0 \underleftarrow{\lim} X_n \to \underleftarrow{\lim} \pi_0 X_n$ is onto. ■

We now return to computing $\pi_* \underleftarrow{\lim} X_n$ by the homotopy spectral sequence. Because of $E_r^{s,t}$ given in (6.1)–(6.4) and $E_\infty^{s,t}$ in (6.9) we can think of $E_\infty^{s,t}$ as an equivalence class of elements $x \in \pi_{t-s}F_s$ so that for each $r \geq 0$ there is an element $y_r \in \pi_{t-s}X_{s+r}$ mapping to $i_*x \in \pi_{t-s}X_s$. (Notice we do not say y_r maps to y_{r-1}; see Example 6.17.2 below.) If $z \in \pi_{t-s} \underleftarrow{\lim} X_n$ maps to zero in $\pi_{t-s}X_{s-1}$, then one gets such an equivalence class. To make this precise, let

$$G_s\pi_i X = \operatorname{Im}(\pi_i X \to \pi_i X_s)$$

and

$$e_\infty^{s,t} = \ker\{G_s\pi_{t-s}X_s \to G_s\pi_{t-s}X_{s-1}\}. \tag{6.15}$$

If $y \in e_\infty^{s,t}$, let $x \in \pi_{t-s}F_s$ be any element so that $i_*x = y$.

Lemma 6.16.
1) The assignment $y \mapsto x$ induces a well-defined monomorphism $e_\infty^{s,t} \to E_\infty^{s,st}$.

2) The inclusion maps $G_s\pi_*X \to \pi_*X_s$ induce an isomorphism

$$\underleftarrow{\lim}_s G_s\pi_*X \xrightarrow{\cong} \underleftarrow{\lim}_s \pi_*X_s.$$

Proof: This is an exercise in unraveling the definitions. ■

Notice that any element of $\underleftarrow{\lim}^1 \pi_*X$ maps to zero in all $G_s\pi_*X$, so will not be detected in any $e_\infty^{s,t}$. It also turns out that $e_\infty^{s,t} \to E_\infty^{s,t}$ need not be an isomorphism.

Here are two examples of the potential difficulties.

EXAMPLE 6.17.
1) Consider the tower of abelian groups

$$\cdots \xrightarrow{\times p} \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \xrightarrow{\times p} \mathbb{Z}$$

where $p \in \mathbb{Z}$ is a prime number. The resulting tower of Eilenberg-MacLane spaces

$$\to K(\mathbb{Z},n) \xrightarrow{\times p} K(\mathbb{Z},n) \xrightarrow{\times p} \cdots \xrightarrow{\times p} K(\mathbb{Z},n)$$

can be transformed into a tower of fibrations. One easily calculates that $E_1^{0,n} = \mathbb{Z}$ and $E_1^{s,s+n-1} = \mathbb{Z}/p\mathbb{Z}$ and $E_1^{s,t} = 0$ otherwise. Next $E_r^{0,n} = p^{r-1}\mathbb{Z} \subseteq \mathbb{Z} = E_1^{0,n}$ and $d_r : E_r^{0,n} \to E_r^{r,r+n-1} = \mathbb{Z}/p\mathbb{Z}$ is onto.. Thus $E_\infty^{s,t} = 0$ for all $t - s \geq 0$. However

$$\pi_{n-1} \varprojlim X_s = \mathbb{Z}_p/\mathbb{Z} \cong \varprojlim_r{}^1 (p^r\mathbb{Z}) \neq 0.$$

Note that in this case $\varprojlim^1 E_r^{0,n} \neq 0$.

2) Consider the tower of abelian groups, where a is the addition map,

$$\to \bigoplus_{s\geq 3} \mathbb{Z} \to \bigoplus_{s\geq 2} \mathbb{Z} \to \bigoplus_{s\geq 1} \mathbb{Z} \xrightarrow{a} \mathbb{Z}$$

and let $\{X_n\}$ be the resulting tower of fibrations obtained by applying $K(\cdot,n)$ and the converting tower of fibrations. Now

$$\pi_{n-1} \varprojlim X_m \cong \lim{}^1 \pi_n X_m \cong \prod_{s\geq 1} \mathbb{Z} / \bigoplus_{s\geq 1} \mathbb{Z}$$

and $\pi_k \varprojlim X_m = 0$ otherwise. Then

$$E_1^{s,t} = \begin{cases} \mathbb{Z} & s = 0, t = n \\ \bigoplus_{i\geq 1} \mathbb{Z} & s = 1, t = n+1 \\ \mathbb{Z} & s \geq 2, t = s+n-1 \end{cases}$$

and $d_r : E_r^{1,n+1} \to E_r^{r+1,n+r}$ is onto. Thus $E_\infty^{0,n} = \mathbb{Z}$ and $E_\infty^{s,t} = 0$ otherwise. Thus $e_\infty^{0,n} \neq E_\infty^{0,n}$. Note also $\varprojlim^1 E_r^{1,n+1} \neq 0$. ■

The following definition now seems appropriate.

DEFINITION 6.18. *The spectral sequence of the pointed tower of fibrations $\{X_n\}$ converges completely if*

1) $\varprojlim^1 \pi_i X_n = *$ *for* $i \geq 1$, *and*
2) $e_\infty^{s,t} \cong E_\infty^{s,t}$ *for all* (s,t) *with* $t - s \geq 1$.

When 6.18.2 holds, the spectral sequence effectively computes $\varprojlim \pi_* X_n$, subject to the fringing effect when $t - s = 0$, and when 6.18.1 holds, we have $\pi_*(\varprojlim X_n) \cong \varprojlim \pi_* X_n$. One would like to be able to decide when one has complete convergence from knowledge of the spectral sequence, rather than using knowledge of $\pi_* \varprojlim X_n$, which Definition 6.18 demands — after all one is trying to compute $\pi_* \varprojlim X_n$. The next lemma is the best possible result. Compare the examples of 6.17.

LEMMA 6.19 (COMPLETE CONVERGENCE LEMMA). *The spectral sequence of the pointed tower of fibrations $\{X_n\}$ converges completely if and only if*

$$\varprojlim_r{}^1 E_r^{s,t} = *, \qquad t - s \geq 1.$$

This will be proved below, after some examples and preliminaries.

COROLLARY 6.20. *Suppose that for each integer $n \geq 0$ there are only finitely many s, so that $E_2^{s,s+n} \neq *$. Then the spectral sequence converges completely.*

More generally we have

COROLLARY 6.21. *Suppose that for each integer $n \geq 0$ and each integer s, there is an integer N so that*

$$E_\infty^{s,s+n} \cong E_N^{s,s+n}.$$

Then the spectral sequence converges completely.

PROOF: For $\varprojlim^1 E_r^{s,t} = *$. See 6.22.1 below. ■

The phenomenon encountered in Corollary 6.21 is called *Mittag-Leffler convergence.* The wider implications are explored in [14, p.264].

As a preliminary to proving the complete convergence lemma, we discuss derived towers of group. Suppose $\{G_n\}$ is a tower of groups. Define $G_n^{(r)} = \operatorname{Im}\{G_{n+r} \to G_n\}$. We already encountered $\pi_* X_s^{(r)} = (\pi_* X_s)^{(r)}$ in (6.1)–(6.4). There are maps

$$G_n^{(r)} \to G_{n-1}^{(r+1)} \to G_{n-1}^{(r)}$$

and one has two new towers of groups

$$\{G_n^{(r)}\}_{n\geq 0} \qquad\qquad \{G_n^{(r)}\}_{r\geq 0}.$$

LEMMA 6.22.
1) The inclusions $G_n^{(r)} \subseteq G_n$ induce isomorphisms

$$\varprojlim_n G_n^{(r)} \cong \varprojlim_n G_n \quad \textit{and} \quad \varprojlim_n{}^1 G_n^{(r)} \cong \varprojlim_n{}^1 G_n.$$

2) There is a natural isomorphism

$$\varprojlim_n \varprojlim_r G_n^{(r)} \cong \varprojlim_n G_n$$

and a natural short exact sequence

$$* \to \varprojlim_n{}^1 \varprojlim_r G_n^{(r)} \to \varprojlim{}^1 G_n \to \varprojlim_n \varprojlim_r{}^1 G_n^{(r)} \to *.$$

PROOF: For 1) note that the functor $\{G_n\}_{n\geq 0} \mapsto \{G_n^{(r)}\}_{n\geq 0}$ is exact and takes tower of surjections to surjections. Since $\varprojlim_n G_n^{(n)} \cong \varprojlim_n G_n$ by inspection, the result follows from the characterization of $\varprojlim^1$ given in Lemma 6.11: the functor

$$\{G_n\} \to \varprojlim{}^1 G_n^{(r)}$$

satisfies the requirements there.

For 2), note that if the groups were abelian, what we would have is a degenerate form of a composite functor spectral sequence. In the non-abelian case, it is better to proceed topologically. For fixed r define a tower $Y_r = \{Y_{r,n}\}$ with $Y_{r,n} = BG_n^{(r)}$. There are maps of towers $Y_{r+1} \to Y_r$; convert the resulting tower of towers into a tower of fibrations in $\mathbf{tow}(\mathbf{S}_*)$:

$$\cdots \to Z_2 \to Z_1 \to Z_0.$$

Thus $Y_0 \to Z_0$ is a weak equivalence in $\mathbf{tow}(\mathbf{S}_*)$ with Z_0 fibrant (i.e., a tower of fibrations) and $Y_r \to Z_r \to Z_{r-1}$ factors $Y_r \to Y_{r-1} \to Z_{r-1}$ as a weak equivalence followed by a fibrations. Then

$$\cdots \to \varprojlim_n Z_2 \to \varprojlim_n Z_1 \to \varprojlim_n Z_0$$

is a tower of fibrations and for all r, $\pi_1 \varprojlim_n Z_r \cong \varprojlim_n G_n$ and $\pi_0 \varprojlim_n Z_r \cong \varprojlim_n {}^1 G_r$, by part 1). Hence

$$\pi_1(\varprojlim_r \varprojlim_n Z_{r,n}) \cong \varprojlim_n G_n$$
$$\pi_0(\varprojlim_r \varprojlim_n Z_{r,n}) \cong \varprojlim_n{}^1 G_n.$$

Now one takes limits in the other direction: for fixed n, $\{Z_{r,n}\}_{r\geq 0}$ is a tower fibration, so

$$\pi_1 \varprojlim_r Z_{r,n} \cong \varprojlim_r G_n^{(r)}$$
$$\pi_0 \varprojlim_r Z_{r,n} \cong \varprojlim_r{}^1 G_n^{(r)}$$

Now we calculate $\pi_* \varprojlim_n \varprojlim_r Z_{r,n}$ and use that limits commute to finish the result. ∎

PROOF OF THE COMPLETE CONVERGENCE LEMMA 6.19: We prove only that if $\varprojlim_r{}^1 E_r^{s,t} = *$, then we have complete convergence. The reverse implication is left as an exercise—it is a matter of reversing the arguments.

