Chapter V Simplicial groups

This is a somewhat complex chapter on the homotopy theory of simplicial groups and groupoids, divided into seven sections. Many ideas are involved. Here is a thumbnail outline:

Section 1, *Skeleta*: Skeleta for simplicial sets were introduced briefly in Chapter I, and then discussed more fully in the context of the Reedy closed model structure for bisimplicial sets in Section IV.3.2. Skeleta are most precisely described as Kan extensions of truncated simplicial sets. The current section gives a general description of such Kan extensions in a more general category $\mathcal{C}$, followed by a particular application to a description of the skeleta of almost free morphisms of simplicial groups. The presentation of this theory is loosely based on the Artin-Mazur treatment of hypercovers of simplicial schemes [3], but the main result for applications that appear in later sections is Proposition 1.9. This result is used to show in Section 5 that the loop group construction outputs cofibrant simplicial groups.

Section 2, *Principal Fibrations I: Simplicial G-spaces*: The main result of this section asserts that the category $\mathbf{S}_G$ of simplicial sets admitting an action by a fixed simplicial group G admits a closed model structure: Theorem 2.3. Principal G-fibrations in the classical sense may then be identified with cofibrant objects of $\mathbf{S}_G$, by Corollary 2.10, and an equivariant map between two such objects is an isomorphism if and only if it induces an isomorphism of coinvariants (Lemma 2.9).

Section 3, *Principal Fibrations II: Classifications*: This section shows (Theorem 3.9) that isomorphism classes of principal G-fibrations $p : E \to B$ can be classified by homotopy classes of maps $B \to BG$, where $BG = EG/G$, and EG is an arbitrary cofibrant object of $\mathbf{S}_G$ admitting a trivial fibration $EG \to *$, all with respect to the the closed model structure for $\mathbf{S}_G$ of Section 2.

Section 4, *Universal cocycles and $\overline{W}G$*: It is shown here that the classical model $\overline{W}G$ for the classifying object BG of Section 3 can be constructed as a simplicial set of cocycles taking values in the simplicial group G. This leads to "global" descriptions of the simplicial structure maps for $\overline{W}G$, as well as for the G-bundles associated to simplicial set maps $X \to \overline{W}G$. The total space WG for the (canonical) bundle associated to the identity map on $\overline{W}G$ is contractible (Lemma 4.6).

Section 5, *The loop group construction*: The functor $G \mapsto \overline{W}G$ has a left adjoint $X \mapsto GX$, defined on reduced simplicial sets X (Lemma 5.3). The simplicial group GX is the loop group of the reduced simplicial set X, in the sense that the total space of the bundle associated to the adjunction map $X \to \overline{W}GX$ is contractible: this is Theorem 5.10. The proof of this theorem is a modernized version of the Kan's original geometric proof, in that it involves a reinterpretation of the loop group GX as an object constructed from equivalence classes of loops.

Section 6, *Reduced simplicial sets, Milnor's FK-construction*: This section gives a closed model structure for the category $\mathbf{S}_0$ of reduced simplicial sets. This structure is used to show (in conjunction with the results of Section 1) that the loop group functor preserves cofibrations and weak equivalences, and that $\overline{W}$ preserves fibrations and weak equivalences (Proposition 6.3). In particular, the loop group functor and the functor $\overline{W}$ together induce an equivalence between the homotopy categories associated to the categories of reduced simplicial sets and simplicial groups (Corollary 6.4). Furthermore, any space of the form $\overline{W}G$ is a Kan complex (Corollary 6.8); this is the last piece of the proof of the assertion that $\overline{W}G$ is a classifying space for the simplicial group G, as defined in Section 3. Milnor's FK-construction is a simplicial group which gives a fibrant model for the space $\Omega\Sigma K$: Theorem 6.15 asserts that FK is a copy of $G(\Sigma K)$, by which is meant the loop group of the Kan suspension of K. The Kan suspension was introduced in Section III.5.

Section 7, *Simplicial groupoids*: The main result of Section 5, which leads to the equivalence of homotopy theories between reduced simplicial sets and simplicial groups of Section 6, fails badly for non-reduced simplicial sets. We can nevertheless recover an analogous statement for the full category of simplicial sets if we replace simplicial groups by simplicial groupoids, by a series of results of Dwyer and Kan. This theory is presented in this section. There is a closed model structure on the category $s\mathbf{Gd}$ of simplicial *groupoids* (Theorem 7.6) whose associated homotopy category is equivalent to that of the full simplicial set category (Corollary 7.11). The classifying object and loop group functors extend, respectively, to functors $\overline{W} : s\mathbf{Gd} \to \mathbf{S}$ and $G : \mathbf{S} \to s\mathbf{Gd}$; the object $\overline{W}A$ associated to a simplicial groupoid A is a simplicial set of cocycles in a way that engulfs the corresponding object for simplicial groups, and the extended functor G is its left adjoint.

1. Skeleta.

Suppose that $\mathcal{C}$ is a category having all finite coproducts, and let $s\mathcal{C}$ denote the category of simplicial objects in $\mathcal{C}$. Recall that simplicial objects in $\mathcal{C}$ are contravariant functors of the form $\mathbf{\Delta}^{op} \to \mathcal{C}$, defined on the ordinal number category $\mathbf{\Delta}$.

The ordinal number category contains a full subcategory $\mathbf{\Delta}_n$, defined on the objects $\mathbf{m}$ with $0 \leq m \leq n$. Any simplicial object $X : \mathbf{\Delta}^{op} \to \mathcal{C}$ restricts to a contravariant functor $i_{n*}X : \mathbf{\Delta}_n^{op} \to \mathcal{C}$, called the *$n$-truncation of X*. More generally, an *n-truncated simplicial object* in $\mathcal{C}$ is a contravariant functor $Y : \mathbf{\Delta}_n^{op} \to \mathcal{C}$, and the category of such objects (functors and natural transformations between them) will be denoted by $s_n\mathcal{C}$.

