Chapter VI The homotopy theory of towers

In principle, the Postnikov tower construction gives a method of breaking up a space X into a collection of spaces $X(n)$, $n \geq 0$, such that $X(n)$ carries the homotopy groups of X up to level n, along with a tower of fibrations $X(n) \to X(n-1)$ each of which gives a calculus of adding the n^{th} homotopy group of X to $X(n-1)$ to create the space $X(n)$. Then, in practice but sometimes with a little luck, the space X can be recovered as the inverse limit (or more properly a homotopy inverse limit) of the tower. This gives one of the central methods for recovering qualitative and calculational properties of X from the Eilenberg-Mac Lane spaces corresponding to its homotopy groups and the associated k-invariants.

We've already seen an application, namely the proof of the Hurewicz theorem in Section III.3. The theory of Postnikov towers and related constructions is the central object of study of this chapter. We begin in Section 1 by describing an approach to constructing homotopy inverse limits of towers: a closed model structure of towers of spaces is introduced, for which the homotopy inverse limit of a tower is the inverse limit of an associated fibrant model. This is a special case of a much more general concept which is more fully described in Chapter IX. Section 2 contains a formal introduction to Postnikov towers, both for spaces and maps. The construction is the standard one for simplicial sets, which is due to Moore.

In many applications, say in rational homotopy theory or more generally in the theory of R-completions and localizations, one is presented with a suite of results which says

1) Eilenberg-Mac Lane spaces have property **P**,
2) If $X \to Y$ is a principal $K(A, n)$-fibration and Y has property **P**, then so does X, and
3) If $\cdots \to X_1 \to X_0 \to *$ is a tower of fibrations such that all X_i have property **P**, then so does $\varprojlim X_i$.

This gives a method of inferring that a certain class of spaces X has property **P** which involves "crawling up a Postnikov tower" or some refinement thereof, *provided* that all of the fibrations in the tower are principal fibrations, at least up to homotopy equivalence. There is a good class of spaces for which this yoga works, namely the nilpotent spaces, and the method for establishing it involves a careful analysis of k-invariants.

For us, a k-invariant is a map

$$k_n : X(n-1) \to \underrightarrow{\operatorname{holim}}_{\Gamma} K(\pi_n X, n+1)$$

for a connected space X taking values in a homotopy colimit arising from the action of the fundamental groupoid Γ on the homotopy group $\pi_n X$. This map is implicitly fibred over the classifying space $B\Gamma$ of the fundamental groupoid Γ, and can be interpreted as a representative of a class in a suitably defined equivariant cohomology group $H^{n+1}_{\Gamma}(X(n-1), \pi_n X)$. The main point overall (Proposition 5.1) is that the fibration $X(n) \to X(n-1)$ in the Postnikov tower for X sits in a homotopy cartesian diagram

$$\begin{tikzcd}
X(n) \arrow[r] \arrow[d] & \underrightarrow{\operatorname{holim}}_{\Gamma} L(\pi_n X, n+1) \arrow[d, "p_*"] \\
X(n-1) \arrow[r, "k_n"'] & \underrightarrow{\operatorname{holim}}_{\Gamma} K(\pi_n X, n+1)
\end{tikzcd}$$

where the map p_* is induced by a natural (hence equivariant) contractible covering $L(\pi_n X, n+1) \to K(\pi_n X, n+1)$ of the space $K(\pi_n X, n+1)$. From this, it's pretty much immediate (Corollary 5.3) that the fibration $X(n) \to X(n-1)$ is homotopy equivalent to a principal $K(\pi_n, n)$-fibration if the fundamental group acts trivially on $\pi_n X$. Similarly, if X is nilpotent then the covering $L(\pi_n X, n+1) \to K(\pi_n, n+1)$ has a finite refinement by fibrations that induce principal fibrations after taking homotopy colimit (Proposition 6.1), giving the refined Postnikov tower for a nilpotent space.

This stream of results occupies the last four sections of this chapter. The main techniques involve relating homotopy classes of maps $[X, \underrightarrow{\operatorname{holim}}_{\Gamma} K(A, n)]$ fibred over the classifying space $B\Gamma$ of a groupoid Γ to equivariant cohomology in various forms (Theorems 3.10, 3.11) — this and some general nonsense about equivariant homotopy theory is the subject of Sections 3 and 4. The formal introduction of k-invariants and the proofs of their main properties occupy Section 5, while the application to nilpotent spaces appears in Section 6.

1. A model category structure for towers of spaces.

The purpose of this section is to introduce the structure of a simplicial model category on $\mathbf{tow}(\mathbf{S})$. This will have implications for Postnikov systems as well as allowing us to define homotopy inverse limits for towers of spaces.

Let $\mathcal{C}$ be a category. Then $\mathbf{tow}(\mathcal{C})$ is the category of towers in $\mathcal{C}$. Thus an object in $\mathbf{tow}(\mathcal{C})$ is a diagram in $\mathcal{C}$

$$\cdots \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0$$

and a morphism in $\mathbf{tow}(\mathcal{C})$ is a morphism of diagrams; that is, a "commutative ladder". It will be convenient to write $\mathbf{X}$ for a tower $\{X_n\}$

First notice that if $\mathcal{C}$ is a simplicial category, then $\mathbf{tow}(\mathcal{C})$ is a simplicial category. Indeed if $\mathbf{X} \in \mathbf{tow}(\mathcal{C})$ and $K \in \mathbf{S}$, let

$$\mathbf{X} \otimes K = \{X_n \otimes K\}$$

and

$$\mathbf{hom}(K, \mathbf{X}) = \{\mathbf{hom}_{\mathcal{C}}(K, X_n)\}.$$

Then $\mathbf{tow}(\mathcal{C})$ is a simplicial category with

$$\mathbf{Hom}(\mathbf{X}, \mathbf{Y})_n = \hom(\mathbf{X} \otimes \Delta^n, \mathbf{Y}).$$

The subscript in the last equation is the simplicial degree.

DEFINITION 1.1. *Define a morphism $f : \mathbf{X} \to \mathbf{Y}$ in $\mathbf{tow}(\mathbf{S})$ to be*

1) *a weak equivalence if $f_n : X_n \to Y_n$ is a weak equivalence for all $n \geq 0$;*
2) *a cofibration if $f_n : X_n \to Y_n$ is a cofibration for all $n \geq 0$; and*
3) *a fibration if $f_0 : X_0 \to Y_0$ is a fibration and for all $n \geq 1$, the induced map*

$$X_n \to Y_n \times_{Y_{n-1}} X_{n-1}$$

is a fibration.

A useful preliminary lemma is:

LEMMA 1.2. *Let $q : \mathbf{X} \to \mathbf{Y}$ be a fibration in $\mathbf{tow}(\mathcal{C})$. Then for all $n \geq 0$, $q_n : X_n \to Y_n$ is a fibration.*

PROOF: This is true if $n = 0$. If it is true for $n - 1$, contemplate the induced diagram

$$\begin{tikzcd}
X_n \arrow[r, "p_n"] \arrow[dr, "q_n"'] & Y_n \times_{Y_{n-1}} X_{n-1} \arrow[r] \arrow[d, "\overline{q}_n"] & X_{n-1} \arrow[d, "q_{n-1}"] \\
& Y_n \arrow[r] & Y_{n-1}
\end{tikzcd}$$

The morphism p_n is a fibration by hypothesis; the morphism $\overline{q}_n$ is a fibration because the pullback of a fibration is a fibration. So q_n is a fibration. ∎

PROPOSITION 1.3. *With these definitions,* $\mathbf{tow}(\mathcal{C})$ *becomes a simplicial model category.*

PROOF: Axioms **CM1–CM3** are obvious. Suppose given a lifting problem in $\mathbf{tow}(\mathcal{C})$

$$\begin{array}{ccc} \mathbf{A} & \longrightarrow & \mathbf{X} \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle q} \\ \mathbf{B} & \longrightarrow & \mathbf{Y} \end{array}$$

with j a cofibration and q a fibration. If j is also a weak equivalence, one can recursively solve the lifting problem

$$\begin{tikzcd} A_n \arrow[r] \arrow[d, "j"'] & X_n \arrow[d] \\ B_n \arrow[r] \arrow[ur, dashed] & Y_n \times_{Y_{n-1}} X_{n-1} \end{tikzcd} \tag{1.4}$$

to solve the lifting problem in $\mathbf{tow}(\mathcal{C})$. If q is a weak equivalence, the pullback diagram

$$\begin{array}{ccc} Y_n \times_{Y_{n-1}} X_{n-1} & \longrightarrow & X_{n-1} \\ {\scriptstyle q'_n}\downarrow & & \downarrow{\scriptstyle q_{n-1}} \\ Y_n & \longrightarrow & Y_{n-1} \end{array}$$

shows q'_n is a trivial fibration, since q_{n-1} is a fibration and a weak equivalence. Hence $X_n \to Y_n \times_{Y_{n-1}} X_{n-1}$ is a trivial fibration and we can again recursively solve the lifting problem of (1.4). This proves **CM4**.

To prove **CM5** fix a morphism $f : \mathbf{X} \to \mathbf{Y}$. To factor f as a cofibration followed by a trivial fibration, proceed inductively as follows. First, factor $f_0 : X_0 \to Y_0$ as a cofibration followed by a trivial fibration

$$X_0 \xrightarrow{j_0} Z_0 \xrightarrow{q_0} Y_0.$$

Then, having factored through level $n-1$, consider the induced maps

$$\begin{array}{ccccc} X_n & \xrightarrow{p_n} & Z_{n-1} \times_{Y_{n-1}} Y_n & \xrightarrow{\overline{q}_{n-1}} & Y_n \\ & & \downarrow & & \downarrow \\ & & Z_{n-1} & \xrightarrow{q_{n-1}} & Y_{n-1}. \end{array}$$

Since q_{n-1} is a trivial fibration so is $\overline{q}_{n-1}$. Factor p_n as

$$X_n \xrightarrow{j_n} Z_n \xrightarrow{q'_n} Z_{n-1} \times_{Y_{n-1}} Y_n$$

where j_n is a cofibration and q'_n is a trivial fibration. The other factoring is similar, but easier.

Finally, **SM7** follows from **SM7b**, which is obvious in this case. ■

REMARK 1.5.

1) If every object of $\mathcal{C}$ is cofibrant, then every object of $\mathbf{tow}(\mathcal{C})$ is cofibrant. This applies, for example, if $\mathcal{C} = \mathbf{S}$ or $\mathbf{S}_*$.
2) The fibrant objects of $\mathbf{tow}(\mathcal{C})$ are the ones where X_0 is fibrant in $\mathcal{C}$ and every $q_n : X_n \to X_{n-1}$ is a fibration.

REMARK 1.6. The model structure of Proposition 1.3 extends easily to bigger towers. Suppose that β is a limit ordinal, and identify it with a poset. A β-tower is a simplicial set-valued functor $X : \beta^{op} \to \mathbf{S}$ which is contravariant on β. Say that a map $f : X \to Y$ between β-towers is a cofibration (respectively weak equivalence) if the maps $f : X_s \to Y_s$ are cofibrations (respectively weak equivalences) for $s < \beta$. A map $g : Z \to W$ is a fibration if the following conditions hold:

1) the map $g : Z_0 \to W_0$ is a fibration,
2) for all ordinals $s < \beta$ the induced map $Z_{s+1} \to W_{s+1} \times_{W_s} Z_s$ is a fibration, and
3) for all limit ordinals $\alpha < \beta$ the map

$$Z_\alpha \to W_\alpha \times_{(\varprojlim_{s<\alpha} W_s)} (\varprojlim_{s<\alpha} Z_s)$$

is a fibration.