For each pair of positive integers $r \geq n$, let us define K_n^r by the exact sequence of groups

$$* \to K_n^r \to \pi_{t-s} X_{s+n}^{(r-n)} \to \pi_{t-s} X_s^{(r)} \to *. \tag{6.23}$$

If $n = 1$ and $r > s$, $K_n^r = E_r^{s+1,t+1}$; therefore, if $r > s+n$, there is a composition series for K_n^r with all of the horizontal maps surjections

$$\begin{tikzcd}
K_n^r \arrow[r] & K_{n-1}^{r+1} \arrow[r] & \cdots \arrow[r] & K_1^{r+n-1} \\
E_r^{s+n,t+n} \arrow[u] & E_{r+1}^{s+n-1,t+n-1} \arrow[u] & & E_{r+n-1}^{s+1,t+1} \arrow[u, equal]
\end{tikzcd}$$

An induction argument shows $\varprojlim_r{}^1 K_n^r = *$ and that each of the maps $\varprojlim_r K_n^r \to \varprojlim_r K_{n-1}^{r+1}$ is onto. Taking the limit over r in (6.23) yields a short exact sequence

$$* \to \varprojlim_r K_n^r \to \varprojlim_r \pi_{t-s} X_{s+n}^{(r-n)} \to \varprojlim_r \pi_{t-s} X_s^{(r)} \to *.$$

Now we take the limit over n. By the previous lemma

$$\varprojlim_n \varprojlim_r \pi_{t-s} X_{s+n}^{(r-n)} \cong \varprojlim_n \pi_{t-s} X_n,$$

and since $\varprojlim^1$ vanishes on towers of surjections, we get a short exact sequence

$$* \to \varprojlim_n \varprojlim_r K_n^r \to \varprojlim_n \pi_{t-s} X_n \to \varprojlim_r \pi_{t-s} X_s^{(r)} \to * \tag{6.24}$$

and an injection

$$\varprojlim_n{}^1 \varprojlim_r \pi_{t-s} X_{s+n}^{(r-n)} \to \varprojlim_n{}^1 \pi_{t-s} X_s^{(r)}. \tag{6.25}$$

In particular, (6.24) yields that the natural map $\varprojlim_n \pi_{t-s} X_n \to \varprojlim_r \pi_{t-s} X_s^{(r)}$ is a surjection. Thus $G_s \pi_{t-s} \varprojlim X_n \cong \varprojlim_r \pi_{t-s} X_s^{(r)}$ and, by induction, $e_\infty^{s,t} = E_\infty^{s,t}$.

Also, since n does not appear in $\varprojlim^1_n \pi_{t-s} X_s^{(r)}$, this term is trivial, so we have that the term $\varprojlim^1_n \varprojlim_r \pi_{t-s} X_{s+n}^{(r-n)} = *$. If $\varprojlim_n \varprojlim^1_r \pi_{t-s} X_{s+n}^{(r-n)} = *$, the previous lemma implies $\varprojlim^1 \pi_{t-s} X_n = *$ and we'll be done. So we'll close by showing $\varprojlim^1_r \pi_{t-s} X_{s+n}^{(r-n)} = *$.

As with K_n^r, there is a composition series for $\pi_{t-s} X_s^{(r)}$: if $r > s$

$$\begin{tikzcd}
\pi_{t-s} X_s^{(r)} \arrow[r] & \pi_{t-s} X_{s-1}^{(r+1)} \arrow[r] & \cdots \arrow[r] & \pi_{t-s} X_0^{(r+s)} \\
E_{r+1}^{s,t} \arrow[u] & E_{r+2}^{s-1,t-1} \arrow[u] & & E_{r+s+1}^{0,t-s} \arrow[u]
\end{tikzcd}$$

A similar inductive argument as with K_n^r, shows $\varprojlim^1_r \pi_{t-s} X_s^{(r)} = *$. ∎

It is worth pointing out that even without convergence one has a comparison result.

PROPOSITION 6.26. *Suppose $f : X = \{X_n\} \to Y = \{Y_n\}$ is a map of towers of pointed fibrations and suppose there is a (finite) $N \geq 1$ so that*

$$f_* : E_N^{s,t} X \to E_N^{s,t} Y$$

is an isomorphism for all (s,t). If $E_r^{s,s} X = *$ for all s, then $\varprojlim X_n \to \varprojlim Y_n$ is a weak equivalence of connected spaces.

Proof: Note that $f_* : E_r^{s,t}X \to E_r^{s,t}Y$ is an isomorphism for all $r \geq N$. Then induction on n and the exact sequence

$$E_r^{s+t,t+1} \to \pi_{t-s}X_{s+1}^{(r-1)} \to \pi_{t-s}X_s^{(r)}$$

(see (6.23)) implies $\pi_k X_n^{(n)} \to \pi_k Y_n^{(r)}$ is an isomorphism for all $r \geq N-1$. Finally, the hypotheses on $E_r^{s,s}$ imply $\pi_0 X_n^{(r)} = * = \pi_0 Y_n^{(r)}$ and $\pi_1 X_n^{(r)} \to \pi_1 X_{n-1}^{(r)}$ is onto. Thus the result follows from Lemmas 6.22.1, 6.11, and Proposition 6.14.■

7. The homotopy spectral sequence of a cosimplicial space.

If X is a pointed cosimplicial space, then the tower $\{\mathrm{Tot}_n X\}_{n\geq 0}$ has a homotopy spectral sequence. In this section, we work out the E_1 and E_2 terms of the spectral sequence, and give an extended example—the original Bousfield-Kan spectral sequence, which is a kind of unstable Adams spectral sequence.

Fix a fibrant pointed cosimplicial space X, and form the tower $\{\mathrm{Tot}_n X\}_{n\geq 0}$, where $\mathrm{Tot}_n X = \mathbf{Hom}_{c\mathbf{S}}(\mathrm{sk}_n \Delta, X)$. The constant map is the basepoint of $\mathrm{Tot}_n X$ and by the diagram (5.6), the fiber of $\mathrm{Tot}_n X \to \mathrm{Tot}_{n-1} X$ is $\Omega^n N^n X$, where $N^n X$ is the fiber of $s : X^n \to M^{n-1}X$. Thus the E_1 term of the associate spectral sequence is, for $t - s \geq 0$,

$$E_1^{s,t} = \pi_{t-s}\Omega^s N^s X \cong \pi_t N^s X.$$

If G is a cosimplicial (not necessarily abelian) group, define

$$N^s G = \bigcap_{i=0}^{j-1} \ker\{s^i : G^s \to G^{s-1}\}.$$

If G is a cosimplicial abelian group, then $NG = \{N^s G, \Sigma(-1)^i d^i\}$ becomes a cochain complex. Define $N\pi_0 X = \pi_0 X^0$.

Lemma 7.1. *For a fibrant pointed cosimplicial space X, there is a natural isomorphism*

$$E_1^{s,t} \cong \pi_t N^s X \cong N^s \pi_t X, t - s \geq 0$$

Proof: First assume $t \geq 2$. The cases $t = 0, 1$ will be handled at the end. We use the pullback square

(7.2)
$$\begin{tikzcd}
M^{n-1,k+1}X \arrow[r] \arrow[d] & X^{n-1} \arrow[d] \\
M^{n-1,k}X \arrow[r] & M^{n-2,k}X
\end{tikzcd}$$

of Proposition 1.25, and a double induction on the following two statements. Notice that $M^{n,k}$ is a functor defined as $c\mathcal{C}$ where $\mathcal{C}$ is any category with finite limits.

$H(n,k)$: The natural map $\pi_t M^{n-1,k}X \to M_{n-1,k}\pi_t X$ is an isomorphism.

$I(n,k)$: Let $N^{n,k}X$ be the fiber of $s : X^n \to M_{n-1,k}X$. Then

$$0 \to \pi_t N^{n,k}X \to \pi_t X^n \to \pi_t M_{n-1,k}X \to 0$$

is short exact.

The result follows from setting $n = k = s$ in $I(n,k)$ and $H(n,k)$.

First notice that $I(0,0)$ is trivial since $M^{-1,0}X = *$. Also $H(n,0)$ and $I(n,0)$ are trivial since $M^{n-1,0}X = *$ and $M^{n-1,0}\pi_t X = 0$.

Next, $H(n,k), H(n-1,k)$, and $I(n-1,k)$ yield $H(n,k+1)$, since $\pi_t(\cdot)$ applied to (7.2) yields a pullback diagram

$$\begin{tikzcd}
\pi_t M^{n-1,k+1}X \arrow[r] \arrow[d] & \pi_t X_{n-1} \arrow[d] \\
M^{n-1,k}\pi_t X \arrow[r] & M_{n-2,k}\pi_t X.
\end{tikzcd}$$

Finally, $H(n,k) \Rightarrow I(n,k)$ because $\pi_t X^n \to M^{n-1,k}\pi_t X$ is surjective. Indeed, as in Example 5.8,

$$M^{n-1,k}\pi_t X \cong \bigoplus_{\phi:\mathbf{n}\to\mathbf{m}} N^m \pi_t X$$

where ϕ runs over the objects of $\mathcal{M}_{n,k}$.

For the cases $t = 0$ and 1 of Lemma 7.1, note that if $t = 0, s = 0$, so one need only check $\pi_0 N^0 X = \pi_0 X^0 = N\pi^0 X$ which is true since $N^0 X = X^0$. If $t = 1, s = 0$ or 1. If $s = 0$, one has $\pi_1 N^0 X = \pi_1 X^0$ as required. If $s = 1$, $* \to \pi_1 N^1 X \to \pi_1 X^1 \xrightarrow{s^0} \pi_1 X^0 \to *$ is exact, since $s^0 d^0 = 1$ and $X^0 = M^0 X$. ■

We use this calculation as the input for the calculation of the E_2 term. In fact, we wish to claim that $E_2^{s,t} = \pi^s \pi_t X$, but we need to take some care with the definitions, especially for $t = 1$.

If A is a cosimplicial abelian group, we can make A into a cochain complex

$$\cdots \to A_2 \xrightarrow{\partial} A_1 \xrightarrow{\partial} A_0$$

with $\partial = \Sigma(-1)^i d^i$. Then $\pi^s A = H^s(A)$. This is equivalent to computing $H^s N^* A$, where $N^* A$ is the normalized cochain complex with

$$N^* A = \{N^s A, \Sigma(-1)^i d^i\}.$$

One has

$$\pi^s A = H^s A \cong H^s N^* A. \tag{7.3}$$

If G is a cosimplicial (not necessarily abelian) group, one still has $\pi^0 G$ and $\pi^1 G$. The group $\pi^0 G$ is the equalizer of d^0, d^1:

$$\pi^0 G \to G^0 \underset{d^1}{\overset{d^0}{\rightrightarrows}} G^1. \tag{7.4}$$

The pointed set $\pi^1 G$ is defined by the formula

$$\pi^1 G = Z^1 G / G^0 \tag{7.5}$$

where $Z^1 G$ is the pointed set of cocycles

$$Z^1 G = \{x \in G^1 | (d^2 x)(d^1 x)^{-1}(d^0 x) = e\}$$

and G^0 acts on $Z^1 G$ by

$$g \circ x = d^0 g \cdot x (d^1 g)^{-1}.$$

Notice that as G is a cosimplicial abelian group, the two definitions of $\pi^1 G$ agree. One can also normalize in the non-abelian case; indeed, the situation is already normalized in the sense that if $x \in Z^1 G$, then $x \in N^1 G$ because

$$e = s^0[(d^2 x)(d^1 x)^{-1}(d^0 x)] = d^1 s^0 x$$

so

$$e = s^0 d^1 s^0 x = s^0 x.$$

Thus $N^1 G \cap Z^1 G = Z^1 G$ and

$$\pi^1 G = (N^1 G \cap Z^1 G)/G^0. \tag{7.6}$$

If X is a cosimplicial set we define $\pi^0 X$ to be the equalizer of $d^0, d^1 : X^0 \to X^1$.

PROPOSITION 7.7. *Let X be a pointed fibrant cosimplicial space. Then the E_2 term of the spectral sequence of the tower of fibrations $\{\operatorname{Tot}_n X\}$ can be calculated by natural isomorphism*

$$E_2^{s,t} \cong \pi^s \pi_t X, \qquad t - s \geq 0.$$

Because of this result, one often writes

$$\pi^s \pi_t X \Rightarrow \pi_{t-s} \operatorname{Tot} X$$

for the homotopy spectral sequence arising from a pointed cosimplicial space. The double arrow is not meant to prejudice anyone for or against convergence of the spectral sequence; for this, one must appeal to the results of Section 6.