The n-truncation functor $s\mathcal{C} \to s_n\mathcal{C}$ defined by $X \mapsto i_{n*}X$ has a left adjoint $i_n^* : s_n\mathcal{C} \to s\mathcal{C}$, on account of the exactness assumption on the category $\mathcal{C}$. Explicitly, the theory of left Kan extensions dictates that, for an n-truncated object Y, $i_n^*Y_m$ should be defined by

$$i_n^*Y_m = \varinjlim_{\mathbf{m} \xrightarrow{\theta} \mathbf{i},\ i \leq n} Y_i.$$

As the notation indicates, the colimit is defined on the finite category whose objects are ordinal number morphisms $\theta : \mathbf{m} \to \mathbf{i}$ with $i \leq n$, and whose morphisms $\gamma : \theta \to \tau$ are commutative diagrams

$$\begin{tikzcd} & \mathbf{i} \arrow[dd, "\gamma"] \\ \mathbf{m} \arrow[ur, "\theta"] \arrow[dr, "\tau"'] & \\ & \mathbf{j} \end{tikzcd}$$

in the ordinal number category. The simplicial structure map $\omega^* : i_n^*Y_m \to i_n^*Y_k$ is defined on the index category level in the obvious way, by precomposition with the morphism $\omega : \mathbf{k} \to \mathbf{m}$.

The functor $Y \mapsto i_n^*Y$ is left adjoint to the n-truncation functor: this can be seen by invoking the theory of Kan extensions, or directly.

If $m \leq n$, then the index category of arrows $\mathbf{m} \to \mathbf{i}$, $i \leq n$, has an initial object, namely $1_{\mathbf{m}} : \mathbf{m} \to \mathbf{m}$, so that the canonical map

$$Y_m \xrightarrow{in_{1_{\mathbf{m}}}} \varinjlim_{\mathbf{m} \xrightarrow{\theta} \mathbf{i},\ i \leq n} Y_i$$

is an isomorphism by formal nonsense. Furthermore, maps of this form in $\mathcal{C}$ are the components of the adjunction map

$$Y \xrightarrow{\eta} i_{n*}i_n^*Y,$$

so that this map is an isomorphism of $s_n\mathcal{C}$.

The objects $i_n^*Y_m$, $m > n$, require further analysis. The general statement that is of the most use is the following:

LEMMA 1.1. *There is a coequalizer diagram*

$$\bigsqcup_{i<j} Y_{n-1} \rightrightarrows \bigsqcup_{i=0}^{n} Y_n \xrightarrow{s} i_n^* Y_{n+1},$$

where the maps in the coequalizer are defined by the commutativity of the following diagram:

$$\begin{tikzcd}
Y_{n-1} \arrow[r, "s_i"] \arrow[d, "in_{i<j}"'] & Y_n \arrow[d, "in_j"'] \arrow[dr, "in_{sj}"] & \\
\bigsqcup_{i<j} Y_{n-1} \arrow[r, shift left] \arrow[r, shift right] & \bigsqcup_{i=0}^{n} Y_n \arrow[r, "s"'] & i_n^* Y_{n+1}. \\
Y_{n-1} \arrow[r, "s_{j-1}"'] \arrow[u, "in_{i<j}"] & Y_n \arrow[u, "in_i"] &
\end{tikzcd}$$

PROOF: Write $\mathbf{D}$ for the category of ordinal number morphisms $\theta : \mathbf{n+1} \to \mathbf{j}$, $j \leq n$. Suppose that $t : \mathbf{n+1} \to \mathbf{i}$ is an ordinal number epimorphism, where $i \leq n$, and write $\mathbf{D}_t$ for the category of ordinal number morphisms $\theta : \mathbf{n+1} \to \mathbf{j}$, $j \leq n$, which factor through t. Then $\mathbf{D}_t$ has an initial object, namely t, so that the canonical map in_t induces an isomorphism

$$Y_i \xrightarrow[\cong]{in_t} \varinjlim_{\mathbf{n+1} \xrightarrow{\theta} \mathbf{j} \ \in \mathbf{D}_t} Y_j$$

Furthermore, if t has a factorization

$$\begin{tikzcd}
\mathbf{n+1} \arrow[rr, "t"] \arrow[dr, "r"'] & & \mathbf{i} \\
& \mathbf{m} \arrow[ur, "s"'] &
\end{tikzcd}$$

where r and s are ordinal number epimorphisms, the inclusion $\mathbf{D}_t \subset \mathbf{D}_r$ induces a morphism s^* of colimits which fits into a commutative diagram

$$\begin{tikzcd}
Y_i \arrow[r, "s^*"] \arrow[d, "in_t"', "\cong"] & Y_m \arrow[d, "\cong"', "in_r"] \\
\varinjlim_{\mathbf{n+1} \to \mathbf{j} \in \mathbf{D}_t} Y_j \arrow[r, "s^*"'] & \varinjlim_{\mathbf{n+1} \to \mathbf{j} \in \mathbf{D}_r} Y_j .
\end{tikzcd}$$

Write $\mathbf{D}_j$ for the category $\mathbf{D}_{s^j}$, $0 \leq j \leq n$.

For $i < j$, the diagram

$$\begin{tikzcd}
\mathbf{n+1} \arrow[r, "s^j"] \arrow[d, "s^i"'] & \mathbf{n} \arrow[d, "s^i"] \\
\mathbf{n} \arrow[r, "s^{j-1}"'] & \mathbf{n-1}
\end{tikzcd} \tag{1.2}$$

is a pushout in the ordinal number category: this is checked by fiddling with simplicial identities. Now, suppose given a collection of maps

$$f_j : \varinjlim_{\mathbf{n+1} \xrightarrow{\theta} \mathbf{i} \in \mathbf{D}_j} Y_i \to X,$$

$0 \leq j \leq n$, such that the diagrams

$$\begin{tikzcd}
\varinjlim_{\mathbf{n+1} \to \mathbf{i} \in \mathbf{D}_t} Y_i \arrow[r, "s_{j-1}"] \arrow[d, "s_i"'] & \varinjlim_{\mathbf{n+1} \to \mathbf{i} \in \mathbf{D}_{s^i}} Y_i \arrow[d, "f_i"] \\
\varinjlim_{\mathbf{n+1} \to \mathbf{i} \in \mathbf{D}_{s^j}} Y_i \arrow[r, "f_j"'] & X
\end{tikzcd} \tag{1.3}$$

commute, where $t = s^i s^j = s^{j-1} s^i$. Let $\theta : \mathbf{n+1} \to \mathbf{k}$ be an object of $\mathbf{D}$. Then $\theta \in \mathbf{D}_i$ for some i, and we define a morphism $f_\theta : Y_k \to X$ to be the composite

$$Y_k \xrightarrow{in_\theta} \varinjlim_{\mathbf{n+1} \to \mathbf{k}\ \in \mathbf{D}_i} Y_k \xrightarrow{f_i} X.$$

It's easily seen, using the pushout diagram (1.2) and the commutativity conditions (1.3), that the definition of f_θ is independent of i. The collection of maps f_θ, $\theta \in \mathbf{D}$, determine a unique map

$$f_* : \varinjlim_{\mathbf{n+1} \to \mathbf{k}\ \in \mathbf{D}} Y_k$$

which restricts to the maps f_i, for $0 \le i \le n$, and the lemma is proved. ■

Write $\operatorname{sk}_n Y = i_n^* i_{n*} Y$, and write $\epsilon : \operatorname{sk}_n Y \to Y$ for the counit of the adjunction. $sk_n Y$ is the n-skeleton of Y.