With these definitions, the category of β-towers satisfies the conditions for a simplicial model category. This model structure could be arrived at in a different way, by using the model structure for β^{op}-diagrams of simplicial sets which appears in Section IX.5 below, but that method does not produce the completely explicit description of fibration that you see here.

An extremely flexible notion is the following idea of a homotopy inverse limit.

DEFINITION 1.7. *Let* $\mathbf{X} \in \mathbf{tow}(\mathcal{C})$. *Choose a weak equivalence* $\mathbf{X} \to \mathbf{Y}$ *where* $\mathbf{Y}$ *is fibrant in* $\mathbf{tow}(\mathcal{C})$. *Then*

$$\underleftarrow{\mathrm{holim}}\, \mathbf{X} = \underleftarrow{\lim}\, \mathbf{Y}.$$

As usual, $\underleftarrow{\mathrm{holim}}\, \mathbf{X}$ is well-defined and functorial up to homotopy; indeed, $\underleftarrow{\mathrm{holim}}(\cdot)$ is the total right derived functor of $\underleftarrow{\lim}(\cdot)$. Notice that $\underleftarrow{\lim}(\cdot) : \mathbf{tow}(\mathcal{C}) \to \mathcal{C}$ is right adjoint to the constant tower functor, which preserves cofibrations and weak equivalences, so $\underleftarrow{\lim}(\cdot)$ preserves fibrations and trivial fibrations. Furthermore if $\mathbf{X} \to \mathbf{X}'$ is a weak equivalence, $\underleftarrow{\mathrm{holim}}\, \mathbf{X} \to \underleftarrow{\mathrm{holim}}\, \mathbf{X}'$ is a weak equivalence. It is this invariance property that justifies the name homotopy inverse limit.

EXERCISE 1.8 (HOMOTOPY PULLBACKS). Let I be the category with objects 1, 2, and 12 and non-identity morphisms as follows:

$$1 \to 12 \leftarrow 2.$$

If $\mathcal{C}$ is a category, let $\mathcal{C}^I$ denote the resulting diagram category. An object in $\mathcal{C}^I$ is a diagram

$$\begin{array}{ccc} & & Y_2 \\ & & \downarrow \\ Y_1 & \longrightarrow & Y_{12} \end{array}$$

in $\mathcal{C}$. Suppose $\mathcal{C}$ is a simplicial model category. Then $\mathcal{C}^I$ becomes a simplicial model category with, for $K \in S$,

$$\left(\begin{array}{ccc} & & Y_2 \\ & & \downarrow \\ Y_1 & \longrightarrow & Y_{12} \end{array}\right) \otimes K = \begin{array}{ccc} & & Y_2 \otimes K \\ & & \downarrow \\ Y_1 \otimes K & \longrightarrow & Y_{12} \otimes K. \end{array}$$

The techniques of this section can be adapted to prove the following:

THEOREM 1.9. *The category* $\mathcal{C}^I$ *is a simplicial model category with a morphism*

$$\begin{array}{ccccc} X_1 & \longrightarrow & X_{12} & \longleftarrow & X_2 \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_{12}}\downarrow & & \downarrow{\scriptstyle f_2} \\ Y_1 & \longrightarrow & Y_{12} & \longleftarrow & Y_2 \end{array}$$

1) *a weak equivalence if* f_1, f_2, *and* f_{12} *are;*
2) *a cofibration if* f_1, f_2, *and* f_{12} *are; and*
3) *a fibration if* $f_{12} : X_{12} \to Y_{12}$ *is a fibration and for* $i = 1, 2$ *the induced maps*
$$X_i \to X_{12} \times_{Y_{12}} Y_i$$
are fibrations.

A crucial lemma is:

LEMMA 1.10. *If a morphism* $f : X_\bullet \to Y_\bullet$ *in* $\mathcal{C}^I$ *is a trivial fibration, then the induced maps*
$$X_i \to X_{12} \times_{Y_{12}} Y_i$$
are trivial fibrations.

Note the fibrant objects in $\mathcal{C}^I$ are those diagrams
$$Y_1 \to Y_{12} \leftarrow Y_2$$
with Y_{12} fibrant and $Y_i \to Y_{12}$ a fibration.

If $X_\bullet \in \mathcal{C}^I$ is a diagram, define the homotopy pullback by taking the actual pullback of a fibrant replacement. This has the usual homotopy invariance properties, much as after Definition 1.7. Prove that if $\mathcal{C}$ is a proper model category and
$$Y_1 \to Y_{12} \leftarrow Y_2$$
is a diagram with Y_{12} fibrant and one of $Y_1 \to Y_{12}$ or $Y_2 \to Y_{12}$ a fibration, then $Y_1 \times_{Y_{12}} Y_2$ is weakly equivalent to the homotopy pullback.

Here is another description of the homotopy inverse limit that is often useful for computations. Fix a simplicial model category $\mathcal{C}$ and write, for $X \in \mathcal{C}$,
$$D : \mathbf{hom}_{\mathcal{C}}(\Delta^1, X) \to \mathbf{hom}_{\mathcal{C}}(\partial\Delta^1, X) \cong X \times X.$$
Then define, for a tower $\mathbf{X} = \{X_n\} \in \mathbf{tow}(\mathcal{C})$, the object $T(\mathbf{X}) \in \mathcal{C}$ by the pullback diagram

(1.11)
$$\begin{tikzcd} T(\mathbf{X}) \arrow[r] \arrow[d] & \prod_n \mathbf{hom}(\Delta^1, X_n) \arrow[d, "\prod D_n"] \\ \prod_n X_n \arrow[r, "(1,q)"'] & \prod_n X_n \times X_n \end{tikzcd}$$

where $(1,q)$ is the product of the maps

$$\prod_n X_n \to X_n \xrightarrow{1\times q_n} X_n \times X_{n-1}.$$

LEMMA 1.12. *Let $\mathbf{X} = \{X_n\} \in \mathbf{tow}(\mathcal{C})$ be a tower so that each X_n is fibrant in $\mathcal{C}$. Then there is a weak equivalence*

$$T(\mathbf{X}) \simeq \underleftarrow{\operatorname{holim}} \mathbf{X}.$$

PROOF: The functor $T : \mathbf{tow}(\mathcal{C}) \to \mathcal{C}$ has a left adjoint defined as follows. Note that specifying a map $f : Y \to T(\mathbf{X})$ is equivalent to specifying a sequence of maps $g_n : Y \to X_n$ and right homotopies

$$Y \to \mathbf{hom}(\Delta^1, X_n)$$

between g_n and $q_{n+1} \cdot g_{n+1}$. Define $[0,\infty) \in \mathbf{S}$ to be the simplicial "half-line"; thus $[0,\infty)$ has non-degenerate 1-simplices $[n,n+1]$, $0 \leq n < \infty$, and $d_0[n-1,n] = d_1[n,n+1]$. Let $[n,\infty) \subseteq [0,\infty)$ be the evident sub-complex. Then the left adjoint to T is given by the functor $F(\cdot)$:

$$F(Y) = \{Y \otimes [n,\infty)\}.$$

Since this left adjoint preserves cofibrations and weak equivalences among cofibrant objects, $F(\cdot)$ has a total left derived functor $\mathbf{L}F$. But the unique maps $[n,\infty) \to *$ induces a weak equivalence $F(Y) \to \{Y\}_{n\geq 0}$ from $F(Y)$ to the constant tower, so $\mathbf{L}F : \operatorname{Ho}(\mathcal{C}) \to \operatorname{Ho}(\mathbf{tow}(\mathcal{C}))$ is the total left derived functor of the constant tower functor.

The functor T preserves fibrations, since it is right adjoint to a functor that preserves trivial cofibrations. T also preserves weak equivalences between objects of the form $\mathbf{X} = \{X_n\}$ where X_n is fibrant for each n. This is because

$$\prod D_n : \prod_n \mathbf{hom}(\Delta^1, X_n) \to \prod_n X_n \times X_n$$

will be a fibration, so we may apply the definition and homotopy invariance property of homotopy pullback given in Exercise 1.8. Thus T has a total right derived functor $\mathbf{R}T : \operatorname{Ho}(\mathbf{tow}(\mathcal{C})) \to \operatorname{Ho}(\mathcal{C})$, right adjoint to $\mathbf{L}F$. However, $\mathbf{L}F$ is the total left derived functor of the constant diagram functor; hence, by uniqueness of adjoints

$$\mathbf{R}T \cong \underleftarrow{\operatorname{holim}} : \operatorname{Ho}(\mathbf{tow}(\mathcal{C})) \to \operatorname{Ho}(\mathcal{C}).$$

Since T preserves weak equivalences among objects $\mathbf{X} = \{X_n\}$ with all X_n fibrant one has for such X, $\underleftarrow{\operatorname{holim}} \mathbf{X} \simeq \mathbf{R}T(\mathbf{X}) \simeq T(\mathbf{X})$. ∎

2. Postnikov towers.

This section presents basic facts about Postnikov towers, including Moore's construction. The material on k-invariants, which uses cohomology with twisted coefficients is presented in Section 4.

DEFINITION 2.1. *Let X be a space. A Postnikov tower $\{X_n\}$ for X is a tower of spaces*

$$\cdots \to X_2 \xrightarrow{q_1} X_1 \xrightarrow{q_0} X_0$$

equipped with maps $i_n : X \to X_n$ so that $q_n i_n = i_{n-1} : X \to X_{n-1}$ and so that for all vertices $v \in X$, $\pi_i X_n = 0$ for $i > n$ and

$$(i_n)_* : \pi_i X \xrightarrow{\cong} \pi_i X_n$$

for $i \leq n$.

The purpose of the next few sections is to construct such and to prove a uniqueness theorem.

REMARK 2.2. If X is not connected we may write $X = \bigsqcup_\alpha X_\alpha$ as the disjoint union of its components. If $\{(X_\alpha)_n\}$ is a Postnikov tower for X_α, then, by setting $X_n = \bigsqcup_\alpha (X_\alpha)_n$ we create a Postnikov tower for X. Conversely, if $\{X_n\}$ is a Postnikov tower for X and $v \in X_\alpha \subset X$ is a vertex of X_α, then if $(X_\alpha)_n \subseteq X_n$ is the component of $i_n(v)$, $\{(X_\alpha)_n\}$ is a Postnikov tower for X_α. Hence without loss of generality, we may assume X is connected.

REMARK 2.3. *We also may as well assume X is fibrant. For if $X \to Y$ is a weak equivalence with Y fibrant, then a Postnikov tower for Y is a Postnikov tower for X.*

We now give a specific model for the Postnikov tower, due to Moore. For this reason it is called the Moore-Postnikov tower. It is pleasing to note that it is functorial in fibrant X.

DEFINITION 2.4. *Let X be a fibrant simplicial set. Define, for each integer $n \geq 0$, an equivalence relation $\sim_n$ on the simplices of X as follows: two q-simplices*

$$f, g : \Delta^q \to X$$

are equivalent if

$$f = g : sk_n \Delta^q \to X;$$

that is, the classifying maps f and g agree on the n-skeleton. Define $X(n) = X/\sim_n$.

Then there are evident maps $q_n : X(n) \to X(n-1)$ and $i_n : X \to X(n)$ yielding a map of towers $\iota : \{X\} \to \{X(n)\}$. The principal result of this section is

THEOREM 2.5. *The tower $\{X(n)\}$ is a Postnikov tower for X, and it is a tower of fibrations. Furthermore, the evident map*

$$X \to \varprojlim X(n)$$

is an isomorphism.