Before proving Proposition 7.7, we line up notation and ideas. We are interested in the tower $\{\operatorname{Tot}_n X\}$, where the tower maps

$$\operatorname{Tot}_s X = \mathbf{Hom}_{c\mathbf{S}}(\operatorname{sk}_s \Delta, X) \to \mathbf{Hom}_{c\mathbf{S}}(\operatorname{sk}_{s-1} \Delta, X) = \operatorname{Tot}_{s-1} X$$

are induced by the inclusion $\operatorname{sk}_{s-1} \Delta \to \operatorname{sk}_s \Delta$. Thus the fiber at the constant map is the space of pointed maps

$$\mathbf{Hom}_{c\mathbf{S}_*}(\operatorname{sk}_s \Delta / \operatorname{sk}_{s-1} \Delta, X).$$

If $f : \operatorname{sk}_s \Delta / \operatorname{sk}_{s-1} \Delta \to X$ is a pointed map one gets a diagram

$$\begin{tikzcd} \Delta^s/\partial\Delta^s \arrow[r, "f"] \arrow[d] & X^s \arrow[d] \\ * \arrow[r] & M_s X \end{tikzcd}$$

and hence a map of spaces

$$\mathbf{Hom}_{c\mathbf{S}_*}(\operatorname{sk}_s \Delta / \operatorname{sk}_{s-1} \Delta, X) \to \mathbf{Hom}_*(\Delta^s/\partial\Delta^s, N_s X).$$

Diagram (5.6) that this map is an isomorphism. To calculate $E_2^{s,t} = Z_2^{s,t}/B_2^{s,t}$ (see (6.1)–(6.4)) we calculate $Z_2^{s,t}$, $B_2^{s,t}$, and the action of $B_2^{s,t}$. These are given in Lemmas 7.8, 7.14, and 7.15 below. Proposition 7.7 will follow.

Recall that for $t \geq 2$, we may define

$$\partial = \sum_{i=0}^{s+1} d^i : \pi_t N^s X \to \pi_t N^{s+1} X.$$

This defines the sub-group $\ker(\partial) \subseteq \pi_t N^s X$, $t \geq 2$. If $t = 1$, define

$$\ker(\partial) \subseteq \pi_1 N^1 X$$

by

$$\ker(\partial) = Z^1 \pi_1 N^* X$$

(see (7.4)–(7.5)). This is a pointed set. If $t = 0$ or 1 define $\mathrm{Ker}(\partial) \subseteq \pi_t N^0 X - \pi_t X^0$ to be the equalizer of

$$d^0, d^1 : \pi_t X^0 \to \pi_t X^1.$$

If $t = 1$, $\ker(\partial)$ is a group; if $t = 0$, $\ker(\partial)$ is a pointed set.

LEMMA 7.8. *For all $t - s \geq 0$, there is a natural isomorphism*

$$Z_2^{s,t} \cong \ker(\partial) \subseteq \pi_t N^s X.$$

PROOF: First take $s \geq 1$. According to (6.2),

$$f \in \pi_{t-s} \mathbf{Hom}_{c\mathbf{S}_*}(\mathrm{sk}_s \Delta / \mathrm{sk}_{s-1} \Delta, X)$$

is in $Z_2^{s,t}$ if and only if there is a lift

$$\begin{tikzcd} & & \mathrm{Tot}_{s+1} X \arrow[d] \\ S^{t-s} \arrow[r, "f"] \arrow[urr, dashed] & \mathbf{Hom}_{c\mathbf{S}_*}(\mathrm{sk}_s \Delta / \mathrm{sk}_{s-1} \Delta, X) \arrow[r] & \mathrm{Tot}_s X. \end{tikzcd}$$

Here we have confused f with a representative. We can do this as all objects are fibrant. Thus, by adjointness, $f \in Z_2^{s,t}$ if and only if there is an extension (with $+$ denoting a disjoint basepoint)

$$\begin{tikzcd} S^{t-s} \wedge (\mathrm{sk}_s \Delta)_+ \arrow[r] \arrow[d] & S^{t-s} \wedge (\mathrm{sk}_s \Delta / \mathrm{sk}_{s-1} \Delta) \arrow[r, "g"] & X \\ S^{t-s} \wedge (\mathrm{sk}_{s+1} \Delta)_+ \arrow[urr, dashed] & & \end{tikzcd}$$

in the category of pointed cosimplicial spaces. Here g is adjoint to f. By Diagram (5.6) and adjointnetss, $f \in Z_2^{s,t}$ if and only if there is a lift

$$
\begin{tikzcd}
S^{t-s} \wedge (\partial\Delta^{s+1})_+ \arrow[r] \arrow[d] & S^{t-s} \wedge (\partial\Delta^{s+1}/\operatorname{sk}_{s-1}\Delta^{s+1}) \arrow[r, "g"] & X^{s+1} \arrow[d] \\
S^{t-s} \wedge (\Delta^{s+1})_+ \arrow[rr] \arrow[urr, dashed] & & M_s X
\end{tikzcd}
\tag{7.9}
$$

in the category of pointed spaces. Let $\iota : \Delta^s/\partial\Delta^s \to \Delta^s/\partial\Delta^s$ be the identity. Then the collapse map $\partial\Delta^{s+1} \to \partial\Delta^{s+1}/\operatorname{sk}_{s-1}\Delta^{s-1}$ represents, when we choose $0 \in \partial\Delta^{s+1}$ as the basepoint,

$$\sum_{i=0}^{s+1}(-1)^i d^i\iota \in \pi_s(\partial\Delta^{s+1}/\operatorname{sk}_{s-1}\Delta^{s+1}) \qquad \text{if } s+1>2$$

and

$$[d^2\iota][d^1\iota]^{-1}[d^0\iota] \in \pi_1(\partial\Delta^2/\operatorname{sk}_0\Delta^2) \qquad \text{if } s+1=2.$$

Thus if the original map f was adjoint to $\alpha \in \pi_t N^s X$, the morphism across the top of (7.9) represents $\partial\alpha$ (if $s+1>2$) and $[d^2\alpha][d^1\alpha]^{-1}[d^0\alpha]$ (if $s+1=2$). Thus $f \in Z_2^{s,t}$ if and only if the morphism across the top is null-homotopic; that is, if and only if $\alpha \in \ker(\partial)$.

If $s = 0$, one is asking whether there is a lift

$$
\begin{tikzcd}
& \operatorname{Tot}_1 X \arrow[d] \\
S^t \arrow[r, "f"'] \arrow[ur, dashed] & \operatorname{Tot}_0 X = X^0
\end{tikzcd}
$$

or, again by (5.6) and adjointness, whether the following diagram can be completed

$$
\begin{tikzcd}
S^t \wedge (\partial\Delta^1)_+ \arrow[r] \arrow[d] & X_1 \arrow[d] \\
S^t \wedge (\Delta^1)_+ \arrow[r] \arrow[ur, dashed] & X_0.
\end{tikzcd}
$$

This happens if and only if $d^0 f = d^1 f \in \pi_t X^1$. ■

In order to calculate the boundaries $B_2^{s,t}$ and the quotient $E_2^{s,t}$, we must be explicit about the action of the fundamental group on the components of the fiber. This requires a short digression.

Let $\mathcal{C}$ be any simplicial model category. Suppose A is pointed cofibrant and $A \to B$ is a pointed cofibration. Then if B/A is define the pushout diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ * \arrow[r] & B/A, \end{tikzcd}$$

the object B/A is pointed and there is a pointed fibration sequence. If $A \in \mathcal{C}$ is pointed cofibrant and K is a pointed simplicial set

$$A \wedge K = A \otimes K / A \otimes * \vee * \otimes K.$$

If $X \in \mathcal{C}$ is pointed and fibrant, there is a pointed fibration sequence

$$\mathbf{Hom}_*(B/A, X) \to \mathbf{Hom}_*(B, X) \to \mathbf{Hom}_*(A, X)$$

where we take the constant to be the basepoint. We describe the action of $\pi_t \mathbf{Hom}_*(A, X) \cong [A \wedge S^t, X]_*$ on $\pi_{t-s}\mathbf{Hom}_*(B/A, X) \cong [B/A \wedge S^{t-1}, X]_*$.

One model for the pointed cone on A is $C^1 A = A \wedge \Delta^1_+ / A \wedge \{1\}_+$. For another model let $[0,2] \in \mathbf{S}$ be the simplicial set with two non-degenerate one simplices $[0,1]$ and $[1,2]$ and $d^0[0,1] = \{1\} = d^1[1,2]$. (We can think of Δ^1 as $[0,1]$.) Then $C^2 A = A \wedge [0,2]_+ / A \wedge \{2\}_+$ is another model for the pointed cone on A. Think of $A \subseteq C^1 A$ or $A \subseteq C^2 A$ as the subspace $A \wedge \{0\}_+$. Then there are weak equivalences

$$B \cup_A C^2 A \simeq B/A \simeq B \cup_A C^1 A$$

and an isomorphism $C^1 A / A \cong A \wedge S^1$. Thus one gets a map

$$\psi : B \cup_A C^2 A \to B \sqcup C^2 A / A \wedge \{1\}_+ = A \wedge S^1 \vee B \cup_A C^1 A \tag{7.10}$$

If $\alpha \in \pi_t \mathbf{Hom}_*(A, X)$ and $\beta \in \pi_{t-1}\mathbf{Hom}_*(B/A, X)$ are represented by $f : A \wedge S^t \to X$ and $g : (B \cup_A C^1 A) \wedge S^{t-1} \to X$ respectively $\alpha \cdot \beta \in \pi_{t-1}\mathbf{Hom}_*(B/A, X)$ is represented by

$$(B \cup_A C^2 A) \wedge S^{t-1} \xrightarrow{\psi \wedge 1} (A \wedge S^1) \wedge S^{t-1} \vee (B \cup_A C^1 A) \wedge S^{t-1} \xrightarrow{f \vee g} X. \tag{7.11}$$

If $t \geq 2$, $\pi_{t-1}\mathbf{Hom}_*(B/A, X)$ is a group, the action is given by $\alpha \cdot \beta = (j_*\alpha)\beta$ where j_* is the connecting homomorphism of the long exact sequence defined by $j_*\alpha = \alpha \cdot e$.

If $A \to B$ is not a pointed cofibration, we still have a fibration sequence

$$\mathbf{Hom}_*(B/A, X) \to \mathbf{Hom}(B, X) \to \mathbf{Hom}(A, X).$$

We can calculate the action $\pi_t\mathbf{Hom}(A, X)$ on $\pi_{t-1}\mathbf{Hom}_*(B/A, X)$ via the device of the disjoint basepoint. Define $A_+ = A \sqcup *$. Then $\mathbf{Hom}(A, X) \cong \mathbf{Hom}_*(A_+, X)$, so if $\alpha \in \pi_t\mathbf{Hom}(A, X)$ is represented by $f : A_+ \wedge S^t \to X$ and $\beta \in \pi_{t-1}\mathbf{Hom}(B/A, X)$ is represented by $g : (B \cup_A C^1A) \wedge S^{t-1} \to X$, then $\alpha \cdot \beta$ is represented by

$$(7.12)\quad (B \cup_A C^2A) \wedge S^{t-1} \to (A_+ \wedge S^1) \wedge S^{t-1} \vee (B \cup_A C^1A) \wedge S^{t-1} \xrightarrow{f \vee g} X.$$

The disjoint basepoint is crucial in the following.

EXAMPLE 7.13. *Let $X \in \mathbf{S}$ be pointed and fibrant and consider the cofibration $\partial\Delta^1 \to \Delta^1$. Then, the associated fiber sequence is*

$$\Omega X \to \mathbf{Hom}(\Delta^1, X) \to X \times X$$

and the action of $\pi_t(X \times X) \cong \pi_t X \times \pi_t X$ on $\pi_{t-1}\Omega X \cong \pi_t X$ is given by

$$(\alpha_1, \alpha_2) \cdot \beta = \alpha_1 \beta \alpha_2^{-1}. \qquad \blacksquare$$

We are now in a position to calculate with $B_2^{s,t}$. Recall from (6.1)–(6.4), that $B_2^{s,t} = \operatorname{Ker}\{\pi_{t-s+1} \operatorname{Tot}_{s-1} X \to \pi_{t-s+1} \operatorname{Tot}_{s-2} X\}$, and, in order to calculate $E_2^{s,t}$ we need to know the action of $B_2^{s,t}$ on

$$Z_2^{s,t} \subseteq \pi_{t-s}\Omega^{s-1}N^{s-1}X.$$

We separate out the cases $s = 1$ and $s > 1$.