Lemma 1.4. *Let Y be a simplicial object in the category $\mathcal{C}$, and suppose that there is a morphism $f : N \to Y_{n+1}$ such that the canonical map $\epsilon : \operatorname{sk}_n Y \to Y$ and f together induce an isomorphism*

$$\operatorname{sk}_n Y_{n+1} \bigsqcup N \xrightarrow[\cong]{(\epsilon, f)} Y_{n+1}.$$

Then an extension of a map $g : \operatorname{sk}_n Y \to Z$ to a map $g' : \operatorname{sk}_{n+1} Y \to Z$ corresponds to a map $\tilde{g} : N \to Z_{n+1}$ such that $d_i \tilde{g} = g d_i f$ for $0 \le i \le n+1$.

Proof: Given such a map $\tilde{g}$, define a map

$$g' : Y_{n+1} \cong \operatorname{sk}_n Y_{n+1} \bigsqcup N \to Z_{n+1}$$

by $g' = (g, \tilde{g})$. In effect, we are looking to extend a map $g : i_{n*} Y \to i_{n*} Z$ to a map $g' : i_{(n+1)*} Y \to i_{(n+1)*} Z$. The truncated map g' will be the map g' in degree $n+1$ and will coincide with the map g in degrees below $n+1$, once we show that g' respects simplicial identities in the sense that the following diagram commutes:

$$\begin{tikzcd} Y_{n+1} \arrow[r, "g'"] \arrow[d, shift right, "\gamma^*"'] & Z_{n+1} \arrow[d, shift right, "\gamma^*"'] \\ Y_m \arrow[r, "g"'] \arrow[u, shift right, "\theta^*"'] & Z_m \arrow[u, shift right, "\theta^*"'] \end{tikzcd}$$

for all ordinal number maps $\gamma : \mathbf{m} \to \mathbf{n+1}$ and $\theta : \mathbf{n+1} \to \mathbf{m}$, where $m < n+1$. The canonical map $\epsilon : \mathrm{sk}_n Y \to Y$ consists of isomorphisms

$$\mathrm{sk}_n Y_i \xrightarrow[\cong]{\epsilon} Y_i$$

in degrees $i \leq n$, so that $\theta^* : Y_m \to Y_{n+1}$ factors through the map $\epsilon : \mathrm{sk}_n Y_{n+1} \to Y_{n+1}$; the restriction of g' to $\mathrm{sk}_n Y_{n+1}$ is a piece of a simplicial map, so that g' respects θ^*. The map γ^* factors through some face map d_i, so it's enough to show that g' respects the face maps, but this is automatic on $\mathrm{sk}_n Y_{n+1}$ and is an assumption on $\tilde{g}$.

The converse is obvious. ■

LEMMA 1.5. *Supppose that $i : A \to B$ is a morphism of $s_{n+1}\mathcal{C}$ which is an isomorphism in degrees $j \leq n$. Suppose further that there is a morphism $f : N \to B_{n+1}$ such that the maps i and f together determine an isomorphism*

$$A_{n+1} \bigsqcup N \xrightarrow[\cong]{(i,f)} B_{n+1}.$$

Suppose that $g : A \to Z$ is a morphism of $s_{n+1}\mathcal{C}$. Then extensions

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "i"'] & Z \\ B \arrow[ur, "g'"'] & \end{tikzcd}$$

of the morphism g to morphisms $g' : B \to Z$ are in one to one correspondence with morphisms $\tilde{g} : N \to Z_{n+1}$ of $\mathcal{C}$ such that $d_i\tilde{g} = gd_if$ for $0 \leq i \leq n$.

PROOF: This lemma is an abstraction of the previous result. The proof is the same. ■

A morphism $j : G \to H$ of simplicial groups is said to be *almost free* if there is a contravariant set-valued functor X defined on the epimorphisms of the ordinal number category $\mathbf{\Delta}$ such that there are isomorphisms

$$G_n * F(X_n) \xrightarrow[\cong]{\theta_n} H_n$$

which

(1) are compatible with the map j in the sense that $\theta_n \circ in_{G_n} = j_n$ for all n, and

(2) respect the functorial structure of X in the sense that the diagram

$$\begin{tikzcd} G_n * F(X_n) \arrow[r, "\theta_n"] \arrow[d, "t^* * F(t^*)"'] & H_n \arrow[d, "t^*"] \\ G_m * F(X_m) \arrow[r, "\theta_m"'] & H_m \end{tikzcd}$$

commutes for every ordinal number epimorphism $t : \mathbf{m} \to \mathbf{n}$.

The *n-skeleton* $\mathrm{sk}_n(j)$ *of the simplicial group homomorphism* $j : G \to H$ is defined by the pushout diagram

$$\begin{tikzcd} \mathrm{sk}_n\, G \arrow[r, "j_*"] \arrow[d] & \mathrm{sk}_n\, H \arrow[d] \\ G \arrow[r] & \mathrm{sk}_n(j) \end{tikzcd}$$

in the category of groups. There are maps $\mathrm{sk}_n(j) \to \mathrm{sk}_{n+1}(j)$ and morphisms $\mathrm{sk}_n(i) \to H$ such that the diagrams

$$\begin{tikzcd} & G \arrow[dl] \arrow[dr] & \\ \mathrm{sk}_n(j) \arrow[rr] \arrow[dr] & & \mathrm{sk}_{n+1}(j) \arrow[dl] \\ & H & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G \arrow[r] \arrow[dr, "j"'] & \mathrm{sk}_n(j) \arrow[d] \\ & H \end{tikzcd}$$

commute, and such that $j : G \to H$ is a filtered colimit of the maps $G \to \mathrm{sk}_n(j)$ in the category of simplicial groups under G. The maps $\mathrm{sk}_n(j)_i \to H_i$ are group isomorphisms for $i \leq n$, so the map $\mathrm{sk}_n(j) \to \mathrm{sk}_{n+1}(j)$ consists of group isomorphisms in degrees up to n.