Note that it is a simple exercise that $\{X(n)\}$ is a tower of fibrations. Thus it suffices to prove

LEMMA 2.6. *For any choice of base point in X, $\pi_k X(n) = 0$ for $k > n$ and $(i_n)_* : \pi_k X \to \pi_k X(n)$ is an isomorphism for $k \leq n$.*

Before proving this, let us recall a combinatorial definition of $\pi_k X$, with X pointed and fibrant. An element in $\pi_k X$ can be represented by a pointed map

$$f : S^k = \Delta^k / \partial\Delta^k \to X$$

Two such maps yield the same element in $\pi_k X$ if they are related by a pointed homotopy

$$H : S^k \wedge \Delta^1_+ \to X.$$

PROOF OF LEMMA 2.6: First notice that $\pi_k X(n) = 0$ for $k > n$. This is because any representative of an element in $\pi_k X(n)$

$$f : S^k \to X(n)$$

has the property that the composite

$$\Delta^k \to S^k \xrightarrow{f} X(n)$$

is the constant map, by the definition of $\sim_n$. Next, let $E(n) \subseteq X$ be the fiber of the projection $X \to X(n)$ at some vertex v. Then $E(n)$ consists of those simplices $f : \Delta^q \to X$ so that

$$f : sk_n \Delta^q \to X$$

is constant. In particular $E(n)$ is fibrant and $E(n)_f = \{v\}$ for $k \leq n$, where v is the chosen base point. Hence $\pi_k E(n) = 0$ for $k \leq n$. The result now follows from the long exact sequence of the fibration

$$E(n) \to X \to X(n). \qquad \blacksquare$$

The complex $E(n) \subseteq X$ of this proof is called, by Moore, the n-th Eilenberg subcomplex of X. It depends on the choice of base point. It is worth pointing out that $X(0)$ has contractible components; however, it is not necessarily true that these componets have a single vertex. Indeed $X(0)_0 = X_0$, so that, for example, $X(0)$ is a one-point space if and only if $X_0 = \{v\}$.

COROLLARY 2.7. *Let X be fibrant and connected, and $v \in X$ a vertex. Let $v \in X(n)$ be the image of v under $X \to X(n)$ and $K(n)$ the fiber at v of the projection $q_n : X(n) \to X(n-1)$. Then there is a weak equivalence*

$$K(n) \to K(\pi_n X, n).$$

PROOF: This follows from the long exact sequence of the fibration sequence

$$K(n) \to X(n) \to X(n-1).$$ ∎

In particular, since X is assumed to be connected,

$$K(1) \to X(1)$$

is a weak equivalence, so there is a weak equivalence

$$X(1) \simeq K(\pi_1 X, 1) = B\pi_1 X.$$

The following is also worth noting:

PROPOSITION 2.8. *Suppose X is connected, fibrant, and minimal. Then $q_n : X(n) \to X(n-1)$ is a minimal fibration and $X(0) = \{v\}$ where $v \in X$ is the unique vertex. Furthermore $K(n)$ is a minimal complex and there are isomorphisms, $n \geq 1$,*

$$K(n) \cong K(\pi_n X, n)$$

and

$$K(1) \cong B\pi_1 X.$$

Next, let us remark that there is a relative version of the Moore construction.

DEFINITION 2.9. *Let $f : X \to B$ be a morphism of simplicial sets. Then a Postnikov tower for f is a tower of space $\{X_n\}$ equipped with a map*

$$i : \{X\} \to \{X_n\}$$

from the constant tower and a map

$$p : \{X_n\} \to \{B\}$$

to the constant tower so that

1) *for all n, the composite $X \xrightarrow{i_n} X(n) \xrightarrow{p_n} B$ is f.*
2) *for any choice of vertex of X, the map $(i_n)_* : \pi_k X \to \pi_k X(n)$ is an isomorphism for $k \leq n$;*
3) *for any choice of vertex of X, the map $(p_n)_* : \pi_k X(n) \to \pi_k B$ is an isomorphism for $k > n+1$.*
4) *for any choice of vertex v of X, there is an exact sequence*

$$0 \to \pi_{n+1} X(n) \to \pi_{n+1} B \xrightarrow{\partial} \pi_n F$$

where F_v is the homotopy fiber of f at v and ∂ is the connecting homomorphism in the long exact sequence of the homotopy fibration $F_v \to X \to B$.

DEFINITION 2.10. *Suppose $f : X \to B$ is a fibration and B is fibrant. Define an equivalence relation $\sim_n$ as simplices*

$$\alpha, \beta : \Delta^q \to X$$

by saying $\alpha \sim_n \beta$ if and only if

1) *$f\alpha = f\beta$ and*
2) *$f = g : sk_n \Delta^q \to X$.*

Then there are maps $i_n : X \to X(n)$ and $p_n : X(n) \to B$ and the appropriate generalization of Theorem 2.5 is:

THEOREM 2.11. *The tower $\{X(n)\}$ is a Postnikov tower for f, and it is a tower of fibrations. Furthermore, the evident map*

$$X \to \underleftarrow{\operatorname{holim}}\, X(n)$$

is an isomorphism.

PROOF: Again we need only check the statements about homotopy groups. Choose a vertex in X and let $F(n)$ be the fiber of the fibration $X(n) \to B$. Then a moment's thought shows the notation $F(n)$ is unambiguous — $F(n)$ really is the n-th stage in the Moore-Postnikov tower of the fiber F of $f : X \to B$. The result thus follows from Lemma 2.6 and the long-exact sequence of the fibration $F(n) \to X(n) \to B$. ■

Note that we have used

LEMMA 2.12. *Let $f : X \to B$ be a fibration with be fibrant and fiber F, for some vertex of B. Then the fiber of $p_n : X(n) \to B$ is the n-th stage $F(n)$ in the Moore-Postnikov tower for F.*

COROLLARY 2.13. *If F is connected $p_0 : X(0) \to B$ is a weak equivalence.*

3. Local coefficients and equivariant cohomology.

The purpose of this section is to demonstrate that cohomology with local coefficients is representable in an appropriate homotopy category, and to identify the homotopy type of the representing object. We develop a calculation scheme in terms of equivariant cohomology on the universal cover in Section 4.

Let X be a fibrant space (meaning fibrant simplicial set) and $\Gamma = \pi_f X$ its fundamental groupoid (see Sections I.8, III.1). There is a canonical simplicial map $\phi : X \to B\Gamma$ which associates to the simplex $\tau : \Delta^n \to X$ the string

$$\tau(0) \to \tau(1) \to \cdots \to \tau(n)$$

in Γ arising from the image of the string

$$0 \to 1 \to \cdots \to n$$

in the 1-skeleton of Δ^n.

We shall, more generally, consider the category of spaces $\mathbf{S} \downarrow B\Gamma$ over $B\Gamma$. The objects of this category are the simplicial set maps $Y \to B\Gamma$, and the morphisms are commutative diagrams of simplicial set maps

$$\begin{tikzcd} Y \arrow[rr] \arrow[dr] & & Z \arrow[dl] \\ & B\Gamma & \end{tikzcd}$$

It is trivial to observe the following:

LEMMA 3.1. *Suppose that X is an object of a closed model category $\mathcal{C}$. Then the category $\mathcal{C} \downarrow X$ has a closed model structure, in which a morphism*

$$\begin{tikzcd} Z \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & X & \end{tikzcd}$$

is a weak equivalence, fibration, or cofibration if the same is true for the map $f : Z \to Y$ of $\mathcal{C}$.

In particular, the category $\mathbf{S} \downarrow B\Gamma$ inherits a closed model category from the category of simplicial sets. Furthermore, the fibrant objects in $\mathbf{S} \downarrow B\Gamma$ are fibrations $Y \to B\Gamma$. At this level of generality, the following result gives us an good selection of fibrations and fibrant objects (compare with Lemma IV.5.7):

LEMMA 3.2. *Suppose that $p : Z \to Y$ is a natural transformation of functors $\Gamma \to \mathbf{S}$ such that each map $p : Z_v \to Y_v$ is a fibration. Then the induced map*

$$p_* : \underrightarrow{\mathrm{holim}}_{\Gamma} Z \to \underrightarrow{\mathrm{holim}}_{\Gamma} Y$$

is a fibration. If all fibrations p_v are minimal then p_* is minimal.

PROOF: Consider a commutative solid arrow diagram

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\alpha"] \arrow[d] & \underrightarrow{\mathrm{holim}}_{\Gamma} Z \arrow[d, "p_*"] \\ \Delta^n \arrow[r, "\beta"'] \arrow[ur, dashed] & \underrightarrow{\mathrm{holim}}_{\Gamma} Y. \end{tikzcd} \tag{3.3}$$

We need to show that the dotted arrow exists, making the diagram commute.

The pullback of the canonical map $\pi : \underrightarrow{\mathrm{holim}}_{\Gamma} Y \to B\Gamma$ along the composite

$$\Delta^n \xrightarrow{\beta} \underrightarrow{\mathrm{holim}}_{\Gamma} Y \xrightarrow{\pi} B\Gamma$$

is the homotopy colimit of the functor $\pi(\beta) : \mathbf{n} \to \Gamma$. The category Γ is a groupoid, so that there is a natural isomorphism of functors on the category $\mathbf{n}$ of the form

$$\begin{tikzcd} X(\pi(\beta)(0)) \arrow[r, "1"] \arrow[d, "\cong"'] & X(\pi(\beta)(0)) \arrow[r, "1"] \arrow[d, "\cong"'] & \dots \arrow[r, "1"] & X(\pi(\beta)(0)) \arrow[d, "\cong"] \\ X(\pi(\beta)(0)) \arrow[r] & X(\pi(\beta)(1)) \arrow[r] & \dots \arrow[r] & X(\pi(\beta)(n)). \end{tikzcd}$$

It follows that the induced map $\Delta^n \times_{B\Gamma} \underrightarrow{\mathrm{holim}}_{\Gamma} Y \to \Delta^n$ is isomorphic over Δ^n to $\Delta^n \times X(\pi(\beta)(0)) \to \Delta^n$. Similarly, the map $\Delta^n \times_{B\Gamma} \underrightarrow{\mathrm{holim}}_{\Gamma} Z \to \Delta^n$

is isomorphic over Δ^n to the projection $\Delta^n \times Z(\pi(\beta)(0)) \to \Delta^n$. The original diagram (3.3) therefore factors (uniquely) through a diagram of the form

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\alpha_*"] \arrow[d] & \Delta^n \times Z(\pi(\beta)(0)) \arrow[d, "1 \times p"] \\ \Delta^n \arrow[r, "\beta_*"'] \arrow[ur, dashed] & \Delta^n \times Y(\pi(\beta)(0)) \end{tikzcd}$$

The map $1 \times p$ is a fibration since p is a fibration by assumption, so the dotted arrow exists. Observe further that if all fibrations $p : Z_v \to Y_v$ are minimal, then $1 \times p$ is minimal, and so p_* is minimal. ∎

Corollary 3.4. *Suppose that $K : \Gamma \to \mathbf{S}$ is a functor such that K_v is fibrant for each v, then the canonical map $\pi : \underrightarrow{\operatorname{holim}}_{\Gamma} K \to B\Gamma$ is a fibration.*

A *local coefficient system* is a functor $A : \Gamma \to \mathbf{Ab}$. We shall write A_x for $A(x)$. Note there is nothing special about abelian groups. If $\mathcal{C}$ is a category, a local system of objects in $\mathcal{C}$ is simply a functor $\Gamma \to \mathcal{C}$. In particular, the functors of Lemma 3.2 are local systems of simplicial sets.