LEMMA 7.14. *If $s = 1$, $B_2^{s,t} = \pi_t \operatorname{Tot}_0 X \cong \pi_t X$ and if $\alpha \in \pi_t X$ and $\beta \in \pi_{t-1}\Omega^1 N^1 X \cong \pi_t N^1 X$, then*

$$\alpha \cdot \beta = (d^0\alpha)\beta(d^1\alpha)^{-1}.$$

PROOF: One is concerned with the fibration sequence in $c\mathbf{S}$

$$\mathbf{Hom}_*(\operatorname{sk}_1 \Delta / \operatorname{sk}_0 \Delta, X) \to \mathbf{Hom}(\operatorname{sk}_1 \Delta, X) \to \mathbf{Hom}(\operatorname{sk}_0 \Delta, X).$$

Thus if $\alpha \in \pi_t \mathbf{Hom}(\operatorname{sk}_0 \Delta, X) \cong \pi_t X$ and $\beta \in \pi_{t-1}\mathbf{Hom}_*(\operatorname{sk}_1 \Delta / \operatorname{sk}_0 \Delta, X) \cong \pi_{t-1}\Omega^1 N^1 X$, then $\alpha \cdot \beta$ is represented by (see (7.12))

$$\begin{array}{c} (\operatorname{sk}_1 \Delta \cup_{\operatorname{sk}_0 \Delta} C^2 \operatorname{sk}_0 \Delta) \wedge S^{t-1} \\ \downarrow \\ (\operatorname{sk}_0 \Delta)_+ \wedge S^1 \wedge S^{t-1} \vee (\operatorname{sk}_1 \Delta \cup_{\operatorname{sk}_0 \Delta} C^1 \operatorname{sk}_0 \Delta) \wedge S^{t-1} \xrightarrow[\beta \vee \alpha]{} X. \end{array}$$

Specializing to cosimplicial degree 1, we have

$$\Delta^1 \cup_{\partial\Delta^1} C^2 \partial\Delta^1 \wedge S^{t-1} \to (\partial\Delta^1)_+ \wedge S^1 \wedge S^{t-1} \vee (\Delta^1 \cup_{\partial\Delta^1} C^2 \partial\Delta^1) \wedge S^{t-1} \xrightarrow{\beta^1 \vee \alpha^1} X^1.$$

However $\alpha^1 = d^0\alpha \vee d^1\alpha : (\partial\Delta^1)_+ \wedge S^t \to X^1$ and β^1 is a representative for $\beta \in \pi_{t-1}\Omega N^1 X$. Hence Example 7.13 implies the result. ∎

Lemma 7.15. *If $s > 1$, then*

$$i_* \pi_{t-s+1} \Omega^{s-1} N^{s-1} X = B_2^{s,t} \subseteq \pi_{t-s+1} \operatorname{Tot}_{s-1} X$$

and $j_* B_2^{s,t} \cong \operatorname{Im}(\sum (-1)^i d^i) \subseteq \pi_{t-s}\Omega^s N^s X \cong \pi_t N^s X$.

Proof: The equality $\iota_* \pi_* \Omega^{s-1} N^{s-1} X = B_2^{s,*}$ follows from the long exact of the fibration

$$\Omega^{s-1} N^{s-1} X \to \operatorname{Tot}_{s-1} X \to \operatorname{Tot}_{s-2} X.$$

For the calculation of $j_* : \pi_* \operatorname{Tot}_{s-1} X \to \pi_{*-1}\Omega^s N^s X$ we are concerned with the fibration sequence

$$\mathbf{Hom}_*(\operatorname{sk}_s \Delta / \operatorname{sk}_{s-1} \Delta, X) \to \mathbf{Hom}(\operatorname{sk}_s \Delta, X) \to \mathbf{Hom}(\operatorname{sk}_{s-1} \Delta, X).$$

If $\alpha \in \pi_{t-s+1}\mathbf{Hom}(\operatorname{sk}_{s-1} \Delta, X)$, then $j_*\alpha$ is the composite (where f represents α)

$$(\operatorname{sk}_s \Delta \cup_{\operatorname{sk}_{s-1} \Delta} C^2 \operatorname{sk}_{s-1} \Delta) \wedge S^{t-s} \to (\operatorname{sk}_{s-1} \Delta_+) \wedge S^{t-s+1} \xrightarrow{f} X.$$

At cosimplicial degree s, this yields

$$\Delta^s \cup_{\partial\Delta^s} C^2 \partial\Delta^s \wedge S^{t-s} \to (\partial\Delta^s_+) \wedge S^{t-s+1} \xrightarrow{f^s} X^s. \tag{7.16}$$

If $\alpha \in B_2^{s,t}$ then f^s factors as

$$
\begin{tikzcd}
(\partial\Delta^s_+) \wedge S^{t-s+1} \arrow[rr] \arrow[d] & & X^s \\
(\partial\Delta^s_+/\operatorname{sk}_{s-2}\Delta^s_+) \wedge S^{t-s+1} \arrow[urr] & &
\end{tikzcd}
\tag{7.17}
$$

Now $\Delta^s \cup_{\partial\Delta^s} C^2\partial\Delta^s \simeq \Delta^s/\partial\Delta^s \simeq S^s$, $\partial\Delta^s \simeq S^{s-1}$ and if $i : \Delta^{s-1} \to \Delta^{s-1}$ is the identity then the collapse map

$$\partial\Delta^s \to \partial\Delta^s/\operatorname{sk}_{s-2}\Delta^s$$

represents $\Sigma(-1)^i d^i i$. Combining these facts with (7.16) and (7.17) yields the result. ■

EXAMPLE 7.18. Let X be a pointed space and let $X \to \overline{\mathbb{F}}_p^\bullet X$ be the Bousfield-Kan resolution of X. Then the total space $\operatorname{Tot}(\overline{\mathbb{F}}_p^\bullet X) = X_p^\wedge$ is called the Bousfield-Kan p-completion of X, and $\pi_* X_p^\wedge$ may be calculated by the spectral sequence

$$\pi^s\pi_t\overline{\mathbb{F}}^\bullet X \Rightarrow \pi_{t-s}X_p^\wedge.$$

The point of this example is to rewrite the E_2 term in terms of homological algebra. We assume the reader is familiar with the category $\mathcal{CA}$ of unstable coalgebras over the Steenrod algebra. See [71, Sec. 1]. If X is a space, $H_*X \subset \mathcal{CA}$. Let $\mathcal{CA}_*$ be the category of augmented (or pointed) coalgebras in $\mathcal{CA}$; thus an object in $\mathcal{CA}_*$ is an unstable coalgebra C over the Steenrod algebra equipped with a morphism of such coalgebras $\mathbb{F}_p \to C$. The homology of a pointed space is in $\mathcal{CA}_*$.

First notice that if V is any simplicial vector space, then, choosing $0 \in V$ as the basepoint,

$$\pi_t V \cong \hom_{\mathcal{CA}_*}(H_*S^t, H_*V). \tag{7.19}$$

One way to see this is to note that there is a (non-canonical) weak equivalence $V \simeq \bigoplus_\alpha K(\mathbb{Z}/p, n_\alpha)$ for some set of non-negative integers $\{n_\alpha\}$, and to use the calculations of Serre and Cartan on $H_*K(\mathbb{Z}/p, n_\alpha)$. Now the fact that V is a simplicial abelian group implies (7.19) holds for any choice of basepoint. Since a choice of basepoint for X induces a basepoint for $\overline{\mathbb{F}}_p^\bullet X$,

$$\pi^s\pi_t\overline{\mathbb{F}}_p^\bullet X \cong \pi^s \hom_{\mathcal{CA}_*}(H_*S^t, H_*\overline{\mathbb{F}}_p^\bullet X). \tag{7.20}$$

Now, the forgetful functor $\mathcal{CA} \to n\mathbb{F}_p$ to the category of graded vector spaces has a right adjoint G. The functor G is characterized by the fact that it commutes with products and filtered colimits, and by the fact that if $W \in n\mathbb{F}_p$ is dimension 1 concentrated in degree n, then

$$G(W) \cong H_*K(\mathbb{Z}/p, n). \tag{7.21}$$

(Note that G is the functor of a cotriple on $n\mathbb{F}_p$ and $\mathcal{CA}$ is the category of coalgebras over G.)

Let $\overline{G} : \mathcal{CA} \to \mathcal{CA}$ be the composite of G and the forgetful functor. Then $\overline{G}$ is a triple on $\mathcal{CA}$, so if $C \in \mathcal{CA}$, we have an augmented cosimplicial object (see Example 4.1).

$$C \to \overline{G}^\bullet C. \tag{7.22}$$

This is a resolution in the sense that as an augmented cosimplicial graded vector space it has a retraction, so $\pi^*\overline{G}^\bullet C \cong C$ via the augmentation. Thus we define, for $C \in \mathcal{CA}_*$,

$$\mathrm{Ext}^s_{\mathcal{CA}_*}(H_*S^t, C) = \pi^s \hom_{\mathcal{CA}_*}(H_*S^t, \overline{G}^\bullet C). \tag{7.23}$$

Now if X is a space, $H_*\overline{\mathbb{F}}X \cong \overline{G}(H_*X)$. This follows from (7.19), (7.21) and the properties of $\overline{G}$. Thus combining (7.20) with (7.23) we have

$$\pi^s\pi_t\overline{\mathbb{F}}_p^\bullet X \cong \mathrm{Ext}_{\mathcal{CA}_*}(H_*S^t, H_*X)$$

and the homotopy spectral sequence of $\overline{\mathbb{F}}_p^\bullet X$ is the Bousfield-Kan spectral sequence

$$\mathrm{Ext}^s_{\mathcal{CA}_*}(H_*S^t, H_*X) \Rightarrow \pi_{t-s}X_p^\wedge.$$

It is an unstable Adams spectral sequence.

The reader might object that, for pointed coalgebras, one should define the Ext groups differently. If $C \in \mathcal{CA}_*$, define the "coaugmentation ideal" by

$$JC = \mathrm{coker}\{\mathbb{F}_p \to C\}.$$

Then $J : \mathcal{CA}_* \to n\mathbb{F}_p$ has a right adjoint G_* with $G_*V = GV$ augmented by $G(0) \cong \mathbb{F}_p \to G(V)$. Let $\overline{G}_* = G_* \circ J : \mathcal{CA}_* \to \mathcal{CA}_*$ and one might demand that the Ext object be

$$\pi^s \hom_{\mathcal{CA}_*}(H_*S^t, \overline{G}_*^\bullet C).$$

However, a bicomplex argument with

$$\hom_{\mathcal{CA}_*}(H_*S^t, \overline{G}^\bullet\overline{G}_*C)$$

shows that this definition agrees with the previous one (7.23); furthermore, (7.23) offers additional flexibility with basepoints.

We finish this section by constructing a spectral sequence for computing the homotopy groups of a pointed homotopy inverse limit.