Write DX_n for the degenerate part of X_{n+1}. This subset can be described (as usual) as the union of the images of the functions $s_i : X_n \to X_{n+1}$, $0 \leq i \leq n$. For $i < j$ the diagram of group homomorphisms

$$\begin{tikzcd} H_{n-1} \arrow[r, "s_i"] \arrow[d, "s_{j-1}"'] & H_n \arrow[d, "s_j"] \\ H_n \arrow[r, "s_i"'] & H_{n+1} \end{tikzcd} \tag{1.6}$$

is a pullback, by manipulating the simplicial identities. Pullback diagrams are closed under retraction, so the diagram of group homomorphisms

$$\begin{tikzcd} F(X_{n-1}) \arrow[r, "s_i"] \arrow[d, "s_{j-1}"'] & F(X_n) \arrow[d, "s_j"] \\ F(X_n) \arrow[r, "s_i"'] & F(X_{n+1}) \end{tikzcd} \tag{1.7}$$

is also a pullback. All the homomorphisms in (1.7) are monomorphisms (since they are retracts of such), so an argument on reduced words shows that (1.7) restricts on generators to a pullback

$$\begin{tikzcd} X_{n-1} \arrow[r, "s_i"] \arrow[d, "s_{j-1}"'] & X_n \arrow[d, "s_j"] \\ X_n \arrow[r, "s_i"'] & X_{n+1} \end{tikzcd} \tag{1.8}$$

in the set category. It follows that the degenerate part DX_n of the set X_{n+1} can be defined by a coequalizer

$$\bigsqcup_{i<j} X_{n-1} \rightrightarrows \bigsqcup_{i=0}^{n} X_n, \xrightarrow{s} DX_n$$

such as one would expect if X were part of the data for a simplicial set, in which case DX_n would be a copy of $\mathrm{sk}_n X_{n+1}$.

It therefore follows from Lemma 1.1 that the diagram of group homomorphisms

$$\begin{tikzcd} \mathrm{sk}_n G_{n+1} \arrow[r] \arrow[d] & \mathrm{sk}_n H_{n+1} \arrow[d] \\ G_{n+1} \arrow[r] & \mathrm{sk}_n(i)_{n+1} \end{tikzcd}$$

can be identified up to canonical isomorphism with the diagram

$$\begin{tikzcd} \mathrm{sk}_n G_{n+1} \arrow[r] \arrow[d] & \mathrm{sk}_n G_{n+1} * F(DX_n) \arrow[d] \\ G_{n+1} \arrow[r] & G_{n+1} * F(DX_n). \end{tikzcd}$$

The map $\mathrm{sk}_n(i)_{n+1} \to \mathrm{sk}_{n+1}(i)_{n+1}$ can therefore be identified up to isomorphism with the monomorphism

$$G_{n+1} * F(DX_n) \to G_{n+1} * F(X_{n+1})$$

which is induced by the inclusion $DX_n \subset X_{n+1}$.

Let $NX_{n+1} = X_{n+1} - DX_n$ be the non-degenerate part of X_{n+1}. The truncation at level $n+1$ of the map $\mathrm{sk}_n(j) \to \mathrm{sk}_{n+1}(j)$ is an isomorphism in degrees up to n, and is one of the components of an isomorphism

$$\mathrm{sk}_n(j)_{n+1} * F(NX_{n+1}) \cong \mathrm{sk}_{n+1}(j)_{n+1}.$$

in degree $n+1$.

PROPOSITION 1.9. *Suppose that $j : G \to H$ is an almost free simplicial group homomorphism, with H generated over G by the functor X as described above. Let NX_{n+1} be the non-degenerate part of X_{n+1}. Then there is pushout diagram of simplicial groups of the form*

$$\begin{tikzcd} \displaystyle\mathop{*}_{x \in NX_{n+1}} F(\partial\Delta^{n+1}) \arrow[r] \arrow[d] & \mathrm{sk}_n(j) \arrow[d] \\ \displaystyle\mathop{*}_{x \in NX_{n+1}} F(\Delta^{n+1}) \arrow[r] & \mathrm{sk}_{n+1}(j) \end{tikzcd}$$

for each $n \geq -1$.

Corollary 1.10. *Any almost free simplicial group homomorphism $j : G \to H$ is a cofibration of simplicial groups.*

Proof of Proposition 1.9: Any commutative diagram

$$\begin{tikzcd} \mathop{*}\limits_{x \in NX_{n+1}} F(\partial\Delta^{n+1}) \arrow[r] \arrow[d] & \mathrm{sk}_n(j) \arrow[d, "\alpha"] \\ \mathop{*}\limits_{x \in NX_{n+1}} F(\Delta^{n+1}) \arrow[r, "\beta"'] & K \end{tikzcd}$$

of simplicial group homomorphisms uniquely determines an morphism of $(n+1)$-truncations $\tilde{\gamma} : i_{(n+1)*}\,\mathrm{sk}_{n+1}(j) \to i_{(n+1)*}K$ which makes the following diagram commute:

$$\begin{tikzcd} & i_{(n+1)*}G \arrow[dl, "j"'] \arrow[d] \arrow[dr, "i_{(n+1)*}(\alpha|_G)"] & \\ & i_{(n+1)*}\,\mathrm{sk}_n(j) \arrow[d] \arrow[dr, "i_{(n+1)*}\alpha"] & \\ i_{(n+1)*}H & i_{(n+1)*}\,\mathrm{sk}_{n+1}(j) \arrow[l, "\cong"] \arrow[r, "\tilde{\gamma}"'] & i_{(n+1)*}K \end{tikzcd}$$

The homomorphisms $\alpha|_G : G \to K$ and $\tilde{\gamma}$ therefore uniquely determine a morphism $\gamma : \mathrm{sk}_{n+1}(j) \to K$. ∎

2. Principal fibrations I: simplicial G-spaces.

A principal fibration is one in which the fibre is a simplicial group acting in a particular way on the total space. They will be defined completely below and we will classify them, but it simplifies the discussion considerably if we discuss more general actions first.