The cohomology of X with local coefficients in A can be defined via cochains. The group $C^n_\Gamma(X, A)$ of n-cochains with coefficients in A is defined to be the collection of commutative diagrams

$$\begin{tikzcd} X_n \arrow[rr, "\alpha"] \arrow[dr, "\phi"'] & & \underrightarrow{\operatorname{holim}}_{\Gamma} A_n \arrow[dl] \\ & B\Gamma^n & \end{tikzcd}$$

or rather $C^n_\Gamma(X, A)$ is the collection of all assignments $x \mapsto \alpha(x) \in A_{\phi(x)(0)}$, $x \in X_n$.

All calculations in such groups of cochains depend (initially) on the following observation:

Lemma 3.5. *Suppose that $\psi : Y \to B\Gamma$ is a second space over $B\Gamma$, $\theta : \mathbf{m} \to \mathbf{n}$ is an ordinal number map, and that $f : Y_n \to X_m$ is a function such that the* *diagram*

$$\begin{tikzcd} Y_n \arrow[r, "f"] \arrow[d, "\psi"'] & X_m \arrow[d, "\phi"] \\ B\Gamma_n \arrow[r, "\theta^*"'] & B\Gamma_m \end{tikzcd}$$

commutes. Then, given any cochain $\alpha : X_m \to \underrightarrow{\mathrm{holim}}_\Gamma A_m$, *there is a unique cochain* $f^*(\alpha) : Y_n \to \underrightarrow{\mathrm{holim}}_\Gamma A_n$ *such that the diagram*

$$\begin{tikzcd} Y_n \arrow[r, "f^*(\alpha)"] \arrow[d, "f"'] & \underrightarrow{\mathrm{holim}}_{\Gamma} A_n \arrow[d, "\theta^*"] \\ X_m \arrow[r, "\alpha"'] & \underrightarrow{\mathrm{holim}}_{\Gamma} A_m \end{tikzcd}$$

commutes.

PROOF: Recall that the simplicial structure map $\theta^* : \underrightarrow{\mathrm{holim}}_\Gamma A_n \to \underrightarrow{\mathrm{holim}}_\Gamma A_m$ is defined on the component $A_{\sigma(0)}$ of

$$\underrightarrow{\mathrm{holim}}_{\Gamma} A_n = \coprod_{\sigma(0)\to\sigma(1)\to\cdots\to\sigma(n)} A_{\sigma(0)}$$

corresponding to the simplex $\sigma : \mathbf{n} \to \Gamma$ by the composite

$$A_{\sigma(0)} \xrightarrow{\sigma(0,\theta(0))_*} A_{\sigma(\theta(0))} \xrightarrow{in_{\sigma\theta}} \underrightarrow{\mathrm{holim}}_{\Gamma} A_m,$$

where $\sigma(0, \theta(0))$ denotes the composite of the morphisms

$$\sigma(0) \to \sigma(1) \to \cdots \to \sigma(\theta(0))$$

in Γ. For $y \in Y_n$, $f^*(\alpha)(y)$ is therefore the unique element in $A_{\psi(y)(0)}$ which maps to $\alpha(f(y)) \in A_{\phi(f(y))(0)} = A_{\theta^*\psi(y)(0)} = A_{\psi(y)\theta(0)}$ under the isomorphism

$$A_{\psi(y)(0)} \xrightarrow[\cong]{\psi(y)(0,\theta(0))} A_{\psi(y)\theta(0)}. \qquad \blacksquare$$

Lemma 3.5 gives a cosimplicial object $C^*_\Gamma(X, A)$ — write $H^*_\Gamma(X, A)$ for the corresponding cohomology groups.

Suppose that y is an object of the groupoid Γ. Recall that the groupoid $\Gamma \downarrow y$ has for objects all morphisms $\tau : x \to y$, and has morphisms given by commutative diagrams in Γ of the form

$$\begin{tikzcd} x \arrow[rr, "\alpha"] \arrow[dr, "\tau"'] & & x' \arrow[dl, "\tau'"] \\ & y & \end{tikzcd}$$

Write $E\Gamma_y$ for the nerve of the category $\Gamma \downarrow y$. The functor $\Gamma \downarrow y \to \Gamma$ which forgets y induces a simplicial set map $\pi_y : E\Gamma_y \to B\Gamma$, whereas composition with $\beta : y \to z$ induces a functor $\beta_* : \Gamma \downarrow y \to \Gamma \downarrow z$ which commutes with forgetful functors. It follows that there is a functor $\Gamma \to \mathbf{S}$ defined by $y \mapsto E\Gamma_y$ which is fibred over $B\Gamma$. Note that each of the categories $\Gamma \downarrow y$ has a terminal object, namely the identity $1_y : Y \to y$, so that all spaces $E\Gamma_y$ are contractible. Observe finally that the maps $\pi_y : E\Gamma_y \to B\Gamma$ are Kan fibrations.

Suppose that $\phi : X \to B\Gamma$ is a simplicial set map, and define a collection of spaces $\tilde{X}_y$ by forming pullbacks

$$\begin{tikzcd} \tilde{X}_y \arrow[r] \arrow[d] & E\Gamma_y \arrow[d, "\pi_y"] \\ X \arrow[r, "\phi"'] & B\Gamma \end{tikzcd}$$

In this way, we define a functor $\tilde{X} : \Gamma \to \mathbf{S}$. We shall call $\tilde{X}$ the *covering system* for ϕ; the construction specializes to a Γ-diagram of covering spaces for X in the case where Γ is the fundamental groupoid of X and $\phi : X \to B\Gamma$ is the canonical map.

Write $\mathbf{S}_\Gamma$ for the category of diagrams $X : \Gamma \to \mathbf{S}$ taking values in simplicial sets. There is plainly a functor $\underrightarrow{\mathrm{holim}}_\Gamma : \mathbf{S}_\Gamma \to \mathbf{S} \downarrow B\Gamma$ defined by taking homotopy colimits. The covering system construction $X \mapsto \tilde{X}$ defines a functor $\mathbf{S} \downarrow B\Gamma \to \mathbf{S}_\Gamma$. These two functors are adjoint:

LEMMA 3.6. *There is a natural bijection*

$$\hom_{\mathbf{S}^\Gamma}(\tilde{X}, Y) \cong \hom_{\mathbf{S}\downarrow B\Gamma}(X, \underrightarrow{\operatorname{holim}}_{\Gamma} Y).$$

PROOF: For each object $y \in \Gamma$, the simplicial set $\tilde{X}_y$ has n-simplices $(\tilde{X}_y)_n$ consisting of all pairs of the form (x, α), where $x \in X_n$ and $\alpha : \phi(x)(0) \to y$ is a morphism of Γ. The simplicial structure map $\theta^* : (\tilde{X}_y)_n \to (\tilde{X}_y)_m$ associated to $\theta : \mathbf{m} \to \mathbf{n}$ is defined by

$$\theta^*(x, \alpha) = (\theta^*(x), \alpha\phi(x)(0, \theta(0))^{-1}).$$

There is a natural bijection

$$\hom_{\mathbf{Sets}^\Gamma}(\tilde{X}_n, A) \cong \hom_{\mathbf{Sets}\downarrow B\Gamma_n}(X_n, \underrightarrow{\operatorname{holim}}_{\Gamma} A_n) \tag{3.7}$$

in each degree n. In effect, if $A : \Gamma \to \mathbf{Sets}$ is an arbitrary set valued functor, then a natural transformation

$$g_y : (\tilde{X}_y)_n \to A_y \ \ y \in \Gamma,$$

is completely determined by its effect on the elements $(x, 1_{\phi(x)(0)})$, $x \in X_n$, and $g(x, 1_{\phi(x)(0)}) \in A_{\phi(x)(0)}$, so the assignment $x \mapsto g(x, 1_{\phi(x)(0)})$ determines an element of $\hom_{\mathbf{Sets}\downarrow B\Gamma_n}(X_n, \underrightarrow{\operatorname{holim}}_{\Gamma} A_n)$.

One checks to see that the isomorphisms (3.7) assemble to give the desired adjunction on the simplicial set level. ∎

If $Y : \Gamma \to \mathbf{S}$ is a Γ-diagram of simplicial sets and $A : \Gamma \to \mathbf{Ab}$ is a local coefficient system, there is an obvious way to form a cochain complex (aka. cosimplicial abelian group) $\hom_\Gamma(Y, A)$, having n-cochains given by the group $\hom_\Gamma(Y_n, A)$ of all natural transformations from Y_n to A.

COROLLARY 3.8. *There is a natural isomorphism of cosimplicial abelian groups*

$$C^*_\Gamma(X, A) \cong \hom_\Gamma(\tilde{X}, A)$$

for all spaces X over $B\Gamma$ and all local coefficient systems A.

PROOF: Use the calculation in the proof of Lemma 3.5 to show that the natural bijections (3.7) from the proof of Lemma 3.6 preserve cosimplicial structure. ∎

The category $\mathbf{S}_\Gamma$ has a simplicial model structure for which a map $f : Z \to W$ is a weak equivalence (respectively fibration) if and only if each of the components $f : Z_v \to W_v$, $v \in \Gamma$, is a weak equivalence (respectively fibration) of simplicial sets. This is the Bousfield-Kan structure for Γ-diagrams — see Example II.7.9. The homotopy colimit construction takes weak equivalences of $\mathbf{S}_\Gamma$ to weak equivalences of $\mathbf{S} \downarrow B\Gamma$ (Proposition IV.1.9), whereas the covering system functor $\mathbf{S} \downarrow B\Gamma \to \mathbf{S}_\Gamma$ preserves weak equivalences by the coglueing lemma (Lemma II.9.12). There are therefore induced functors

$$\mathrm{Ho}(\mathbf{S}_\Gamma) \leftrightarrows \mathrm{Ho}(\mathbf{S} \downarrow B\Gamma)$$

which are adjoint, and so the natural isomorphism of Lemma 3.6 extends to a natural bijection

$$[\tilde{X}, Y]_{\mathbf{S}_\Gamma} \cong [X, \underrightarrow{\mathrm{holim}}_{\Gamma}\, Y]_{\mathbf{S}\downarrow B\Gamma}$$

on the homotopy category level.

To go further, we need to know the following:

LEMMA 3.9. *For any object $\phi : X \to B\Gamma$, the corresponding covering system $\tilde{X}$ is a cofibrant object of $\mathbf{S}_\Gamma$.*

PROOF: Suppose that the groupoid Γ is connected, and pick an object $v \in \Gamma$. Write $\pi = \hom_\Gamma(v, v)$, and pick a morphism $\eta_x : x \to v$ in Γ for each object x, such that $\eta_v = 1_v$.

There is an inclusion functor $i : \pi \to \Gamma$ which identifies the group π with the group of automorphisms of v in Γ. There is also a functor $r : \Gamma \to \pi$ which is defined by sending $\omega : x \to y$ to the composite $\eta_y \omega \eta_x^{-1} : v \to v$, and the functors r and i are the component functors of an equivalence of the (connected) groupoid Γ with the group π: $r \cdot i = 1_\pi$, and the isomorphisms $\eta_x : x \to v$ determine a natural isomorphism $1_\Gamma \cong i \cdot r$.