We begin with a discussion of the derived functors of limit on I diagrams of abelian groups. Let $\mathbf{Ab}^I$ be the category of these diagrams. Then the functor $G_i : \mathbf{Ab}^I \to \mathbf{Ab}$ with $G_i A = Ai$ has both a left and a right adjoint; in particular, the right adjoint is given by

$$(T_i B)_j = \prod_{j \to i} B. \tag{7.24}$$

If $B \in \mathbf{Ab}$ is injective, then $T_i B$ is injective in $\mathbf{Ab}^I$ and this implies $\mathbf{Ab}^I$ has enough injectives; indeed, if we embed each i in an injective abelian group B_i, then $A \to \prod_i T_i B_i$ is an embedding of A into an injective in $\mathbf{Ab}^I$. We define the functors $\varprojlim^s = \varprojlim^s_I$ to be the right derived functors of $\varprojlim_I$. Since limit is left exact, $\varprojlim = \varprojlim^0$. The standard observation is that one need not go all the way to injectives to compute the derived functors. As with sheaf cohomology one can resolve by "flabby" objects rather than injective ones. The crucial result is:

LEMMA 7.25. *Let B be an abelian group. Then for all i we have $\varprojlim T_i B \cong B$ and $\varprojlim^s T_i B = 0$ if $s > 0$.*

PROOF: To prove $\varprojlim T_i B \cong B$, let $A \in \mathbf{Ab}^I$ be a constant diagram on an abelian group A. Then

$$\hom_{\mathbf{Ab}}(A, \varprojlim T_i B) \cong \hom_{\mathbf{Ab}^I}(A, T_i B) \cong \hom_{\mathbf{Ab}}(A, B).$$

To obtain the vanishing of the higher derived functors, let $B \to J^*$ be an injective resolution of B. Then (see (7.24)), $T_i B \to T_i J^*$ is an injective resolution of $T_i B$. Applying $\varprojlim$ to this resolution, one recovers $B \to J^*$, which is acyclic by construction. ∎

We use this to build a cosimplicial resolution of $A \in \mathbf{Ab}^I$ suitable for computing $\varprojlim^s$. For our category I let dI be the associated discrete category; i.e., dI has the same objects as I but no non-identity morphisms. There is a functor $dI \to I$, inducing a forgetful functor $\mathbf{Ab}^I \to \mathbf{Ab}^{dI}$. The category $\mathbf{Ab}^{dI}$ has objects indexed families of abelian groups $A = \{A_i\}_{i \in I}$. This forgetful functor has a right adjoint given by

$$TA = \prod_i T_i A_i.$$

We also write $T : \mathbf{Ab}^I \to \mathbf{Ab}^I$ be the composite functor. This is the same functor T as in Example 4.2. Then T is the functor of a triple on $\mathbf{Ab}^I$ and one, therefore, obtains a natural augmented cosimplicial object in $\mathbf{Ab}^I$

$$A \to T^{\cdot} A. \tag{7.26}$$

If one forgets to $\mathbf{Ab}^{dI}$ this cosimplicial object has a retraction and so is acyclic. This implies it is acyclic in $\mathbf{Ab}^I$ and Lemma 7.25 implies:

LEMMA 7.27. *For all $A \in \mathbf{Ab}^I$ there is a natural isomorphism*

$$\varprojlim{}^s A \cong \pi^s \varprojlim T^{\cdot} A.$$

By analogy we use this construction to define $\varprojlim^1 G$ where G is an I diagram of non-abelian groups. The reader will have noticed that there is nothing special about abelian group in the construction of the cosimplicial object $A \to T^{\cdot} A$ of (7.26). Indeed, if $\mathcal{C}$ is any category and $C \in \mathcal{C}^I$ is an I diagram, we obtain a cosimplicial object $C \to T^{\cdot} C$. In particular if G is an I-diagram of (not necessarily abelian) groups, one has a cosimplicial I-diagram of groups

$$G \to T^{\cdot} G$$

and we define

$$\varprojlim{}^1 G = \pi^1 \varprojlim T^{\cdot} G. \tag{7.28}$$

Thus $\varprojlim^1 G$ is a pointed set and is a group of cycles modulo a group of boundaries. See (7.5). At this point we have potentially two different definitions of $\varprojlim^1 G_n$ for a tower of groups, one from (7.28) and the other from Lemma 6.11; this difficulty will be resolved by Proposition 7.30 below.

Now let $X \in \mathbf{S}_*^I$ be an I-diagram of pointed spaces. We wish to calculate with $\operatorname{holim} X$. We use the cosimplicial space $T^{\cdot} X$ of Example 4.2. We will see in Section IX.5 that $\operatorname{Tot} T^{\cdot} X$ is naturally weakly equivalent $\underleftarrow{\operatorname{holim}}_I X$. Note that if $t \geq 0$,

$$\pi_t T^{\cdot} X \cong T^{\cdot} \pi_t X.$$

Hence the spectral sequence

$$\pi^s \pi_t T^{\cdot} X \Longrightarrow \pi_{t-s} \operatorname{Tot} T^{\cdot} X$$

is naturally isomorphic to a spectral sequence

$$\varprojlim_I{}^s \pi_t X \Longrightarrow \pi_{t-s} \operatorname{holim}_{\longleftarrow I} X. \tag{7.29}$$

This follows from Lemma 7.25 and a simple calculation in the case $t = s = 0$.

Finally, we show that our two potentially different definitions of $\varprojlim^1 G_n$ agree up to natural isomorphism.

For the next few paragraphs, let $\{G_n\}$ be a tower of groups. We define three functors. First, there is $\varprojlim^1 G_n$, as in Lemma 6.7. Then there is $\underline{\lim}^1 G_n$, which is a temporary name for $\varprojlim^1 G_n$ as defined in (7.28). Finally, define a functor by

$$\{G_n\} \mapsto \pi_0 \operatorname{holim}_{\longleftarrow} BG_n.$$

Proposition 7.30. *There are natural isomorphisms*

$$\pi_0 \operatorname{holim}_{\longleftarrow} BG_n \cong \underline{\lim}^1 G_n \cong \varprojlim{}^1 G_n.$$

Proof: We will show $\pi_0 \operatorname{holim}_{\longleftarrow} BG_n \cong \underline{\lim}^1 G_n$ and $\pi_0 \operatorname{holim}_{\longleftarrow} BG_n \cong \varprojlim^1 G_n$. The second of these isomorphisms was already proved in Proposition 6.14. For the first, we have that

$$\pi_0 \operatorname{holim}_{\longleftarrow} BG_n \cong \pi_0 \operatorname{Tot} \varprojlim T^{\cdot}\{BG_n\}.$$

By (5.7) and Example 5.9 we have an equality

$$\pi_0 \operatorname{Tot} \varprojlim T^{\cdot}\{BG_n\} = \pi_0 \operatorname{Tot}_1 \varprojlim T^{\cdot}\{BG_n\}.$$

Since $\pi_0 \operatorname{Tot}_0 \varprojlim T^{\cdot}\{BG_n\} = \pi_0 \prod_n BG_n = *$, we have

$$\pi_0 \operatorname{Tot} \varprojlim T^{\cdot}\{BG_n\} = E_2^{1,1} \cong \underline{\lim}^1 G_n$$

by (6.1)–(6.4). ■

8. Obstruction theory.

The purpose of this section is to develop a small amount of the theory of Bousfield's paper [10], and to discuss to some extent the meaning of the elements on the fringe of the spectral sequence of a cosimplicial space. We close the section with an extended example intended to make the theory concrete.

In order to make the exposition flow more smoothly we will make the following assumption: we have a fibrant cosimplicial space X with the property that for all $n \geq 0$ and every choice of basepoint $v \in X^n$, Whitehead products in $\pi_*(X^n, v)$ vanish. Thus $\pi_1(X^n, v)$ is abelian and the action of $\pi_1(X^n, v)$ on $\pi_m(X^n, v)$, $m \geq 1$, is trivial. We do not assume $\pi_0(X^n, v) = *$, as one of our main examples will be $X = \mathbf{Hom}(Y, R^\bullet Z)$, where $Y \in \mathbf{S}$, and $R^\bullet Z$ is the R-resolution of $Z \in \mathbf{S}$, as in Example 4.1.

We wish to address the following question: when is an element $x \in \pi^0\pi_0 X$ in the image of $\pi_0 \operatorname{Tot} X \to \pi_0 \operatorname{Tot}_0 X = \pi_0 X$. We will try to lift x in successive stages up the Tot tower. The following three results are our goal.

PROPOSITION 8.1. *An element $x \in \pi_0 X^0 = \pi_0 \operatorname{Tot}_0 X$ lifts to $\pi_0 \operatorname{Tot}_1 X$ if and only if $x \in \pi^0\pi_0 X$.*

An element $x \in \pi^0\pi_0 X$ determines elements $x_n \in \pi_0 X^n$ by

$$x_n = d_*^{i_n} \cdots d_*^{i_0} X$$

where $d_*^{i_t} : \pi_0 X^t \to \pi_0 X^{t+1}$ is any of the coface operators. We write $\pi_t(X, x)$ for the resulting cosimplicial group. Recall

$$N^s\pi_t(X, x) = \bigcap_{i=0}^{s-1} \ker\{s_*^i : \pi_t(X^s, x_s) \to \pi_t(X^{s-1}, x_{s-1})\}$$

and $Z^s\pi_t(X, s)$ is the group of cocycles in degree s in $N^*\pi_t(X, x)$.

PROPOSITION 8.2. *Let $n \geq 1$, $y \in \pi_0 \operatorname{Tot}_n X$, and $x \in \pi^0\pi_0 X$ the image of y under the projection $\pi_0 \operatorname{Tot}_n X \to \pi^0\pi_0 X \subseteq \pi_0 \operatorname{Tot}_0 X$. Then there is an element*

$$\gamma(y) \in Z^{n+1}\pi_n(X, x)$$

so that y lifts to $\pi_0 \operatorname{Tot}_{n+1} X$ if and only if $\gamma_n(x) = 0$.

Next recall $\pi^s\pi_t(X, x) = H^s N^*\pi_t(X, x)$.

PROPOSITION 8.3. *Let $n \geq 1$ and suppose $z \in \pi_0 \operatorname{Tot}_{n-1} X$ lifts to $\pi_0 \operatorname{Tot}_n X$. Let $x \in \pi^0\pi_0 X$ be the image of z under the projection $\pi_0 \operatorname{Tot}_{n-1} X \to \pi^0\pi_0 X \subseteq \pi_0 \operatorname{Tot}_0 X$. Then there is an element*

$$\epsilon(z) \in \pi^{n+1}\pi_n(X, x)$$

so that z lifts to $\pi_0 \operatorname{Tot}_{n+1} X$ if and only if $\epsilon(z) = 0$.

We will prove these three results in order. Before starting we write down the obvious corollary. Recall our assumptions on X made in the second paragraph of this section.

PROPOSITION 8.4. *Let $x \in \pi^0\pi_0 X$, and suppose $\pi^{n+1}\pi_n(X,x) = 0$ for $n \geq 1$. Then x lifts to $\pi_0 \operatorname{Tot} X$.*

PROOF: This follows from Propositions 8.1 and 8.3 since

$$\pi_0 \operatorname{Tot}(X) \to \varprojlim \pi_0 \operatorname{Tot}_n X$$

is surjective. ∎

We begin the proofs of Proposition 8.1 to 8.3. To pass from $\pi_0 \operatorname{Tot}_n X$ to $\pi_0 \operatorname{Tot}_{n-1} X$ we will have to investigate the pullback square

$$\begin{tikzcd}
\operatorname{Tot}_{n+1} X \arrow[r] \arrow[d] & \mathbf{Hom}(\Delta^{n+1}, X^{n+1}) \arrow[d] \\
\operatorname{Tot}_n X \arrow[r] & \mathbf{Hom}(\Delta^{n+1}, M^n X) \times_{\mathbf{Hom}(\partial\Delta^{n+1}, M^n X)} \mathbf{Hom}(\partial\Delta^{n+1}, X^{n+1}).
\end{tikzcd}$$

Let us make an abbreviation

$$\mathbf{Hom}(i_{n+1}, s_{n+1})$$

for the lower right-hand corner of this square. The following technical lemma is crucial to calculations. Let

$$I\pi_0 \operatorname{Tot}_n X = Im\{\pi_0 \operatorname{Tot}_n X \to \pi_0\mathbf{Hom}(i_{n+1}, s_{n+1})\}.$$

LEMMA 8.5. *The composition of the map $I\pi_0 \operatorname{Tot}_n X \to \pi_0\mathbf{Hom}(i_{n+1}, s_{n+1})$ with*

$$\begin{tikzcd}
\pi_0\mathbf{Hom}(i_{n+1}, s_{n+1}) \arrow[d] \\
\pi_0\mathbf{Hom}(\Delta^{n+1}, M^n X) \times_{\pi_0\mathbf{Hom}(\partial\Delta^{n+1}, M^n X)} \pi_0\mathbf{Hom}(\partial\Delta^{n+1}, X^{n+1})
\end{tikzcd}$$

is injective.

Proof: Take $f \in \pi_0\mathbf{Hom}(i_{n+1}, s_{n+1})$, and let $f_1 \in \mathbf{Hom}(\partial\Delta^{n+1}, X^{n+1})$, $g \in \mathbf{Hom}(\Delta^{n+1}, M^nX)$, and $f_0 \in \mathbf{Hom}(\partial\Delta^{n+1}, M^nX)$ be the induced vertices. If we can show

$$\pi_1(\mathbf{Hom}(\partial\Delta^{n+1}, X^{n+1}), f_1) \to \pi_1(\mathbf{Hom}(\partial\Delta^{n+1}, M^nX); f_0)$$

is onto for all $f \in I\pi_0 \operatorname{Tot}_n X$, the result will follows from the Barratt-Puppe sequence.