DEFINITION 2.1. *Let G be a simplicial group and X a simplicial set. Then G acts on X if there is a morphism of simplicial sets*

$$\mu : G \times X \to X$$

so that the following diagrams commute:

$$\begin{tikzcd}
G \times G \times X \arrow[r, "1 \times \mu"] \arrow[d, "m \times 1"'] & G \times X \arrow[d, "\mu"] \\
G \times X \arrow[r, "\mu"'] & X
\end{tikzcd}$$

and

$$\begin{tikzcd}
X \arrow[d, "i"'] \arrow[dr, "1_X"] & \\
G \times X \arrow[r, "\mu"'] & X
\end{tikzcd}$$

where m is the multiplication in G and $i(X) = (e, X)$.

In other words, at each level, X_n is a G_n-set and the actions are compatible with the face and degeneracy operators.

Let $\mathbf{S}_G$ be the category of simplicial sets with G-action, hereinafter known as G-spaces. Note that $\mathbf{S}_G$ is a simplicial category. Indeed, if $K \in \mathbf{S}$, then K can be given the trivial G-action. Then for $X \in \mathbf{S}_G$ set

$$X \otimes K = X \times K \tag{2.2.1}$$

with diagonal action,

$$\mathbf{hom}_{\mathbf{S}_G}(K, X) = \mathbf{Hom}_{\mathbf{S}}(K, X) \tag{2.2.2}$$

with action in the target, and for X and Y in $\mathbf{S}_G$,

$$\mathbf{Hom}_{\mathbf{S}_G}(X, Y)_n = \hom_{\mathbf{S}_G}(X \otimes \Delta^n, Y). \tag{2.2.3}$$

Then the preliminary result is:

THEOREM 2.3. *There is a simplicial model category structure on $\mathbf{S}_G$ such that $f : X \to Y$ is*

1) *a weak equivalence if and only if f is a weak equivalence in $\mathbf{S}$;*
2) *a fibration if and only if f is a fibration in $\mathbf{S}$; and*
3) *a cofibration if and only if f has the left lifting property with respect to all trivial fibrations.*

PROOF: The forgetful functor $\mathbf{S}_G \to S$ has a left adjoint given by

$$X \mapsto G \times X.$$

Thus we can apply Theorem II.7.8 once we show that every cofibration having the left lifting property with respect to all fibrations is a weak equivalence. Every morphism $X \to Y$ can be factored as $X \xrightarrow{j} Z \xrightarrow{q} X$ where q is a fibration and j is obtained by setting $Z = \varinjlim_n Z_n$ with $Z_0 = X$ and X_n defined by a pushout diagram

$$\begin{array}{ccc} \bigsqcup_\alpha G \times \Lambda^n_k & \longrightarrow & Z_{n-1} \\ {\scriptstyle i_n}\downarrow & & \downarrow{\scriptstyle j_n} \\ \bigsqcup_\alpha G \times \Delta^n & \longrightarrow & Z_n \end{array}$$

where α runs over all diagrams in $\mathbf{S}_G$

$$\begin{array}{ccc} G \times \Lambda^n_k & \longrightarrow & Z_{n-1} \\ {\scriptstyle i_n}\downarrow & & \downarrow \\ G \times \Delta^n & \longrightarrow & Y. \end{array}$$

Since i_n is a trivial cofibration in $\mathbf{S}$, we have that j_n a trivial cofibration in $\mathbf{S}$ (and also in $\mathbf{S}_G$). So $j : X \to Z$ is a trivial cofibration in $\mathbf{S}$ (and $\mathbf{S}_G$). ∎

A crucial structural fact about $\mathbf{S}_G$ is the following:

LEMMA 2.4. *Let $f : X \to Y$ be a cofibration in $\mathbf{S}_G$. Then f is an inclusion and at each level $Y_k - f(X_k)$ is a free G_k-set.*

PROOF: Every cofibration is a retract of a cofibration $j : X \to Z$ where $Z = \varinjlim Z_n$ and Z_n is defined recursively by setting $Z_0 = X$ and defining Z_n by a pushout diagram

$$\begin{array}{ccc} \coprod_\alpha G \times \partial\Delta^n & \longrightarrow & Z_{n-1} \\ \downarrow & & \downarrow j_n \\ \coprod_\alpha G \times \Delta^n & \longrightarrow & Z_n. \end{array}$$

So it is sufficient to prove the result for these more specialized cofibrations. Now each j_n is an inclusion, so $j : X \to Z$ is an inclusion. Also, at each level, we have a formula for k simplices

$$(Z_n)_k - (Z_{n-1})_k = (\amalg_\alpha G \times \Delta^n)_k - (\amalg_\alpha G \times \partial\Delta^n)_k$$

is free. Hence

$$(Z)_k - (X_k) = \bigcup_n (Z_n)_k - (Z_{n-1})_k$$

is free. ∎

For $X \in \mathbf{S}_G$, let X/G be the quotient space by the G-action. Let $q : X \to X/G$ be the quotient map. If $X \in \mathbf{S}_G$ is cofibrant this map has special properties.

LEMMA 2.5. *Let $X \in \mathbf{S}_G$ have the property that X_n is a free G_n set for all n. Let $x \in (X/G)_n$ be an n-simplex. If $f_x : \Delta^n \to X$ represents x, define F_x by the pullback diagram*

$$\begin{array}{ccc} F_x & \longrightarrow & X \\ \downarrow & & \downarrow q \\ \Delta^n & \xrightarrow[f_x]{} & X/G. \end{array}$$

Then for every $z \in X$ so that $q(z) = x$, there is an isomorphsim in $\mathbf{S}_G$

$$\varphi_z : G \times \Delta^n \to F_x$$

so that the following diagram commutes

$$\begin{array}{ccc} G \times \Delta^n & \xrightarrow{\varphi_z} & F_x \\ {\scriptstyle \pi_2}\downarrow & & \downarrow \\ \Delta^n & \xrightarrow[=]{} & \Delta^n. \end{array}$$

Proof: First note that there is a natural G-action on F_x so that $F_x \to X$ is a morphism of G-spaces. Fix $z \in X_n$ so that $q(z) = x$. Now every element of Δ^n can be written uniquely as $\theta^* c_n$ where $c_n \in \Delta^n_n$ is the canonical n-simplex and $\theta : \mathbf{m} \to \mathbf{n}$ is an ordinal number map. Define φ_z by the formula, for $g \in G_m$:

$$\varphi_z(g, \theta^* c_n) = (\theta^* c_n, g\theta^* z).$$

One must check this is a simplicial G-map. Having done so, diagram (1.5.1) obviously commutes, so we need only check φ_z is a bijection.