The functors i and r induce functors $i^* : \mathbf{S}_\Gamma \to \mathbf{S}_\pi$ and $r^* : \mathbf{S}_\pi \to \mathbf{S}_\Gamma$, just by precomposition. The composite $i^* \cdot r^*$ is the identity functor on $\mathbf{S}_\pi$, while the isomorphisms $\eta_{x*} : X_v \to X_x$ define a natural isomorphism $r^* \cdot i^*(X) \cong X$, so that $\mathbf{S}_\Gamma$ and $\mathbf{S}_\pi$ are equivalent categories.

The functor i^* reflects cofibrations, so it suffices to show that $\tilde{X}_v$ is a cofibrant π-space. To see this, observe that the functor r induces a commutative diagram

$$\begin{tikzcd} E\Gamma_v \arrow[r] \arrow[d] & E\pi \arrow[d] \\ B\Gamma \arrow[r] & B\pi, \end{tikzcd}$$

and this diagram is a pullback. But then there is a π-equivariant isomorphism $(E\Gamma_v)_n \cong \pi \times B\Gamma_n$; in other words, $E\Gamma_v$ is a free π-set in each degree, so it must be a cofibrant π-space, by Corollary V.2.9. ∎

We can now relate elements in the cohomology groups $H^n_\Gamma(X, A)$ arising from fibred spaces $X \to B\Gamma$ and local coefficient systems $A : \Gamma \to \mathbf{Ab}$ to morphisms in some homotopy category. The method is to use a Γ-equivariant version of the Dold-Kan correspondence (Section III.2), in conjunction with Lemma 3.9.

First of all, recall from Corollary 3.8 that $H^n_\Gamma(X, A)$ is the n^{th} cohomology group of the cochain complex $\hom_\Gamma(\tilde{X}, A)$ which is defined in degree n by the set $\hom_\Gamma(\tilde{X}_n, A)$ of natural transformations $\tilde{X}_n \to A$. It follows that $H^n_\Gamma(X, A)$ is canonically isomorphic to the group $\pi(\mathbb{Z}(\tilde{X}), A[n])$ of chain homotopy classes of morphisms of Γ-diagrams of chain complexes from the Γ-diagram of Moore chains $\mathbb{Z}(\tilde{X})$ to $A[n]$, where $A[n]$ is the Γ-diagram of chain complexes which consists of a copy of A concentrated in degree n. The normalized chain complex $N\mathbb{Z}(\tilde{X})$ is naturally chain homotopy equivalent to $\mathbb{Z}(\tilde{X})$, so that there is an isomorphism

$$\pi(\mathbb{Z}(\tilde{X}), A[n]) \xrightarrow[\cong]{i^*} \pi(N\mathbb{Z}(\tilde{X}), A[n])$$

which is induced by the inclusion $i : N\mathbb{Z}(\tilde{X}) \hookrightarrow \mathbb{Z}(\tilde{X})$. But now there are isomorphisms

$$\begin{aligned}\pi(N\mathbb{Z}(\tilde{X}), A[n]) &\cong \pi(\mathbb{Z}(\tilde{X}), K(A, n)) \\ &\cong \pi(\tilde{X}, K(A, n)) \\ &\cong [\tilde{X}, K(A, n)]_\Gamma,\end{aligned}$$

where the first isomorphism is induced by the Dold-Kan correspondence, the second isomorphism relates naive homotopy classes in Γ-diagrams of simplicial abelian groups to naive homotopy classes in Γ-diagrams of simplicial sets, and the last isomorphism relating naive homotopy classes to morphisms in the homotopy category of Γ-spaces results from the fact that $\tilde{X}$ is cofibrant and the Eilenberg-Mac Lane object $K(A, n)$ is fibrant. Subject to chasing an explicit cochain through these identifications (which is left to the reader), we have proved:

THEOREM 3.10. *Suppose that $X \to B\Gamma$ is a space over $B\Gamma$, and let $A : \Gamma \to \mathbf{Ab}$ be a local coefficient system. Then there is an isomorphism*

$$[\tilde{X}, K(A, n)]_\Gamma \xrightarrow{\cong} H^n_\Gamma(X, A),$$

which is defined by sending a class represented by a map $f : \tilde{X} \to K(A, n)$ to the class represented by the cocycle $f_n : X_n \to \underrightarrow{\mathrm{holim}}_\Gamma A_n$.

The equivalence of categories $i^* : \mathbf{S}_\Gamma \leftrightarrow \mathbf{S}_\pi : r^*$ induces an equivalence of associated homotopy categories, giving an isomorphism

$$[\tilde{X}, K(A,n)]_\Gamma \xrightarrow[\cong]{i^*} [\tilde{X}_v, K(A_v,n)]_\pi,$$

while the identifications leading to Theorem 3.10 are preserved by r^* as well as being valid in the category of π-diagrams. This leads to an identification of $H^*_\Gamma(X, A)$ with ordinary π-equivariant cohomology:

THEOREM 3.11. *Suppose that X and A are as in the statement of Theorem 3.10, suppose that Γ is a connected groupoid and let π be the group* $\hom_\Gamma(v,v)$ *of an object $v \in \Gamma$. Then restriction to π-spaces determines a commutative diagram of isomorphisms*

$$\begin{tikzcd}
{[\tilde{X}, K(A,n)]_\Gamma} \arrow[r, "\cong"] \arrow[d, "i^*"', "\cong"] & H^n_\Gamma(X, A) \arrow[d, "\cong"', "i^*"] \\
{[\tilde{X}_v, K(A_v,n)]_\pi} \arrow[r, "\cong"'] & H^n_\pi(\tilde{X}_v, A_v).
\end{tikzcd}$$

Observe, from the proof of Lemma 3.9, that $\tilde{X}_v$ is a copy of the universal cover of X if the underlying map $\phi : X \to B\Gamma$ is the canonical map arising from the fundamental groupoid Γ of X.

4. Generalities: equivariant cohomology.

There is a notion of reduced cohomology with local coefficients. Suppose $\Gamma = \pi_f X$ is the fundamental groupoid and the map $X \to B\Gamma$ has a section $s : B\Gamma \to X$. Such a section is a map from the terminal object to X in $\mathbf{S} \downarrow B\Gamma$, which is the natural notion of base point. For example, we could take

$$X = \underrightarrow{\operatorname{holim}}_{\Gamma} K(A,n)$$

where $s(\sigma) = (0, \sigma)$ A *pointed map* in $\mathbf{S} \downarrow \Gamma$

$$X \to \underrightarrow{\operatorname{holim}}_{\Gamma} K(A,n)$$

is a map commuting with sections.

In gross generality, if $\mathcal{C}$ is a simplicial model category, then the category $\mathcal{C}_*$ of pointed objects in $\mathcal{C}$ (meaning morphisms $t \to X$, where t is terminal) satisfies the axioms for a closed model category, where a map of $\mathcal{C}_*$ is a fibration, cofibration or weak equivalence if and only if it is so as a map of $\mathcal{C}$ (compare Lemma 3.1).

A reduced cochain $\varphi \in C^n_\Gamma(X, A)$ is a cochain so that $\varphi(\sigma(x)) = 0$ for all $x \in B\Gamma$; if $\widetilde{C}^n_\Gamma(X, A)$ is the group of such there is a split short exact sequence of cosimplicial abelian groups

$$0 \to \widetilde{C}^*_\Gamma(X, A) \to C^*_\Gamma(X, A) \xrightarrow{\sigma^*} C^*_\Gamma(B\Gamma, A) \to 0$$

and a split short exact sequence of graded groups

$$0 \to \widetilde{H}^*_\Gamma(X, A) \to H^*_\Gamma(X, A) \to H^*_\Gamma(B\Gamma, A) \to 0.$$

Of course, if X is connected, then for any vertex $v \in X$

$$H^*_\Gamma(B\Gamma, A) \cong H^*(\pi_1(X, v), A_v),$$

by Theorem 3.11.

The following is left as an exercise for anyone who has read the proof of Theorem 3.10. Let $\mathbf{S} \downarrow B\Gamma_*$ denote the category of pointed objects over $B\Gamma$.

Lemma 4.1. *For $X \in \mathbf{S} \downarrow B\Gamma_*$ there are natural isomorphisms*

$$[X, \underset{\overrightarrow{\Gamma}}{\mathrm{holim}}\, K(A, n)]_{\mathbf{S}\downarrow B\Gamma_*} \cong \widetilde{H}^n_\Gamma(X; A).$$

Now suppose $X \subseteq Y$ — that is, X is a subspace of Y — and $\Gamma = \pi_f X \cong \pi_f Y$: This would happen for example, if X and Y have the same 2-skeleton. Define $B\Gamma \cup_X Y$ by the push-out diagram in $\mathbf{S} \downarrow B\Gamma$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ B\Gamma \arrow[r] & B\Gamma \cup_X Y. \end{tikzcd}$$

Then $B\Gamma \cup_X Y \in \mathbf{S} \downarrow B\Gamma_*$ and we define

$$H^*_\Gamma(Y, X; A) = \widetilde{H}^*_\Gamma(B\Gamma \cup_X Y; A).$$

For a different point of view, observe that the covering system construction preserves cofibrations and pushouts, and the covering system $\widetilde{B\Gamma}$ for $B\Gamma$ consists of contractible spaces, so that the covering system for $B\Gamma \cup_X Y$ is equivalent to the homotopy cofibre $\tilde{Y}/\tilde{X}$ of the inclusion $\tilde{X} \hookrightarrow \tilde{Y}$ in the category of Γ-diagrams. It follows that there is an isomorphism

$$H^n_\Gamma(Y, X; A) \cong [\tilde{Y}/\tilde{X}, K(A, n)]_{\Gamma *}.$$

PROPOSITION 4.2. *There is a long exact sequence*

$$\cdots \to H^n_\Gamma(Y, X; A) \to H^n_\Gamma(Y; A) \to H^n_\Gamma(X; A) \to H^{n+1}_\Gamma(Y, X; A) \to \cdots$$

PROOF: One easily checks

$$0 \to \widetilde{C}^*_\Gamma(B\Gamma \cup_X Y; A) \to C^*_\Gamma(Y; A) \to C^*_\Gamma(X; A) \to 0$$

is exact.

Alternatively, one could use Theorem 3.10, Corollary 4.1, and the Puppe sequence. For this, let $\Omega_{\mathcal{C}}$ denote the "loop space" functor in any simplicial model category $\mathcal{C}$. Then Lemma 3.2 implies that there is a natural weak equivalence

$$\Omega_{\mathbf{S}\downarrow B\Gamma}(\underset{\overrightarrow{\Gamma}}{\mathrm{holim}}\, K(A, n)) \simeq \underset{\overrightarrow{\Gamma}}{\mathrm{holim}}\, K(A, n-1). \qquad \blacksquare$$

We shall now write down some variants of a spectral sequence (Proposition 4.5) which computes equivariant cohomology.

Fix a discrete group π, and observe that, if A is a $\mathbb{Z}\pi$ module then for all $n \geq 0$, $K(A, n)$ is a fibrant object in the model structure for $\mathbf{S}_\pi$ of Section V.1.

Let $X \in \mathbf{S}_\pi$ be cofibrant, and suppose that $n \geq 0$. We have already seen that there is a natural isomorphism

$$H^n_\pi(X, A) \cong [X, K(A, n)]_\pi$$

in the proof of Theorem 3.11. If X is not cofibrant we make the following:

DEFINITION 4.3. *For arbitrary π-spaces X, define $H^n_\pi(X, A)$ by setting*

$$H^n_\pi(X, A) = [X, K(A, n)]_\pi.$$

The moral is that if X is cofibrant, then $H^n_\pi(X, A)$ has a cochain description; otherwise, take a weak equivalence $Y \to X$ and Y cofibrant in $\mathbf{S}_\pi$, so that

$$[X, K(A, n)]_\pi \cong [Y, K(A, n)]_\pi \cong H^n_\pi(Y, A).$$

While less important for our applications, this also tells one how to define homology.