Let $v_0 = d_{n+1} \cdots d_1 c_n \in (\Delta^{n+1})_0$ be the basepoint, then let $x_1 = f_1(v_0)$ and $x_0 = f_0(v_0)$ be the basepoints of X^{n+1} and M^nX

Evaluating at the basepoint gives a diagram where the vertical maps are fibrations and the horizontal maps give a fiber sequence

$$\begin{tikzcd}
\mathbf{Hom}_*(\partial\Delta^{n+1}, X^{n+1}) \arrow[r] \arrow[d] & \mathbf{Hom}(\partial\Delta^{n+1}, X^{n+1}) \arrow[r] \arrow[d] & X^{n+1} \arrow[d] \\
\mathbf{Hom}_*(\partial\Delta^{n+1}, M^nX) \arrow[r] & \mathbf{Hom}(\partial\Delta^{n+1}, M^nX) \arrow[r, "p"'] & M^nX.
\end{tikzcd}$$

The $*$ means pointed maps. This leads to short exact sequences in homotopy, $m \geq 1$,

(8.6)

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\pi_m(\mathbf{Hom}_*(\partial\Delta^{n+1}, X^{n+1}); f_1) \arrow[r, "\mathbf{Hom}(\partial\Delta^{n+1}, s)_*"] \arrow[d] & \pi_m(\mathbf{Hom}_*(\partial\Delta^{n+1}, M^nX); f_0) \arrow[d] \\
\pi_m(\mathbf{Hom}(\partial\Delta^{n+1}, X^{n+1}); f_1) \arrow[r] \arrow[d] & \pi_m(\mathbf{Hom}(\partial\Delta^{n+1}, M^nX); f_0) \arrow[d] \\
\pi_m(X^{n+1} \arrow[r] \arrow[d] & \pi_m(M^nX, x_0) \arrow[d] \\
0 & 0
\end{tikzcd}$$

The right sequence is exact because of the existence of $g : \Delta^{n+1} \to M^nX$. Thus f_0 is contractible and the constant map section $M^nX \to \mathbf{Hom}(\partial\Delta^{n+1}, M^nX)$ of p actually shows the right is split exact. The left sequence is short exact because Whitehead products vanish in X^{n+1} : $\alpha : \Delta^m/\partial\Delta^m \to X^{n+1}$ lifts, up to homotopy, to $\mathbf{Hom}(\partial\Delta^{n+1}, X^{n+1})$ with basepoint f_1 if and only if the following factors

$$\begin{tikzcd} \Delta^m/\partial\Delta^m \vee \partial\Delta^{n+1} \arrow[r, "\alpha\vee f_1"] \arrow[d] & X^{n+1}. \\ \Delta^m/\partial\Delta^m \times \partial\Delta^{n+1} \arrow[ur, dashed] & \end{tikzcd}$$

To finish the proof we use that f (which determines f_1, f_0, x_1, and x_0) arises from an element of $y \in \pi_0 \operatorname{Tot}_n X$. By projection, y determines an element $x \in \pi^0\pi_0 X$ and hence a unique element $z_k \in \pi_0 X^k$, via any composition of coface maps, and, as above, we write $\pi_m(X, x)$ for the resulting cosimplicial abelian group. Now z_{n+1} is the class of x_1 in $\pi_0 X^{n+1}$ and the argument of Lemma 7.1 implies

$$\pi_m(M^n X, x_0) \cong M^n \pi_m(X, z).$$

It follows that, in diagram (8.6), that s_* is surjective. Also, since f_0 is homotopic to the constant map, if we let $\overline{f}_1 : \partial\Delta^{n+1} \to X^{n+1}$ be a lift of f_0 which is homotopic to the constant map, then

$$\begin{aligned} \mathbf{Hom}(\partial\Delta^{n+1}, s)_* : \pi_m \mathbf{Hom}_*(\partial\Delta^{n+1}, X^{n+1}); \overline{f}_1) &\cong \pi_{m+n}(X; x_1) \\ \to M^n \pi_{m+n}(X, x) &\cong \pi_m \mathbf{Hom}_*(\partial\Delta^{n+1}, M_n X) \end{aligned}$$

is onto. Now $\overline{f}_1$ need not be homotopic to the constant map. However, if we let $N^{n+1}X$ be the fiber of $s : X^{n+1} \to M^n X$ at X_0, then $\mathbf{Hom}_*(\partial\Delta^{n+1}, N^{n+1}X)$ acts on $\mathbf{Hom}_*(\partial\Delta^{n+1}, X^{n+1})$ over $\mathbf{Hom}_*(\partial\Delta^{n+1}, M^n X)$. Since $sf_1 = f_0$, there is a vertex $v \in \mathbf{Hom}_*(\partial\Delta^{n+1}, N^{n+1})$ so that operation by v which is a homotopy equivalence) induces a diagram

$$\begin{tikzcd} \mathbf{Hom}_*(\partial\Delta^{n+1}, X^{n+1}) \arrow[rr, "v"] \arrow[dr] & & \mathbf{Hom}_*(\partial\Delta^{n+1}, X^{n+1}) \arrow[dl] \\ & \mathbf{Hom}_*(\partial\Delta^{n+1}, M^n X) & \end{tikzcd}$$

and so that $\overline{f}_1 = vf_1$. Thus $\mathbf{Hom}(\partial\Delta^{n+1}, s)_*$, as in (8.6), is surjective, and the result follows. ∎

PROOF OF PROPOSITION 8.1: There is, by (5.6), a pullback diagram (since $\mathrm{Tot}_0 X = X^0$)

$$\begin{tikzcd}
\mathrm{Tot}_1 X \arrow[r] \arrow[d] & \mathbf{Hom}(\Delta^1, X^1) \arrow[d] \\
X^0 \arrow[r] & \mathbf{Hom}(i_1, s_1).
\end{tikzcd}$$

Hence

$$\pi_0 \mathrm{Tot}_1 X \to \pi_0 X^0 \times_{\pi_0 \mathbf{Hom}(i_1, s_1)} \pi_0 \mathbf{Hom}(\Delta^1, X^1)$$

is surjective. By Lemma 8.5, the target of this map on π_0 is isomorphic to the pullback of

$$\begin{tikzcd}
 & \pi_0 X^1 \arrow[d, "q"] \\
\pi_0 X^0 \arrow[r, "p"] & \pi_0 X^0 \times_{\pi_0 X^0 \times \pi_0 X^0} \pi_0 X^1 \times \pi_0 X^1
\end{tikzcd}$$

where p and q are respectively

$$\begin{aligned} p(x) &= (x, d^0 x, d^1 x) \\ q(y) &= (s^0 y, y, y). \end{aligned}$$

But this pullback is exactly $\pi^0 \pi_0 X$. ■

Before proving Proposition 8.2, we define the obstruction cocycle $\gamma(y) \in Z^{n+1}\pi_n(X, x)$. We will use heavily that all Whitehead products vanish in X^s, ≥ 0, so that we needn't worry about the difference between unbased and based maps in computing homotopy groups. Components do matter, however.

The element $y \in \pi_0 \mathrm{Tot}_n(X)$ is determined by a map

$$f : \mathrm{sk}_n \Delta \to X$$

and, hence, determines a unique homotopy class of maps

$$f^{n+1} : \partial\Delta^{n+1} = \mathrm{sk}_n \Delta^{n+1} \to X^{n+1}.$$

Then

$$\gamma(y) = [f^{n+1}] \in \pi_n(X^{n+1}, x). \tag{8.7}$$

Since there are commutative diagrams

$$\begin{tikzcd} \partial\Delta^{n+1} \arrow[r, "f^{n+1}"] \arrow[d, "s^i"'] & X^{n+1} \arrow[d, "s^i"] \\ \Delta^n \arrow[r, "f^n"] & X^n \end{tikzcd}$$

we have that $\gamma(y) \in N^{n+1}\pi_n(X,x)$. Since there are commutative diagrams

$$\begin{tikzcd} \partial\Delta^{n+1} \arrow[r, "f^{n+1}"] \arrow[d, "d^i"'] & X^{n+1} \arrow[d, "d^i"] \\ \operatorname{sk}_n \Delta^{n+2} \arrow[r, "f^{n+2}"] & X^{n+2} \end{tikzcd}$$

and since $\pi_n(X^{n+1},x)$ is abelian for $n \geq 1$, we will have $\gamma(y) \in Z^{n+1}\pi_n(X,x)$ if

$$0 = \sum_{i=0}^{n+1}(-1)^i d^i_* a_n \in H_n(\operatorname{sk}_{n+2}\Delta^{n+2}).$$

Here $a_n \in H_n(\partial\Delta^{n+1})$ is the generator. But this is a standard calculation.

In the next argument we will need the following technical lemma. Recall that X is fibrant in $c\mathbf{S}$ and all Whitehead products vanish.

Lemma 8.8. *Let $x \in \pi^0\pi_0 X$ and $b \in X^n$ any vertex with $[b] = x \in \pi_0 X^n$. Let $s : X^n \to M^{n-1}X$ be the canonical map and $v = s(b)$. Then for $k \geq 1$,*

$$\pi_k(M^{n-1}X, v) \cong M^{n-1}\pi_k(X,x).$$

Furthermore, if $N^n(X,x)$ is the fiber of s at v. Then for $k \geq 1$,

$$\pi_k(N^n(X,x), b) \cong N^n\pi_n(X,x).$$

Proof: The argument of Lemma 7.1 goes through verbatim, provided one is explicit about basepoints. Notice the argument extends to π_1 because of the assumption that $\pi_1(X,x)$ is abelian. ∎

PROOF OF PROPOSITION 8.2: If y lifts to $\pi_0 \operatorname{Tot}_{n+1} X$, then there is a factoring

$$\begin{tikzcd} \mathrm{sk}_n \Delta \arrow[r, "f"] \arrow[d] & X. \\ \mathrm{sk}_{n+1} \Delta \arrow[ur] & \end{tikzcd}$$

In particular, f_{n+1} factors as $\partial\Delta^{n+1} \to \Delta^{n+1} \to X^{n+1}$ and $\gamma(y) = 0$.

The converse is a little more difficult. We use the pullback square

$$\begin{tikzcd} \operatorname{Tot}_{n+1} X \arrow[r] \arrow[d] & \mathbf{Hom}(\Delta^{n+1}, X^{n+1}) \arrow[d] \\ \operatorname{Tot}_n X \arrow[r] & \mathbf{Hom}(i_{n+1}, s_{n+1}). \end{tikzcd}$$

Then

$$\pi_0 \operatorname{Tot}_{n+1} X \to \pi_0 \operatorname{Tot}_n X \times_{\pi_0 \mathbf{Hom}(i_{n+1}, s_{n+1})} \pi_0 \mathbf{Hom}(\Delta^{n+1}, X^{n+1})$$

is surjective. By Lemma 8.5, the target of this map is isomorphic to the pullback of

$$\begin{tikzcd} & \pi_0 \mathbf{Hom}(\Delta^{n+1}, X) \arrow[d, "p"] \\ \pi_0 \operatorname{Tot}_n X \arrow[r, "k"] & \pi_0 \mathbf{Hom}(\Delta^{n+1}, M^n X) \times_{\pi_0 \mathbf{Hom}(\partial\Delta^{n+1}, M^n X)} \pi_0 \mathbf{Hom}(\partial\Delta^{n+1}, X^{n+1}) \end{tikzcd} \tag{8.9}$$

We need to show that $k(y)$ is in the image of p.