To see φ_z is onto, for fixed $(a, b) \in F_x$ one has $f_x a = q(b)$. We can write $a = \theta^* c_n$ for some θ, so

$$f_x a = \theta^* f_x c_n = \theta^* x = q\theta^* z$$

so b is in the same orbit as $\theta^* z$, as required.

To see φ_z is one-to-one, suppose

$$(\theta^* c_n, g\theta^* z) = (\psi^* c_n, h\psi^* z).$$

Then $\theta = \psi$ and, hence, $g\theta^* z = h\theta^* z$. By Lemma 5.4, the action is level-wise free, so $g = h$. ■

Corollary 2.6. *Let $X \in \mathbf{S}_G$ have the property that each X_n is a free G_n set. The quotient map $q : X \to X/G$ is a fibration in $\mathbf{S}$. It is a minimal fibration if G is minimal as a Kan complex.*

Proof: Consider a lifting problem

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d, "q"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & X/G. \end{tikzcd}$$

This is equivalent to a lifting problem

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & F_x \arrow[d] \\ \Delta^n \arrow[r, "="'] \arrow[ur, dashed] & \Delta^n. \end{tikzcd}$$

By Lemma 2.5, this is equivalent to a lifting problem

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & G \times \Delta^n \arrow[d, "\pi_j"] \\ \Delta^n \arrow[r, "="'] \arrow[ur, dashed] & \Delta^n. \end{tikzcd}$$

Because G is fibrant in $\mathbf{S}$ (Lemma I.3.4), π_j is a fibration, so the problem has a solution. If G is minimal, the lifting has the requisite uniqueness property to make q a minimal fibration (see Section I.10). ■

LEMMA 2.7. *Let $X \in \mathbf{S}_G$ have the property that each X_n is a free G_n set. Then $X = \varinjlim X_n$ where $X_{-1} = \emptyset$ and for each $n \geq 0$ there is a pushout diagram*

$$\begin{tikzcd} \bigsqcup_\alpha \partial\Delta^n \times G \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\ \bigsqcup_\alpha \Delta^n \times G \arrow[r] & X_n \end{tikzcd}$$

where α runs over the non-degenerate n-simplices of X/G.

PROOF: Define X_n by the pullback diagram

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{sk}_n(X/G) \arrow[r] & X/G. \end{tikzcd}$$

Then $X_{-1} = \emptyset$ and $\varinjlim X_n = X$. Also, since there is a pushout diagram

$$\begin{tikzcd} \bigsqcup_\alpha \partial\Delta^n \arrow[r] \arrow[d] & \mathrm{sk}_{n-1}(X/G) \arrow[d] \\ \bigsqcup_\alpha \Delta^n \arrow[r] & \mathrm{sk}_n(X/G) \end{tikzcd}$$

there is a diagram

$$
\begin{tikzcd}
\bigsqcup_\alpha F(\alpha)|_{\partial\Delta^n} \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\
\bigsqcup_\alpha F(\alpha) \arrow[r] & X_n,
\end{tikzcd}
\tag{2.8}
$$

where $F(\alpha)$ is defined to be the pullback along α. But Lemma 2.5 rewrites $F(\alpha) \cong \Delta^n \times G$ and, hence, $F(\alpha)|_{\partial\Delta^n} \cong \partial\Delta^n \times G$.

Thus we need only show that the diagram (2.8) is a pushout diagram. To do this let Y be the pushout. Then Y is a free G-space (since X_{n-1} is) and there is a map of G-spaces $Y \to X_n$. Since taking orbits is a left adjoint it preserves pushouts; therefore, $Y/G \to X_n/G$ is an isomorphism and the result follows from the next lemma. ∎

One can alternatively finish the argument for Lemma 2.7 by observing that pullbacks and pushouts in the category of G-spaces are formed by the corresponding construction in the underlying simplicial set category, and pullbacks preserve pushouts.

LEMMA 2.9. *Suppose given a morphism in $\mathbf{S}_G$ $f : Y \to X$ so that*

1) *X_n is a free G_n set for all n*
2) *the induced map $Y/G \to X/G$ is an isomorphism*

then f is an isomorphism.

PROOF: This is a variation on the proof of the 5-lemma. To show f is onto, choose $z \in X$. Let $q_X : X \to X/G$ and $q_Y : Y \to Y/G$ be the quotient maps. Then there is a $w \in Y/G$ so that $(f/G)(w) = q_X(z)$. Let $y \in Y$ be so that $q_Y(y) = w$. Then there is a $g \in G$ so that $gf(y) = f(gy) = z$. To show f is one-to-one suppose $f(y_1) = f(y_2)$. Then $q_X f(y_1) = q_X f(y_2)$ so $q_Y(y_1) = q_Y(y_2)$ or there is a $g \in G$ so that $gy_1 = y_2$. Then

$$gf(y_1) = f(y_2) = f(y_1)$$

Since X is free at each level, $g = e$, so $y_1 = y_2$. ∎

COROLLARY 2.10. *An object $X \in \mathbf{S}_G$ is cofibrant if and only if X_n is a free G_n set for all n.*

PROOF: One implication is Lemma 2.4. The other is a consequence of Lemma 2.7. ■

3. Principal fibrations II: classifications.

In this section we will define and classify principal fibrations. Let G be a fixed simplicial group.

DEFINITION 3.1. *A principal fibration $f : E \to B$ is a fibration in $\mathbf{S}_G$ so that*

1) *B has trivial G-action;*
2) *E is a cofibrant G-space; and*
3) *the induced map $E/G \to B$ is an isomorphism.*

Put another way, $f : E \to B$ is isomorphic to a quotient map

$$q : X \to X/G$$

where $X \in \mathbf{S}_G$ is cofibrant. Such a map q is automatically a fibration by Corollary 2.6. Cofibrant objects can be recognized by Corollary 2.10, and Lemma 2.5 should be regarded as a local triviality condition. Finally, there is a diagram

$$\begin{tikzcd} G \times E \arrow[r, "\mu"] \arrow[d, "* \times f"'] & E \arrow[d, "f"] \\ * \times B \arrow[r, "\cong"'] & B \end{tikzcd}$$

where μ is the action; such diagrams figure in the topological definition of principal fibration.