DEFINITION 4.4. *Let A be a $\mathbb{Z}\pi$ module, $X \in \mathbf{S}_\pi$ and $Y \to X$ a weak equivalence with Y cofibrant. Then define*

$$H_n^\pi(X, A) = \pi_n(\mathbb{Z}Y \otimes_{\mathbb{Z}\pi} A).$$

The standard model category arguments (cf. Section II.1) show that $H_*^\pi(X, A)$ is independent of the choice of Y.

For calculational purposes one has the following result. If $X \in \mathbf{S}_\pi$, then $H_*X = H_*(X, \mathbb{Z})$ is a graded $\mathbb{Z}\pi$ module.

PROPOSITION 4.5. *There is a first quadrant cohomology spectral sequence*

$$\mathrm{Ext}^p_{\mathbb{Z}\pi}(H_qX, A) \Rightarrow H_\pi^{p+q}(X, A).$$

PROOF: We may assume X is cofibrant and, indeed, that X_n is a free π-set for all n, by Lemma V.2.4. Let $P_\bullet(\cdot) = \{\cdots \to P_1(\cdot) \to P_0(\cdot) \to (\cdot)\}$ denote a functorial projective resolution — for example the bar resolution — of $\mathbb{Z}\pi$ modules. We require each $P_s(\cdot)$ to be exact. Form the double complex

$$\hom_{\mathbb{Z}\pi}(P_\bullet(\mathbb{Z}X), A) = \{\hom_{\mathbb{Z}\pi}(P_p(\mathbb{Z}X_q), A\}.$$

Filtering by degree in p we get a spectral sequence with

$$E_1^{p,q} = \hom_{\mathbb{Z}\pi}(P_p(H_qX), A)$$

so that E_2 is as required. To determine what the spectral sequence abuts to, filter by degree in p, whence

$$E_1^{p,q} = \mathrm{Ext}^p_{\mathbb{Z}\pi}(\mathbb{Z}X_q, A).$$

But $\mathbb{Z}X_q$ is a free $\mathbb{Z}\pi$ module, so $E_1^{p,q} = 0$ if $p > 0$ and

$$E_1^{0,q} = \hom_{\mathbb{Z}\pi}(\mathbb{Z}X_q, A).$$

Hence the spectral sequence abuts to $E_2^{0,*} = H_\pi^*(X, A)$, as required. ∎

Proposition 4.5 is a special case of a result that holds in great generality, in the context of homotopy theories of simplicial sheaves and presheaves [45], [46]. We shall confine ourselves here to displaying a few of its close relatives.

An argument similar to the one given in Proposition 4.5 yields a first quadrant homology spectral sequence

$$\mathrm{Tor}_p^{\mathbb{Z}\pi}(H_qX, A) \Rightarrow H^\pi_{p+q}(X, A). \tag{4.6}$$

There are further refinements. For example, if A is an $\mathbb{F}\pi$ module where $\mathbb{F}$ is a field one has

$$\operatorname{Ext}^p_{\mathbb{F}\pi}(H_q(X;\mathbb{F}),A) \Rightarrow H^{p+q}_\pi(X;A). \tag{4.7}$$

EXAMPLE 4.8. Let $X = *$ be a point. Then $H_*X = \mathbb{Z}$ concentrated in degree 0 and

$$H^p_\pi(*;A) \cong \operatorname{Ext}^p_{\mathbb{Z}\pi}(\mathbb{Z},A) = H^q(\pi,A).$$

Note that unlike ordinary homology, $H^*_\pi(*;A)$ need not split off of $H^*_\pi(X;A)$. The reader is invited to calculate $H^*_{\mathbb{Z}/2\mathbb{Z}}(S^n,\mathbb{Z})$ where S^n has the "antipodal point" action and $\mathbb{Z}$ the trivial action. Use the next observation.

EXAMPLE 4.9. If A is a trivial π module, and X is cofibrant,

$$H^n_\pi(X,A) \cong H^n(X/\pi,A).$$

This follows from the definition.

There is a relative version of this cohomology. In fact, let $\mathbf{S}_{\pi *}$ denote pointed π-spaces. For all $\mathbb{Z}\pi$ modules A, $K(A,n)$ has an evident base point.

DEFINITION 4.10. *For $X \in \mathbf{S}_{\pi *}$, let*

$$\widetilde{H}^n_\pi(X;A) = [X,K(A,n)]_{\mathbf{S}_{\pi *}}.$$

Hence one factors $* \to X$ as $* \xrightarrow{i} Y \to X$ where i is a cofibration in $\mathbf{S}_\pi$ and

$$\widetilde{H}^n_\pi(X;A) = \pi_0 \,\mathbf{Hom}_{\mathbf{S}_{\pi *}}(Y,K(A,n)).$$

PROPOSITION 4.11.

1) *For $X \in \mathbf{S}_{\pi *}$ there is a split natural short exact sequence*

$$0 \to \widetilde{H}^*_\pi(X;A) \to H^*_\pi(X;A) \to H^*(\pi;A) \to 0.$$

2) *There is a first quadrant cohomology spectral sequence*

$$\operatorname{Ext}^p_{\mathbb{Z}\pi}(\widetilde{H}_qX,A) \Rightarrow \widetilde{H}^{p+q}_\pi(X;A).$$

We leave the proof of this, and of the homology analogs, as an exercise.

DEFINITION 4.12. *Let $X \to Y$ be an inclusion in $\mathbf{S}_\pi$. Define relative cohomology by*

$$H^n_\pi(Y, X; A) = \widetilde{H}^n_\pi(Y/X; A).$$

PROPOSITION 4.13. *There is a natural long exact sequence*

$$\cdots \to H^n_\pi(Y, X; A) \to H^n_\pi(Y; A) \to H^n_\pi(X; A) \to H^{n+1}_\pi(Y, X; A) \to \cdots .$$

PROOF: Consider the diagram in $\mathbf{S}_\pi$

$$\begin{tikzcd} U \arrow[r, "i"] \arrow[d, "\simeq"'] & V \arrow[d, "\simeq"] \\ X \arrow[r] & Y \end{tikzcd}$$

where $U \to X$ is a weak equivalence with U cofibrant, $U \xrightarrow{i} V \to Y$ factors $U \to X \to Y$ as a cofibration followed by a weak equivalence. Then $* \to V/U$ is a cofibration and $\widetilde{H}_*(Y/X) \cong H_*(V/U)$, so

$$\widetilde{H}^n_\pi(Y/X; A) \cong \widetilde{H}^n_\pi(V/U, A).$$

Now use the chain description of the various cohomology groups. ■

5. On k-invariants.

This section is devoted to k-invariants — their definition and properties. We will end with a discussion of the uniqueness of Postnikov towers.

Let X be a connected fibrant space and $X = \{X_n\}_{n \geq 0}$ a Postnikov tower for X. Let $\pi_f X_{n-1}$ be the fundamental groupoid for X_{n-1} and π_n the local coefficient system on X_{n-1} obtained as follows: factor $X_n \to X_{n-1}$ as

$$X_n \xrightarrow{i} Y \xrightarrow{q} X_{n-1}$$

with i a weak equivalence and q a fibration. Then if $v \in X_{n-1}$ a vertex, let $F_v = q^{-1}(v)$ and

$$\pi_n(v) = \pi_n(F_v).$$

If $\pi_f X_n = \pi_f X_{n-1}$, as in the case of the Moore-Postnikov tower, there is obviously no ambiguity in $\pi_n(v)$, since the fiber has a single vertex. In general, however, the fiber is simply connected, and for any two choice of vertices x and

y in F_v there is a canonical isomorphism $\pi_n(F_v, x) \cong \pi_n(F_v, y)$, so we identify the two groups via this isomorphism and obtain $\pi_n(F_v)$.

Let $p : L(\pi, n+1) \to K(\pi, n+1)$ be a fixed functorial fibration with $L(\pi, n+1)$ contractible. Setting $L(\pi, n+1) = WK(\pi, n)$ would certainly suffice — this has the advantage that the corresponding fibration $p : L(\pi, n+1) \to K(\pi, n+1)$ is minimal (Lemma III.2.19).

The cohomology class k_n in the statement of the following result is known as the n^{th} k-invariant. follows:

PROPOSITION 5.1. *There is a cohomology class $k_n \in H^n_\Gamma(X_{n-1}, \pi_n)$ and a homotopy pullback diagram over $B\Gamma$*

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & \underrightarrow{\operatorname{holim}}_{\Gamma} L(\pi_n, n+1) \arrow[d] \\ X_{n-1} \arrow[r, "k_n"'] & \underrightarrow{\operatorname{holim}}_{\Gamma} K(\pi_n, n+1). \end{tikzcd}$$

This is a consequence of the more general result Theorem 5.9 below. For now, we state some consequences.

COROLLARY 5.2. *Suppose X is pointed with chosen base point v. Let $\pi_1 = \pi_1(X, v)$. Then there is a homotopy pullback diagram of over $B\pi_1$*

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & \underrightarrow{\operatorname{holim}}_{\pi_1} L(\pi_n(v), n+1) \arrow[d] \\ X_{n-1} \arrow[r, "k_n"'] & \underrightarrow{\operatorname{holim}}_{\pi_1} K(\pi_n(v), n+1). \end{tikzcd}$$

PROOF: This result is equivalent to Proposition 5.1, since there is a homotopy cartesian square

$$\begin{tikzcd} \underrightarrow{\operatorname{holim}}_{\Gamma} L(\pi_n, n+1) \arrow[r, "\simeq"] \arrow[d] & \underrightarrow{\operatorname{holim}}_{\pi_1} L(\pi_n(v), n+1) \arrow[d] \\ \underrightarrow{\operatorname{holim}}_{\Gamma} K(\pi_n, n+1) \arrow[r, "\simeq"'] & \underrightarrow{\operatorname{holim}}_{\pi_1} K(\pi_n(v), n+1). \end{tikzcd}$$

∎

COROLLARY 5.3. *If X is simple and pointed, with chosen base point v. Then there is a homotopy pullback diagram*

$$\begin{array}{ccc} X_n & \longrightarrow & L(\pi_n(v), n+1) \\ \downarrow & & \downarrow \\ X_{n-1} & \longrightarrow & K(\pi_n(v), n+1). \end{array}$$

PROOF: Again, let $\pi_1 = \pi_1(X, v)$ and consider the homotopy pullback diagram that appears in Corollary 5.2. Then

$$\underset{\overrightarrow{\pi_1}}{\operatorname{holim}}\, K(\pi_n(v), n+1) = K(\pi_n(v), n+1) \times B\pi_1$$

as spaces over $B\pi_1$ and similarly for $\underrightarrow{\operatorname{holim}}_{\pi_1} L(\pi_n(v)$, and the diagram

$$\begin{array}{ccc} L(\pi_n(v), n+1) \times B\pi_1 & \xrightarrow{pr_L} & L(\pi_n(v), n+1) \\ {\scriptstyle p \times 1}\Big\downarrow & & \Big\downarrow{\scriptstyle p} \\ K(\pi_n(v), n+1) \times B\pi_1 & \xrightarrow[pr_L]{} & K(\pi_n(v), n+1) \end{array}$$

is homotopy cartesian. ∎

To prove Proposition 5.1, we pass to essentials. Let $f : Y \to X$ be a map of spaces over $B\Gamma$, where Γ is the fundamental groupoid of Y. If we factor f as

$$Y \xrightarrow{j} Z \xrightarrow{q} X$$

where j is a cofibration and q is a weak equivalence, then for each vertex v of Y we get a relative homotopy group

$$\pi_n(f, v) = \pi_n(Z, Y, v) = \pi_{n-1}(F_v, v),$$

where F_v is the fibre of q over v, or in other words the homotopy fibre of $Y \to X$. This gives a local system $\pi_n(f)$ over $B\Gamma$ and, if Γ is equivalent to the fundamental groupoid of X, there is a long exact sequence of local systems

$$\cdots \to \pi_{n+1}(f) \to \pi_n(Y) \xrightarrow{f_*} \pi_n(X) \to \pi_n(f) \to \cdots .$$