To calculate $k(y)$, first note that for any (fibrant) space Y

$$\pi_0 \mathbf{Hom}(\partial\Delta^{n+1}, Y) \cong \bigsqcup_{x \in \pi_0 Y} \pi_n^{fr}(Y, x),$$

where $\pi_n^{fr}(Y, x)$ denotes (unbased) homotopy classes of maps $\partial\Delta^{n+1} \to Y$ landing in the component x. If $y \in Y$ is a vertex with $[y] = x$, then there is a quotient

$$\pi_n(Y, y) \to \pi_n^{fr}(Y, x)$$

which is an isomorphism if Whitehead products vanish in $\pi_*(Y, y)$. This later will hold if $Y = X^{n+1}$. If $x \in \pi_0 M_n X$ is a component arising from an element of $\pi^0\pi_0 X$ it also holds as (with $[y] = x$)

$$\pi_n(M_n X, y) = M^n \pi_n(X, x), n \geq 1.$$

For each $x \in \pi_0 X^{n+1}$, let

$$K_x = \{\alpha \in \pi_n(X, y) \mid (s_{n+1})_* \alpha = 0 \in \pi_n^{fr}(M_n X, f(y))\}.$$

Then, since $\pi_0 \mathbf{Hom}(\Delta^{n+1}, M^n X) \cong \pi_0 M^n X$

$$\begin{aligned}\pi_0 \mathbf{Hom}(\Delta^{n+1}, M^n X) \times_{\pi_0 \mathbf{Hom}(\partial\Delta^{n+1} M^n X)} \pi_0 \mathbf{Hom}(\partial\Delta^{n+1}, X^{n+1}) \\ \cong \coprod_{x \in \pi_0 X^{n+1}} K_x.\end{aligned}$$

Now let us calculate $k(y)$, where k is as in (8.9). Since y is a map

$$f : \mathrm{sk}_n \Delta \to X^{n+1}$$

we have $k(y)$ is the class of $f^{n+1} : \mathrm{sk}_n \Delta^{n+1} \to X^{n+1}$; that is

$$k(y) = \gamma(y) \in N^{n+1}\pi_n(X, x) = K_x \subseteq \coprod_{x \in \pi_0 X^{n+1}} K_x.$$

The image of

$$p : \pi_0 X^{n+1} \to \coprod_{x \in \pi_0 X^{n+1}} K_x$$

since $x \in \pi_0 X^{n+1}$ to $0 \in K_x$. Hence $k(y)$ is in the image of p if (and only if) $\gamma(y) = 0$. ■

To prove Proposition 8.3, we first define the obstruction class

$$\epsilon(z) \in \pi^{n+1}\pi_n(X, x).$$

In fact, let $y \in \pi_0 \operatorname{Tot}_n X$ be any lift of z and let

$$\epsilon(z) = [\gamma(y)] \in \pi^{n+1}\pi_n(X, x) \tag{8.10}$$

where $\gamma(y) \in Z^{n+1}\pi_n(X, x)$ is the obstruction cocycle of Proposition 8.2, defined in (8.7).

LEMMA 8.11. *The class $\epsilon(z)$ is independent of the choice of lift $y \in \pi_0 \operatorname{Tot}_n X$.*

PROOF: Let y_0 and y_1 be two lifts of z. Then we choose representative

$$f_0, f_1 : \operatorname{sk}_n \Delta \to X$$

of y_0, y_1 respectively. Since $f_0\big|_{\operatorname{sk}_{n-1}\Delta}$ and $f_1\big|_{\operatorname{sk}_{n-1}\Delta}$ both represent z, they are homotopic; since $\operatorname{sk}_{n-1}\Delta \to \operatorname{sk}_n \Delta$ is a cofibration we may, by altering f_1 by a homotopy, assume

$$f_0\big|_{\operatorname{sk}_{n-1}\Delta} = f_1\big|_{\operatorname{sk}_{n-1}\Delta}.$$

Thus we obtain a map

$$f_0 \sqcup f_1 : \operatorname{sk}_n \Delta \cup_{\operatorname{sk}_{n-1}\Delta} \operatorname{sk}_n \Delta \to X.$$

In cosimplicial degree n

$$[\operatorname{sk}_n \Delta \cup_{\operatorname{sk}_{n-1}\Delta} \operatorname{sk}_n \Delta]^n \cong \Delta^n \cup_{\partial\Delta^n} \Delta^n$$

which is a simplicial model for S^n. Let a_1, a_0 be the two obvious n-simplices of $\Delta^n \cup_{\partial\Delta^n} \Delta^n$ and let the class $[a_1 - a_0] \in H_n(\Delta^n \cup_{\partial\Delta^n} \Delta^n)$ define the orientation. Let $d(f_0, f_1) \in \pi_n(X^n, x)$ be the resulting class.

In degree $n-1$,

$$[\operatorname{sk}_n \Delta \cup_{\operatorname{sk}_{n-1}\Delta} \operatorname{sk}_n \Delta]^{n-1} \cong \Delta^{n-1},$$

so

$$d(f_0, f_1) \in N^n \pi_n(X, x).$$

In degree $n+1$

$$[\operatorname{sk}_n \Delta \cup_{\operatorname{sk}_{n-1}\Delta} \operatorname{sk}_n \Delta]^{n+1} \cong \partial\Delta^{n+1} \cup_{\operatorname{sk}_{n-1}\Delta^{n+1}} \partial\Delta^{n+1}.$$

Let $i_t : \partial\Delta^{n+1} \to [\operatorname{sk}_n \Delta \cup_{\operatorname{sk}_{n-1}\Delta} \operatorname{sk}_n \Delta]^{n+1}$ be the two obvious inclusions and $[i_1] - [i_0] \in H^n(\operatorname{sk}_n \Delta \cup_{\operatorname{sk}_{n-1}\Delta} \operatorname{sk}_n \Delta)$ the resulting homology class. Then

$$\partial[q_1 - a_0] = [i_1] - [i_0].$$

Since $(f_1 \sqcup f_0) \circ i_t = \gamma(y_t)$, it follows from naturality that

$$\partial d(f_0, f_1) = \gamma(y_1) - \gamma(y_0). \qquad \blacksquare$$

To complete the proof of Proposition 8.3, we must argue that if $\epsilon(z) = 0$ there exists a lift y of z to $\pi_0 \operatorname{Tot}_n X$ so that $\gamma(y) = 0$. This follows from the next result.

LEMMA 8.12. *Let $z \in \pi_0 \operatorname{Tot}_{n-1} X$ and $y_0 \in \pi_0 \operatorname{Tot}_n X$ a left of z. Let $\omega \in N^n\pi_n(X,x)$, where $x \in \pi^0\pi_0 X$ is the image of z. Then there exists $y_1 \in \pi_1 \operatorname{Tot}_n X$ a left of z so that*

$$\gamma(y_1) = \gamma(y_0) + \partial\omega.$$

REMARK 8.13. Note that Proposition 8.3 immediately follows, for if $\epsilon(z) = 0$, $\gamma(y) = \partial\omega$ for some ω, and we may choose y_1 so that

$$\gamma(y_1) = \gamma(y) + \partial(-\omega)$$

and apply Proposition 8.2. ■

To prove 8.12 we use a sequence of lemmas. Let

$$i_n^n : \partial\Delta \left[\begin{smallmatrix} n \\ n \end{smallmatrix}\right] \to \Delta \left[\begin{smallmatrix} n \\ n \end{smallmatrix}\right]$$

be the cofibration of (4.13). Then $\Delta \left[\begin{smallmatrix} n \\ n \end{smallmatrix}\right] = p_n^*\Delta^n$, so

$$\hom_{c\mathbf{S}}(\Delta \left[\begin{smallmatrix} n \\ n \end{smallmatrix}\right], X) \cong \hom_{\mathbf{S}}(\Delta^n, X^n).$$

Let $\phi_n : \Delta \left[\begin{smallmatrix} n \\ n \end{smallmatrix}\right] \to \operatorname{sk}_n \Delta$ be adjoint to the identity. Also, since

$$\partial\Delta \left[\begin{smallmatrix} n \\ n \end{smallmatrix}\right] = \rho_n^*\partial\Delta^n \cup_{\mu_{n-1}^*\partial\Delta^n} \mu_{n-1}^*\Delta^n$$

we have that $\hom_{c\mathbf{S}}(\partial\Delta \left[\begin{smallmatrix} n \\ n \end{smallmatrix}\right], X)$ is isomorphic to the pullback

$$\hom_{\mathbf{S}}(\Delta^n, M^{n-1}X) \times_{\hom_{\mathbf{S}}(\partial\Delta^n, M^{n-1}, X)} \hom_{\mathbf{S}}(\partial\Delta^n, X^n).$$

Let $\psi_n : \partial\Delta \left[\begin{smallmatrix} n \\ n \end{smallmatrix}\right] \to \operatorname{sk}_{n-1} \Delta$ be adjoint to the element $(s_n, 1_{\partial\Delta^n})$ of

$$\hom_{\mathbf{S}}(\Delta^n, M^{n-1} \operatorname{sk}_{n-1} \Delta) \times_{\hom_{\mathbf{S}}(\partial\Delta^n, M^{n-1} \operatorname{sk}_{n-1} \Delta)} \hom_{\mathbf{S}}(\partial\Delta^n, \partial\Delta^n).$$

Here we are using that

$$M^{n-1} \operatorname{sk}_{n-1} \Delta \cong M^{n-1} \operatorname{sk}_n \Delta$$

and $s_n : \Delta^n \to M^{n-1} \operatorname{sk}_n \Delta$ is the canonical map. The following diagram commutes

$$\begin{tikzcd} \partial\Delta \left[\begin{smallmatrix} n \\ n \end{smallmatrix}\right] \arrow[r, "\psi_n"] \arrow[d, "i_n^n"'] & \operatorname{sk}_{n-1} \Delta \arrow[d] \\ \Delta \left[\begin{smallmatrix} n \\ n \end{smallmatrix}\right] \arrow[r, "\phi_n"'] & \operatorname{sk}_n \Delta. \end{tikzcd} \tag{8.14}$$

LEMMA 8.15. *The diagram (8.14) is a pushout diagram.*

PROOF: It suffices to show that after applying $\hom_{s\mathbf{C}}(\cdot, X)$, $X \in s\mathbf{C}$, one obtains a natural pullback diagram. This follows from (5.6). ■

Now let $i_j : \mathrm{sk}_n \Delta \to \mathrm{sk}_n \Delta \cup_{\mathrm{sk}_{n-1}\Delta} \mathrm{sk}_n \Delta$ be the inclusion of the left functor. Let $j_n : \Delta \left[\begin{smallmatrix} n \\ n \end{smallmatrix}\right] \to \mathrm{sk}_n \Delta \cup_{\mathrm{sk}_{n-1}\Delta} \mathrm{sk}_n \Delta$ be adjoint to the inclusion $\Delta^n \to \Delta^n \cup_{\partial\Delta^n} \Delta^n$ of the right factor.

LEMMA 8.16. *There is a pushout square*

$$\begin{tikzcd} \partial\Delta \left[\begin{smallmatrix} n \\ n \end{smallmatrix}\right] \arrow[r, "\phi_n \circ i_n^n"] \arrow[d, "i_n^n"'] & \mathrm{sk}_n \Delta \arrow[d, "i_0"] \\ \Delta \left[\begin{smallmatrix} n \\ n \end{smallmatrix}\right] \arrow[r, "j_n"] & \mathrm{sk}_n \Delta \cup_{\mathrm{sk}_{n-1}\Delta} \mathrm{sk}_n \Delta. \end{tikzcd}$$

PROOF: This follows from Lemma 8.15 and the formula

$$(B_2 \cup_A B_1) \cup_{B_1} (B_1 \cup_A B_2) = B_2 \cup_A (B_1 \cup_A B_2).$$ ■

This brings us to our final technical argument.