In the same vein, it is quite common to say that a principal G-fibration is a *G-bundle* .

DEFINITION 3.2. *Two principal fibrations $f_1 : E_1 \to B$ and $f_2 : E_2 \to B$ will be called isomorphic if there is an isomorphism $g : E_1 \to E_2$ of G-spaces making the diagram commute*

$$\begin{tikzcd} E_1 \arrow[rr, "g"] \arrow[dr, "f_1"'] & & E_2 \arrow[dl, "f_2"] \\ & B & \end{tikzcd}$$

REMARK 3.3. By Lemma 2.9 it is sufficient to construct a G-equivariant map $g : E_1 \to E_2$ making the diagram commute. Then g is automatically an isomorphism.

Let $PF_G(B)$ be the set of isomorphism class of principal fibrations over B. The purpose of this section is to classify this set.

To begin with, note that $PF_G(\cdot)$ is a contravariant functor. If $q : E \to B$ is a principal fibration and $f : B' \to B$ is any map of spaces, and if $q' : E(f) \to B'$ is defined by the pullback diagram

$$\begin{tikzcd} E(f) \arrow[r] \arrow[d, "q'"'] & E \arrow[d, "q"] \\ B' \arrow[r, "f"'] & B, \end{tikzcd}$$

then f' is a principal fibration. Indeed

$$E(g) = \{(b,e) \in B' \times E \mid f(b) = q(e)\}$$

has G action given by $g(b,e) = (b, ge)$. Then parts 1) and 3) of Definition 3.1 are obvious and part 2) follows from Corollary 2.10.

But, in fact, $PF_G(\cdot)$ is a homotopy functor. Recall that two maps $f_0, f_1 : B' \to B$ are simplicially homotopic if there is a diagram

$$\begin{tikzcd} B' \coprod B' \arrow[rr, "d^0 \sqcup d^1"] \arrow[dr, "f_0 \sqcup f_1"'] & & B' \times \Delta^1 \arrow[dl] \\ & B & \end{tikzcd}$$

LEMMA 3.4. *If f_0 and f_1 are simplicially homotopic, $PF_G(f_0) = PF_G(f_1)$.*

PROOF: It is sufficient to show that given $q : E \to B$ a principal fibration, the pullbacks $E(f_0) \to B'$ and $E(f_1) \to B'$ are isomorphic. For this it is sufficient to consider the universal example: given a principal fibration $E \to B \times \Delta^1$, the pullbacks $E(d^0) \to B$ and $E(d^1) \to B$ are isomorphic. For this consider the lifting problem in $\mathbf{S}_G$

$$\begin{tikzcd} E(d^0) \arrow[r] \arrow[d, "d^0"'] & E \arrow[d] \\ E(d^0) \times \Delta^1 \arrow[r] \arrow[ur, dashed] & B \times \Delta^1. \end{tikzcd}$$

Since $E(d^0)$ is cofibrant in $\mathbf{S}_G$, d^0 is a trivial cofibration, so the lifting exists and by Lemma 1.8 defines an isomorphism of principal fibrations

$$\begin{tikzcd} E(d^0) \times \Delta^1 \arrow[rr, "\cong"] \arrow[dr] & & E \arrow[dl] \\ & B \times \Delta^1 & \end{tikzcd}$$

Pulling back this diagram along d^1 gives the desired isomorphism. ∎

A similar sort of argument proves the following lemma:

LEMMA 3.5. *Let $B \in \mathbf{S}$ be contractible. Then any principal fibration over B is isomorphic to $\pi_2 : G \times B \to B$.*

PROOF: The isomorphism is given by lifting in the diagram (in $\mathbf{S}_G$).

$$\begin{tikzcd} G \arrow[r] \arrow[d, "j"'] & E \arrow[d] \\ G \times B \arrow[r, "\pi_2"'] \arrow[ur, dashed] & B \end{tikzcd}$$

Here j is induced by any basepoint $* \to B$; since G is cofibrant in $\mathbf{S}_G$, j is a trivial cofibration in $\mathbf{S}_G$. ∎

We can now define the classifying object for principal fibrations.

DEFINITION 3.6. *Let $EG \in \mathbf{S}_G$ be any cofibrant object so that the unique map $EG \to *$ is a fibration and a weak equivalence. Let $BG = EG/G$ and $q : EG \to BG$ the resulting principal fibration.*

Note that EG is unique up to equivariant homotopy equivalence, so $q : EG \to BG$ is unique up to homotopy equivalence.

In other words we require more than that EG be a free contractible G-space; EG must also be fibrant. The extra condition is important for the proof of Theorem 3.9 below. It also makes the following result true.

LEMMA 3.7. *The space BG is fibrant as a simplicial set.*

PROOF: First note that given any map $f : \Lambda^n_k \to BG$, then the pullback $E(f) \to \Lambda^n_k$ of $EG \to BG$ fits into a diagram of principal fibrations

$$\begin{tikzcd} \Lambda^n_k \times G \arrow[rr, "\cong"] \arrow[dr] & & E(f) \arrow[dl] \\ & \Lambda^n_k . & \end{tikzcd}$$

Thus there is a lifting of f to a map $g : \Lambda^n_k \to EG$. Now suppose given an extension problem

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "f"] \arrow[d] & BG \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}.$$

Then, since EG is fibrant there is a factoring

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "g"] \arrow[d] & EG \\ \Delta^n \arrow[ur, "h"'] & \end{tikzcd}$$

and the composite $\Delta^n \xrightarrow{h} EG \to BG$ solves the original extension problem. ■

Note that the same argument proves that if $E \in \mathbf{S}_G$ is cofibrant and fibrant, the resulting principal fibration $E \to E/G$ has fibrant base. This claim and Lemma 3.7 are alternatively consequences of the following:

Exercise 3.8. *Suppose that $p : X \to Y$ is a surjective fibration of simplicial sets, and that X is fibrant. Show that Y is fibrant.*

We now come to the main result.