LEMMA 5.4. *Suppose $f : Y \to X$ is a morphism of spaces over $B\Gamma$, where Γ is the fundamental groupoid of Y. Suppose $n \geq 2$ and for all choices of base point of Y, the map $f_* : \pi_k(Y, v) \to \pi_k(X, fv)$ is an isomorphism for $k < n$ and a surjection for $k = n$. Then for any local system of abelian groups A over $B\Gamma$, there is an isomorphism*

$$f^* : H^k_\Gamma(X, A) \xrightarrow{\cong} H^k_\Gamma(Y, A), \qquad k < n$$

and an exact sequence

$$\begin{aligned} 0 \to H^n_\Gamma(X, A) \xrightarrow{f^*} H^n_\Gamma(Y, A) \to \hom_\Gamma(\pi_{n+1}(f), A) \\ \xrightarrow{d} H^{n+1}_\Gamma(X, A) \xrightarrow{f^*} H^{n+1}_\Gamma(Y, A). \end{aligned}$$

The sequence is natural in maps f satisfying the hypotheses.

PROOF: The hypotheses on homotopy groups in degree 0 and degree 1 imply that Γ is equivalent to the fundamental groupoid on X. Hence, the long exact sequence of local systems above exists, and the groups $H^*_\Gamma(X, A)$ are defined.

We may assume that $f : Y \to X$ is a cofibration over $B\Gamma$ and use the long exact sequence of 4.2:

$$\cdots \to H^k_\Gamma(X, A) \to H^k_\Gamma(Y, A) \to H^{k+1}_\Gamma(X, Y, A) \to H^{k+1}_\Gamma(X, A) \to \cdots .$$

Note that the results of Section 4 imply

$$H^k_\Gamma(Y, X, A) \cong \tilde{H}^k_\Gamma(B\Gamma \cup_Y X, A) \cong \tilde{H}^k_\pi(Z, A_v)$$

where Z is the universal cover of $B\Gamma \cup_Y X$. The latter group may be computed using Proposition 4.11.2, and one gets $\tilde{H}^k_\pi(Z, A_v) = 0$ for $k \leq n$ and

$$\begin{aligned} \tilde{H}^{n+1}_\pi(Z, A_v) &\cong \hom_{\mathbb{Z}\pi}(\tilde{H}_{n+1} Z, A_v) \\ &\cong \hom_{\mathbb{Z}\pi}(\pi_{n+1}(f, v)), A_v) \\ &\cong \hom_\Gamma(\pi_{n+1}(f), A), \end{aligned}$$

by appropriate use of the relative Hurewicz theorem (Theorem III.3.12). ∎

In particular, in Lemma 5.4, one can set $A = \pi_{n+1}(f)$ and define the *k-invariant* of f to be

(5.5) $$k(f) = d(1_{\pi_{n+1}(f)}) \in H^{n+1}_\Gamma(X, \pi_{n+1}(f)).$$

Remark 5.6. Given the identifications that have been made, the k-invariant $k(f)$ is also represented by the composite

$$\tilde{X} \to \tilde{X}/\tilde{Y} \to \tilde{X}/\tilde{Y}(n+1) \simeq K(\pi_{n+1}(\tilde{X}/\tilde{Y}), n+1) \cong K(\pi_{n+1}(f), n+1)$$

in the homotopy category of Γ-diagrams. The object $\tilde{X}/\tilde{Y}(n+1)$ is meant to denote the $(n+1)^{st}$ Postnikov section of a fibrant model of $\tilde{X}/\tilde{Y}$ — this is sensible, because the Postnikov section construction is functorial. The relative Hurewicz isomorphism is also functorial, so that there is an isomorphism $\pi_{n+1}(\tilde{X}/\tilde{Y}) \cong \pi_{n+1}(f)$ of local coefficient systems.

One possible way to build maps $f : Y \to X$ satisfying the hypotheses of Lemma 5.4 is to kill a cohomology class by the following method. Let X be space with fundamental groupoid Γ and let A be a local coefficient system. Fix $x \in H^{n+1}_{\Gamma}(X, A)$ with $n \geq 2$ and form the pullback diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "p"'] & \underrightarrow{\operatorname{holim}}_{\Gamma} L(A, n+1) \arrow[d] \\ X \arrow[r, "\theta"] & \underrightarrow{\operatorname{holim}}_{\Gamma} K(A, n+1) \end{tikzcd} \tag{5.7}$$

where θ represents x. By construction, $\pi_{n+1}(p) \cong A$ and we have the following result.

Lemma 5.8. *There is an equality* $k(p) = x \in H^{n+1}_{\Gamma}(X, A)$.

Proof: By naturality, we may take θ to the identity, so that f is the projection

$$q : \underrightarrow{\operatorname{holim}}_{\Gamma} L(A, n+1) \to \underrightarrow{\operatorname{holim}}_{\Gamma} K(A, n+1)$$

and x is the universal class. Of course, q is not a projection, but the zero section $s : B\Gamma \to \underrightarrow{\operatorname{holim}}_{\Gamma} L(A, n+1)$ is a weak equivalence, and we may calculate using the composite

$$q \cdot s : B\Gamma \to \underrightarrow{\operatorname{holim}}_{\Gamma} K(A, n+1).$$

Then $B\Gamma \cup_{B\Gamma} \underrightarrow{\operatorname{holim}}_{\Gamma} K(A, n+1)$ has as universal cover

$$Z = \underrightarrow{\operatorname{holim}}_{\Gamma} K(A, n+1) \simeq K(A_v, n+1).$$

Thus

$$\tilde{H}^{n+1}_{\pi}(Z, A_v) \cong \hom(A_v, A_v) \cong \hom_{\Gamma}(A, A)$$

and

$$d : \hom_{\Gamma}(A, A) \to H^{n+1}_{\Gamma}(\underset{\Gamma}{\underrightarrow{\mathrm{holim}}}\, K(A, n+1), A)$$

is the standard isomorphism taking 1_A to the universal class. ■

We point out that in the diagram (5.7) the object $\underrightarrow{\mathrm{holim}}_{\Gamma}\, K(A, n)$ is a group object over $B\Gamma$ and it acts on the space Y over $B\Gamma$; that is, there is a map

$$\mu : \underset{\Gamma}{\underrightarrow{\mathrm{holim}}}\, K(A, n) \times_{B\Gamma} Y \to Y$$

over $B\Gamma$ satisfying the usual associativity and unital conditions. Notice further that this action descends to the trivial action on X in the sense that the following diagram commutes

$$\begin{tikzcd} \underset{\Gamma}{\underrightarrow{\mathrm{holim}}}\, K(A, n) \times_{B\Gamma} Y \arrow[r, "\mu"] \arrow[d, "p_2"'] & Y \arrow[d, "p"] \\ X \arrow[r, "="'] & X \end{tikzcd}$$

where $p_2(a, b) = p(b)$.

Now suppose we are given $f : Y \to X$ satisfying the hypotheses of Lemma 5.4. Then we may form the k-invariant $k(f)$ as in (5.5), and the pullback diagram with $A = \pi_{n+1}(f)$,

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "p"'] & \underset{\Gamma}{\underrightarrow{\mathrm{holim}}}\, L(A, n+1) \arrow[d] \\ X \arrow[r, "k(f)"'] & \underset{\Gamma}{\underrightarrow{\mathrm{holim}}}\, K(A, n+1). \end{tikzcd}$$

Notice that we have confused the cohomology class $k(f)$ with a map representing it. This abuse of notation pervades the literature, and is suggestive: the homotopy type of Z does not depend on the choice of a representative for $k(f)$.

Theorem 5.9. *The map $f : Y \to X$ lifts to a map $g : Y \to Z$ inducing an isomorphism*

$$g_* : \pi_{n+1}(f) \to \pi_{n+1}(p).$$

Proof: Since $f^* k(f)$ is in the image of the composite

$$\hom_\Gamma(A, A) \xrightarrow{d} H^{n+1}_\Gamma(X, A) \xrightarrow{f^*} H^{n+1}_\Gamma(Y, A)$$

the composite $k(f) \cdot f$ is null-homotopic, and hence lifts to $\underrightarrow{\mathrm{holim}}_\Gamma L(A, n+1)$. Choose a lifting, and let $h : Y \to Z$ be the induced map. We wish to modify h, if necessary, to a homotopy equivalence.

By the naturality clause of Lemma 5.4, there is a diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^n_\Gamma(X, A) & \longrightarrow & H^n_\Gamma(Z, A) & \longrightarrow & \hom_\Gamma(A, A) & \xrightarrow{d_Z} & H^{n+1}_\Gamma(X, A) \\
 & & {=}\big\downarrow & & {h^*}\big\downarrow & & {h^\sharp}\big\downarrow & & \big\downarrow{=} \\
0 & \longrightarrow & H^n_\Gamma(X, A) & \longrightarrow & H^n_\Gamma(Y, A) & \longrightarrow & \hom_\Gamma(A, A) & \xrightarrow[d_Y]{} & H^{n+1}_\Gamma(X, A).
\end{array}$$

Note that $d_Y(1_A) = k(f)$ by definition and $d_Z(1_A) = k(f)$ by Lemma 4.5. Furthermore $h^\sharp(1_A) = h_*$ where we write h_* for the composite

$$A = \pi_{n+1}(f) \to \pi_{n+1}(p) \cong A.$$

Since $d_Y(1_A - h_*) = k(f) - k(f) = 0$, there is a class $x \in H^n_\Gamma(Y, A)$ mapping to $1_A - h_*$. Let $g : Y \to Z$ be the composite

$$Y \xrightarrow{x \times h} \underrightarrow{\mathrm{holim}}_{\Gamma} K(A, n) \times_{B\Gamma} Z \xrightarrow{\mu} Z. \qquad \blacksquare$$

Remarks 5.10.

1) Note that the map $g : Y \to Z$ of Theorem 5.9 is not unique, but may be modified by any element of $H^n(X, A)$.

2) The proof of Theorem 5.9 can also be carried out entirely within the Γ-diagram category, and writing it down that way would be a good exercise.

Now let X be space and let $\{X_n\}$ be any Postnikov tower for X. Let $q_n : X_n \to X_{n-1}$ be the projection. Then $\pi_k(q_n) = 0$ $k \neq n+1$ and $\pi_{n+1}(q_n) = \pi_n X$. Thus Proposition 5.1 follows immediately from Theorem 5.9. More generally, let $f : X \to B$ be any map and $\{X_n\}$ a Postnikov tower for f (see Definition 2.9).