PROOF OF LEMMA 8.12: Let $f_0 : \mathrm{sk}_n \Delta \to X$ represent y_0 and let

$$\omega \in N^n \pi_n(X, x)_\bullet \subseteq \pi_n(X^n, x).$$

Then

$$f_0 : (\mathrm{sk}_n \Delta)^n = \Delta^n \to X^n$$

can be extended to a map

$$g : (\mathrm{sk}_n \Delta \cup_{sk_{n-1}\Delta} \mathrm{sk}_n \Delta)^n = \Delta^n \cup_{\partial\Delta^n} \Delta^n \to X^n$$

representing ω simply because

$$\pi_0 f_0 : \pi_0 \Delta^n = \{pt\} \to \pi_0 X^n$$

hits x. Let $i_1 : \Delta^n \to \Delta^n \cup_{\partial\Delta^n} \Delta^n$ be the inclusion of the right hand factor and let $h : \Delta \left[\begin{smallmatrix} n \\ n \end{smallmatrix}\right] \to X$ be adjoint to $g \circ i_1 : \Delta^n \to X^n$. The claim is that we may choose h so that the following diagram commutes

$$\begin{tikzcd} \partial\Delta \left[\begin{smallmatrix} n \\ n \end{smallmatrix}\right] \arrow[r, "\phi_n \circ i_n^n"] \arrow[d, "i_n^n"'] & \mathrm{sk}_n \Delta \arrow[d, "f_0"] \\ \Delta \left[\begin{smallmatrix} n \\ n \end{smallmatrix}\right] \arrow[r, "h"'] & X. \end{tikzcd} \tag{8.17}$$

Assuming this for the moment, we would then have, by Lemma 8.16, a map

$$g^* : \mathrm{sk}_n \Delta \cup_{\mathrm{sk}_{n-1}\Delta} \mathrm{sk}_n \Delta \to X$$

extending g. Then if f_1 is the composite

$$\mathrm{sk}_n \Delta \xrightarrow{i_1} \mathrm{sk}_n \Delta \cup_{\mathrm{sk}_{n-1}\Delta} \mathrm{sk}_n \Delta \xrightarrow{g^*} X,$$

the argument for Lemma 8.11 and the fact that g represents ω finishes the argument; in fact, if $y_1 = f_j \in \pi_0 \operatorname{Tot}_n X$, then

$$d(f_0, f_1) = \omega$$

and $\partial d(f_0, f_1) = \gamma(y_1) - \gamma(y_0)$.

To prove the claim about diagram (8.17) we compute in

$$\hom(\Delta^n, M^{n-1}X) \times_{\hom(\partial\Delta^n, M^{n-1}X)} \hom(\partial\Delta^n, X^n).$$

In this pullback, $f_0 \circ i_n^n \circ \phi_n$ is the pair (α, f_0^n) where α is the composite

$$\Delta^n \xrightarrow{s} M^{n-1}\,\mathrm{sk}_n \Delta \xrightarrow{M^{n-1} f_0} M^{n-1}X$$

and f_0^n is short-hand for the composite

$$\partial\Delta^n \subseteq (\mathrm{sk}_n \Delta)^n \xrightarrow{f_0^n} X^n.$$

On the other hand $h \circ i_n^n$ is the pair $(\beta, g^n \circ i_1)$ where β is the composite

$$\Delta^n \xrightarrow{i_1} \Delta^n \cup_{\partial\Delta^n} \Delta^n \xrightarrow{g} X^n \xrightarrow{s} M^{n-1}X$$

and $g \circ i_1$ denotes the composite

$$\partial\Delta^n \xrightarrow{\subseteq} \Delta^n \xrightarrow{i_1} \Delta^n \cup_{\partial\Delta^n} \Delta^n \xrightarrow{g} X^n$$

since $f_0 = g \circ i_1$ when restricted to the right hand factors; the left hand factors are the issue. Thus we need to choose g which provides a lifting in the following diagram

$$\begin{tikzcd}
\Delta^n \arrow[rr, "f_0"] \arrow[d, "i_0"] & & X^n \arrow[d, "g"] \\
\Delta^n \cup_{\mathrm{sk}_{n-1}\Delta^n} \Delta^n \arrow[r, "s \cup S"] \arrow[urr, dashed] & M^{n-1}\,\mathrm{sk}_n \Delta \arrow[r, "M^{n-1} f_0"] & M^{n-1}X.
\end{tikzcd}$$

That is, there is a representative

$$\omega \in N\pi_n(X,x) \subseteq \pi_0\mathbf{Hom}(\Delta^n \cup_{\mathrm{sk}_{n-1}\Delta^n} \Delta^n, X^n)$$

that maps to $(f_0, M^{n-1}f_0 \circ (s \cup S))$ in

$$\pi_0(\mathbf{Hom}(\Delta^n, X) \times_{\mathbf{Hom}(\Delta^n, M^{n-1}X)} \mathbf{Hom}(\Delta^n \cup_{\mathrm{sk}_{n-1}\Delta^n} \Delta^n, M^{n-1}X)).$$

Of course $[f_0^n] = x \in \pi_0\mathbf{Hom}(\Delta^n, X^n) = \pi_0 X^n$.

Thus we need to find a representative g of

$$\omega \in N\pi_n(X,x) \subseteq \pi_0\mathbf{Hom}(\Delta^n \cup_{\partial\Delta^n} \Delta^n, X^n)$$

that maps to the class of $(f_0, M^{n-1}f_0 \circ (s \cup S))$ in

$$\begin{aligned}\pi_0(\mathbf{Hom}(\Delta^n, X^n) \times_{\mathbf{Hom}(\Delta^n, M^{n-1}X)} &\mathbf{Hom}(\Delta^n \cup_{\partial\Delta^n} \Delta^n, M^{n-1}X)) \\ &\cong \coprod_{[b]\in\pi_0 X^n} \pi_n(M^{n-1}X, s(b)).\end{aligned}$$

Now $[f_0] = x \in \pi_0\mathbf{Hom}(\Delta^n, X^n) = \pi_0 X^n$ and

$$[M^{n-1}f_0 \circ (s \cup S)]] = 0 \in \pi_n(M^{n-1}X, x) \subseteq \coprod_{[b]\in\pi_0 X^n} \pi_n(M^{n-1}X, s(b)).$$

By Lemma 8.8,

$$\pi_n(M^{n-1}X, x) \cong M^{n-1}\pi_n(X,x).$$

Since $N^n\pi^n(X,x)$ is exactly the kernel of

$$s_* : \pi_n(X^n, x) \to M^{n-1}\pi_n(X,x)$$

such a g can be found. ∎

Now we come to the example. Let $Y \in \mathbf{S}$ be a space and let $\mathbb{F}_p^{\cdot}Y$ be the Bousfield-Kan resolution of Y, as in Example 4.1. Given another space X we get a cosimplicial space $\mathbf{Hom}(X, \mathbb{F}_p^{\cdot}Y)$ with

$$\begin{aligned}\mathrm{Tot}\,\mathbf{Hom}(X, \mathbb{F}_p^{\cdot}Y) &\cong \mathbf{Hom}(X, \mathrm{Tot}(\mathbb{F}_p^{\cdot}Y)) \\ &\cong \mathbf{Hom}(X, Y_p)\end{aligned}$$

where Y_p is the Bousfield-Kan completion of Y. See Example 7.18. We describe the sets $\pi^s\pi_t\mathbf{Hom}(X, \mathbb{F}_p Y)$.

Let $H^* = H^*(\cdot, \mathbb{F}_p)$ and let $\mathcal{K}$ be the category of unstable algebras over the Steenrod algebra. Then for any space Z there is a Hurewicz homomorphism

$$\pi_0\mathbf{Hom}(X, Z) \to \hom_{\mathcal{K}}(H^*Z, H^*Y) \tag{8.18}$$

send f to f^*. If $f \in \mathbf{Hom}(X, Z)$ is a chosen basepoint, and $\varphi : S^t \to \mathbf{Hom}(X, Z)$ represents a class in $\pi_t\mathbf{Hom}(X, Z)$ with this basepoint, then φ is adjoint to a map $\psi : S^t \times X \to Z$ that fits into a diagram

$$\begin{tikzcd} S^t \times X \arrow[r, "\psi"] & Z \\ * \times X. \arrow[u] \arrow[ur, "f"'] & \end{tikzcd}$$

The morphism $\psi^* : H^*Z \to H^*(S^t \times X) \cong H^*S^t \otimes H^*X$ can be decomposed

$$\psi^*(x) = 1 \otimes f^*(x) + x_t \otimes \partial(x)$$

where $x_t \in H^tS^t$ is a chosen generator. Because ψ^* is a morphism in $\mathcal{K}$, the induced map of graded vector spaces

$$\partial : H^*Z \to \Sigma^t H^*X$$

is a morphism of unstable modules over the Steenrod algebra and a derivation over f^*:

$$\partial(xy) = (-1)^{t|x|} f^*x\partial(y) + \partial(x)f^*y.$$

We write $\mathrm{Der}_{\mathcal{K}}(H^*Z, \Sigma^tH^*X; f^*)$ or simply $\mathrm{Der}_{\mathcal{K}}(H^*Z, \Sigma^tH^*X)$ for the vector space of such derivations. Hence we get a Hurewicz map

$$\pi_t\mathbf{Hom}(X, Z) \longrightarrow \mathrm{Der}_{\mathcal{K}}(H^*Z, \Sigma^tH^*X; f^*). \tag{8.19}$$

If Z is a simplicial $\mathbb{F}_p$ vector space and Z and X are of finite type in the sense that H^nZ and H^nX are finite dimensional for all n, then the Hurewicz maps of (8.18) and (8.19) are isomorphisms. Therefore, if X and Y are of finite type then

$$\pi^0\pi_0\mathbf{Hom}(X, \mathbb{F}_p Y) \cong \hom_{\mathcal{K}}(H^*Y, H^*X) \tag{8.20}$$

and for any $\varphi \in \hom_{\mathcal{K}}(H^*Y, H^*X)$

$$\pi^s\pi_t\mathbf{Hom}(X, \mathbb{F}_p^{\cdot}Y) \cong \pi^s \operatorname{Der}_{\mathcal{K}}(H^*\mathbb{F}_p^{\cdot}Y, \Sigma^t X; \varphi). \tag{8.21}$$

In the last equation we have written φ for any composition of face operators

$$H^*(\mathbb{F}_p^s Y) \to H^*Y \xrightarrow{\varphi} H^*X.$$

The category $s\mathcal{K}$ of simplicial objects in in $\mathcal{K}$ has a simplicial model category structure, as in Example II.6.2. Also the objects $H^*K(\mathbb{Z}/p, n)$ form a set of projective generators for $\mathcal{K}$. If we regard H^*Y as a constant object in $s\mathcal{K}$, then the augmentation

$$H^*\mathbb{F}_p^{\cdot}Y \to H^*Y$$

is a weak equivalence in $s\mathcal{K}$ and $H^*\mathbb{F}_p^{\cdot}Y$ is cofibrant – indeed, almost-free as in Example 1.14. This last assertion follows form Example 5.8. Thus

$$\operatorname{Der}_{\mathcal{K}}(H^*\mathbb{F}_p^{\cdot}Y, \Sigma^t X; \varphi)$$

is a model for the total derived functor of $\operatorname{Der}_{\mathcal{K}}(\cdot, \Sigma^t H^*X; \varphi)$ applied to H^*Y. Thus we may as well write

$$\pi^s \operatorname{Der}_{\mathcal{K}}(H^*\mathbb{F}_p^{\cdot}Y, \Sigma^t X; \varphi) = R^s \operatorname{Der}_{\mathcal{K}}(H^*Y, \Sigma^t X; \varphi). \tag{8.22}$$

This is an example of an André-Quillen cohomology and a great deal of work has gone into learning how to compute this object in special cases. For a general viewpoint one might look at [35]. However, the most successful application of this and related techniques is [62].

Combining (8.22) with (8.20) and (8.21) and appealing to Proposition 8.4 we have

PROPOSITION 8.23. *Suppose X and Y are spaces of finite type. A morphism $\varphi : H^*Y \to H^*X$ in $\mathcal{K}$ can be lifted to a map $X \to Y_p$ if*

$$R^{s+1} \operatorname{Der}_{\mathcal{K}}(H^*Y, \Sigma^s X; \varphi) = 0$$

for $s \geq 1$.

The techniques of Lannes's paper [62], cited above, show that if $X = B(\mathbb{Z}/p)^n$ is the classifying space of an elementary abelian p-group and $A \in \mathcal{K}$ is any unstable algebra, then

$$R^{s+1} \operatorname{Der}_{\mathcal{K}}(A, \Sigma^t X; \varphi) = 0$$

for all $\varphi \in \hom_{\mathcal{K}}(A, H^*B(\mathbb{Z}/p)^n)$ and all $s \geq 0, t \geq 0$.