Again, let $q_n : X_n \to X_{n-1}$ be the projection. Then $\pi_k(q_n) = 0$ for $k \neq n+1$ and

$$\pi_{n+1}(q_n) = \pi_n(F)$$

by the relative Hurewicz theorem, and one obtains a homotopy pullback square

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & \underrightarrow{\operatorname{holim}}_{\Gamma} L(\pi_n, n+1) \arrow[d] \\ X_{n-1} \arrow[r, "k_n"'] & \underrightarrow{\operatorname{holim}}_{\Gamma} K(\pi_n F, n+1). \end{tikzcd}$$

We now examine the consequences for Postnikov towers. Let X be a fibrant connected space and $\{X_n\}$ a Postnikov tower for X. Let $\pi_n = \pi_n X$ regarded as a local coefficient system as X_{n-1}. Then there is a homotopy cartesian square, with $\Gamma = \pi_f X_{n-1}$

(5.11)
$$\begin{tikzcd} X_n \arrow[r] \arrow[d, "q_n"'] & \underrightarrow{\operatorname{holim}}_{\Gamma} L(\pi_n, n+1) \arrow[d, "p_*"] \\ X_{n-1} \arrow[r, "k_n"'] & \underrightarrow{\operatorname{holim}}_{\Gamma} K(\pi_n, n+1) \end{tikzcd}$$

and X_n is weakly equivalent to the pullback. More is true:

COROLLARY 5.12. *If the tower map $q_n : X_n \to X_{n-1}$ is a minimal fibration then the diagram (5.11) is a pullback.*

PROOF: The map

$$p_* : \underrightarrow{\operatorname{holim}}_{\Gamma} L(\pi_n, n+1) \to \underrightarrow{\operatorname{holim}}_{\Gamma} K(\pi_n, n+1)$$

is a minimal fibration (Lemma 3.2), and so the induced map

$$\begin{tikzcd} X_n \arrow[r, "k_{n*}"] \arrow[dr, "q_n"'] & X_{n-1} \times_{\underrightarrow{\operatorname{holim}}_{\Gamma} K(\pi_n, n+1)} \underrightarrow{\operatorname{holim}}_{\Gamma} L(\pi_n, n+1) \arrow[d] \\ & X_{n-1} \end{tikzcd}$$

is a weak equivalence (and hence a homotopy equivalence) of minimal fibrations over X_{n-1}. Lemma I.10.4 implies that the map k_{n*} is an isomorphism. ■

This applies, for example, to the Moore-Postnikov tower $\{X(n)\}$. In that case $\Gamma = \pi_f X = \pi_f X_n$ for all n. If X is minimal, then $q_n : X(n) \to X(n-1)$ is a minimal fibration.

Corollary 5.13. *Let X be a connected fibrant minimal space. Then for all $n \geq 2$ there is a pullback diagram*

$$\begin{tikzcd}
X(n) \arrow[r] \arrow[d, "q_n"'] & \underrightarrow{\operatorname{holim}}_{\pi} L(\pi_n, n+1) \arrow[d] \\
X(n-1) \arrow[r, "k_n"'] & \underrightarrow{\operatorname{holim}}_{\pi} K(\pi_n, n+1)
\end{tikzcd}$$

where $\pi = \pi_1 X$. If the fundamental group $\pi_1(X)$ acts trivially on $\pi_n(X)$, then the map $q_n : X(n) \to X(n-1)$ is a principal $K(\pi_n, n)$-fibration.

Proof: If X is minimal and connected it has a single vertex, so

$$\underrightarrow{\operatorname{holim}}_{\Gamma} K(\pi_n, n+1) = \underrightarrow{\operatorname{holim}}_{\pi} K(\pi_n, n+1),$$

etc. If π_1 acts trivially on π_n, then

$$p_* : \underrightarrow{\operatorname{holim}}_{\pi} L(\pi_n, n+1) \to \underrightarrow{\operatorname{holim}}_{\pi} K(\pi_n, n+1)$$

can be identified up to isomorphism with the map

$$p \times 1 : L(\pi_n, n+1) \times B\pi \to K(\pi_n, n+1)$$

as in the proof of Corollary 5.3, and $p \times 1$ is a principal $K(\pi_n, n)$-fibration. ■

Theorem 5.14. *Let X be a connected fibrant space. Then any two Postnikov towers for X are weakly equivalent as towers under X.*

Proof: Choose a minimal subcomplex $X_0 \subseteq X$ which is a weak equivalence and let $v \in X_0$ be the vertex. Choose a retraction $g : X \to X_0$. Then the Moore-Postnikov tower $\{X_0(n)\}$ is a Postnikov tower under X. We will show that any Postnikov tower $\{X_n\}$ under X is weakly equivalent (under X) to $\{X_0(n)\}$.

Inductively define tower maps $f_n : X_n \to X_0(n)$ as follows. Let $\pi = \pi_1(X, v)$. Then for $n = 1$ choose a weak equivalence $X_1 \to X_0(1) = B\pi$. Suppose $X_k \to X_0(k)$ have been defined, and are compatible and weak equivalences for $k < n$. Then Theorem 5.9 implies that there is a diagram

$$\begin{tikzcd} & X_0(n) \arrow[r] \arrow[d] & \underrightarrow{\operatorname{holim}}_{\pi} L(\pi_n, n+1) \arrow[d] \\ X_n \arrow[r] \arrow[ur, "f_n"] & X_0(n-1) \arrow[r] & \underrightarrow{\operatorname{holim}}_{\pi} K(\pi_n, n+1) \end{tikzcd}$$

where $\pi_n = \pi_n(X, v)$, and f_n is a weak equivalence. Thus we have a weak equivalence of towers $f : \{X_n\} \to \{X_0(n)\}$. To modify this is into a weak equivalence of towers under X, consider the induced map

$$f : \underleftarrow{\operatorname{holim}} X_n \to \underleftarrow{\lim} X_0(n) = X_0.$$

The canonical map $X \to \underleftarrow{\operatorname{holim}} X_n$ is a weak equivalence, and the induced map $\overline{f} : X \to X_0$ is a weak equivalence. There is an isomorphism $\theta : X_0 \to X_0$ so that

$$X \xrightarrow{\overline{f}} X_0 \xrightarrow{\theta} X_0$$

is g. Since the Moore-Postnikov tower is natural we get an induced isomorphism of towers $\theta : \{X_0(n)\} \to \{X_0(n)\}$ and the composite

$$\theta \cdot f : \{X_n\} \to \{X_0(n)\}$$

is a weak equivalence of towers under X. ■

6. Nilpotent spaces.

We now describe how to refine a Postnikov tower for a nilpotent space. A group G is nilpotent if the lower central series eventually stabilizes at the trivial group. Thus, if we define $F_nG \subseteq G$ by $F_0G = G$ and $F_nG = [F_{n-1}G, G]$, we are asking that there be an integer k so that $F_kG = \{e\}$. If G is a group, a G-module M is nilpotent if there is finite filtration of M by G-modules

$$0 = F_kM \subseteq F_{k-1}M \subseteq \cdots F_1M \subseteq M$$

so that G acts trivially on the successive quotients. A simplicial set X is *nilpotent* if X is connected, π_1X is a nilpotent group and π_nX is nilpotent π_1X module for $n \geq 2$.

PROPOSITION 6.1. *Let X be a nilpotent space and $\{X_n\}$ a Postnikov tower for X. Each of the maps $q_n : X_n \to X_{n-1}$ can be refined to a finite composition*

$$X_n = Y_k \to Y_{k-1} \to \cdots Y_1 \to Y_0 = X_{n-1}$$

so that each of the maps $Y_i \to Y_{i-1}$ fits into a homotopy pullback square

$$\begin{tikzcd} Y_i \arrow[r] \arrow[d] & L(A_i, n+1) \arrow[d] \\ Y_{i-1} \arrow[r] & K(A_i, n+1) \end{tikzcd}$$

for some abelian group A_i

PROOF: We begin with the case $n = 1$. Then $X_1 \to X_0$ is weakly equivalent to $B\pi_1 \to *$ where $\pi_1 = \pi_1 X$. Let $\{F_i\}$ be the lower central series of $\pi_1 X$ and let $G_i = \pi_1/F_i$. Let k be an integer so that $F_k = \{e\}$. Then there is a tower

$$B\pi_1 = BG_k \to BG_{k-1} \to \cdots BG_1 \to *.$$

Since

$$\{e\} \to F_{i-1}/F_i \to G_i \to G_{i-1} \to \{e\}$$

is a central extension, there is homotopy pullback diagram, with $A_i = F_{i-1}/F_i$,

$$\begin{tikzcd} BG_i \arrow[r] \arrow[d] & L(A_i, 2) \arrow[d] \\ BG_{i-1} \arrow[r] & K(A_i, 2). \end{tikzcd}$$

In effect, the map $BG_{i-1} \to K(A_i, 2)$ classifies the principal BA_i-fibration $BG_i \to BG_{i-1}$.

Next we assume that $n \geq 2$. Let $\{F_i\}$ be a filtration of $\pi_n X$ by π_1 modules so that each of the successive quotients is a trivial π_1 module. We define Y_i by the homotopy pullback diagram

$$\begin{tikzcd} Y_i \arrow[r] \arrow[d] & \underrightarrow{\operatorname{holim}}_{\pi_1} K(F_i, n+1) \arrow[d] \\ X_{n-1} \arrow[r] & \underrightarrow{\operatorname{holim}}_{\pi_1} K(\pi_n X, n+1) \end{tikzcd}$$

where the bottom map is the k-invariant. Since there is a homotopy pullback diagram

$$\begin{tikzcd}
\underset{\overrightarrow{\pi_1}}{\mathrm{holim}}\, K(F_i, n+1) \arrow[r] \arrow[d] & \underset{\overrightarrow{\pi_1}}{\mathrm{holim}}\, L(F_{i-1}/F_i, n+1) \arrow[d] \\
\underset{\overrightarrow{\pi_1}}{\mathrm{holim}}\, K(F_{i-1}, n+1) \arrow[r] & \underset{\overrightarrow{\pi_1}}{\mathrm{holim}}\, K(F_{i-1}/F_i, n+1).
\end{tikzcd}$$

there is a homotopy pullback diagram

$$\begin{tikzcd}
Y_i \arrow[r] \arrow[d] & \underset{\overrightarrow{\pi_1}}{\mathrm{holim}}\, L(F_{i-1}/F_i, n+1) \arrow[d] \\
Y_{i-1} \arrow[r] & \underset{\overrightarrow{\pi_1}}{\mathrm{holim}}\, K(F_{i-1}/F_i, n+1).
\end{tikzcd}$$

Since π_1 acts trivially on F_{i-1}/F_i, one can finish the proof by arguing as in Proposition 5.3. ■

The previous result has the following consequence.

COROLLARY 6.2. *Let X be a nilpotent space. Then there is a tower of fibrations $\{Z_j\}$ so that $X \simeq \varprojlim Z_j$ and each of the maps $Z_j \to Z_{j-1}$ fits into a homotopy pullback square*

$$\begin{tikzcd}
Z_j \arrow[r] \arrow[d] & L(A_j, n_j+1) \arrow[d] \\
Z_{j-1} \arrow[r] & K(A_j, n_j+1)
\end{tikzcd}$$

where each A_j is an abelian group and $\{n_j\}$ is a non-decreasing sequence of positive integers so that $\varinjlim n_j = \infty$

Further variations on this result are possible. For example, one can remove the hypothesis that X be connected, or consider nilpotent fibrations. It is also possible to construct this refined Postnikov tower by a method similar to the construction of the Moore-Postnikov tower. See Bousfield and Kan [14].