Theorem 3.9. *For all spaces $B \in \mathbf{S}$, the map*

$$\theta : [B, BG] \to PF_G(B)$$

sending the class $[f] \in [B, BG]$ to the pullback of $EG \to BG$ along f is a bijection.

Here, $[B, BG]$ denotes morphisms in the homotopy category $\mathrm{Ho}(\mathbf{S})$ from B to BG. The space BG is fibrant, so this morphism set cat be identified with the set of simplicial homotopy classes of maps from B to BG.

Proof: Note that θ is well-defined by Lemma 3.4. To prove the result we construct an inverse. If $q : E \to B$ is a principal fibration, there is a lifting in the diagram in $\mathbf{S}_G$

$$\begin{tikzcd} \phi \arrow[r] \arrow[d] & EG \arrow[d] \\ E \arrow[r] \arrow[ur] & * \end{tikzcd} \tag{3.10}$$

since E is cofibrant and EG is fibrant, and this lifting is unique up to equivariant homotopy. Let $f : B \to BG$ be the quotient map. Define $\Psi : PF_G(B) \to [B, BG]$, by sending $q : E \to B$ to the class of f.

Note that if $E(f)$ is the pullback of f, there is a diagram

$$\begin{tikzcd} E \arrow[rr] \arrow[dr] & & E(f) \arrow[dl] \\ & B & \end{tikzcd}$$

so Lemma 2.9 implies $\theta\Psi = 1$. On the other hand, given a representative $g : B \to BG$ of a homotopy class in $[B, BG]$, the map g' in the diagram

$$\begin{tikzcd} E(g) \arrow[r, "g'"] \arrow[d] & EG \arrow[d] \\ B \arrow[r, "g"'] & BG \end{tikzcd}$$

makes the diagram (3.10) commute, so by the homotopy uniqueness of liftings $\Psi\theta = 1$. ∎

4. Universal cocycles and $\overline{W}G$.

In the previous sections, we took a simplicial group G and assigned to it a homotopy type BG; that is, the space BG depended on a choice EG of a fibrant, cofibrant contractible G-space. In this section we give a natural, canonical choice for EG and BG called, respectively, WG and $\overline{W}G$. The spaces WG and $\overline{W}G$ are classically defined by letting WG be the simplicial set with

$$WG_n = G_n \times G_{n-1} \times \cdots \times G_0$$

and

$$d_i\,(g_n, g_{n-1}, \dots g_0) = \begin{cases} (d_i g_n, d_{i-1}g_{n-1}, \dots , d_0 g_{n-i} g_{n-i-1}, g_{n-i-2}, \dots g_0) & i < n, \\ (d_n g_n, d_{n-1} g_{n-1}, \dots d_1 g_1) & i = n. \end{cases}$$

$$s_i(g_n, g_{(n-1)} \cdots , g_0) = (s_i g_n, s_{i-1} g_{n-1} \dots s_0 g_{n-1}, e, g_{n-i-1}, \dots g_0)$$

where e is always the unit. Note that WG becomes a G-space if we define $G \times WG \to WG$ by:

$$(h, (g_n, g_{n-1}, \dots , g_0) \longrightarrow (hg, g_{n-1}, \dots g_0).$$

Then $\overline{W}G$ is the quotient of WG by the left G-action; write $q = q_G : WG \to \overline{W}G$ for the quotient map.

LEMMA 4.1. *The map $q : WG \to \overline{W}G$ is a fibration.*

PROOF: This follows from Corollary 1.6 since $(WG)_n$ is a free G_n set. ∎

The functor $G \mapsto \overline{W}G$ takes values in the category $\mathbf{S}_0$ of reduced simplicial sets, where a *reduced simplicial set* is a simplicial set having only one vertex. The salient deeper feature of the functor $\overline{W} : s\mathbf{Gr} \to \mathbf{S}_0$ is that it has a left adjoint $G : \mathbf{S}_0 \to s\mathbf{Gr}$, called the loop group functor, such that the canonical maps $G(\overline{W}G) \to G$ and $X \to \overline{W}(GX)$ are weak equivalences for all simplicial groups G and reduced simplicial sets X. A demonstration of these assertions will occupy this section and the following two. These results are originally due to Kan, and have been known since the late 1950's. The original proofs were calculational — we recast them in modern terms here. Kan's original geometric insights survive and are perhaps sharpened, in the presence of the introduction of a closed model structure for reduced simplicial sets and a theory of simplicial cocycles.

A *segment* of an ordinal number $\mathbf{n}$ is an ordinal number monomorphism $\mathbf{n-j} \hookrightarrow \mathbf{n}$ which is defined by $i \mapsto i+j$. This map can also be variously characterized as the unique monomorphism $\mathbf{n-j} \hookrightarrow \mathbf{n}$ which takes 0 to j, or as the map $(d^0)^j$. This map will also be denoted by $[j,n]$, as a means of identifying its image. There is a commutative diagram of ordinal number maps

$$\begin{tikzcd} \mathbf{n-k} \arrow[dr, "{[k,n]}"] \arrow[dd, "\tau"'] & \\ & \mathbf{n} \\ \mathbf{n-j} \arrow[ur, "{[j,n]}"'] & \end{tikzcd}$$

if and only if $j \leq k$. The map τ is uniquely determined and must be a segment map if it exists: it's the map $(d^0)^{k-j}$. Thus, we obtain a poset $\mathrm{Seg}(\mathbf{n})$ of segments of the ordinal number n. This poset is plainly isomorphic to the poset opposite to the ordinal $\mathbf{n}$.

Suppose that G is a simplicial group. An *n-cocycle* $f : \mathrm{Seg}(\mathbf{n}) \rightsquigarrow G$ associates to each relation $\tau : [k,n] \leq [j,n]$ in $\mathrm{Seg}(\mathbf{n})$ an element $f(\tau) \in G_{n-k}$, such that the following conditions hold:

(1) $f(1_j) = e \in G_{n-j}$, where 1_j is the identity relation $[j,n] \leq [j,n]$,

(2) for any composeable pair of relations $[l,n] \xrightarrow{\zeta} [k,n] \xrightarrow{\tau} [j,n]$, there is an equation
$$\zeta^*(f(\tau))f(\zeta) = f(\tau\zeta).$$

Any ordinal number map $\gamma : \mathbf{r} \to \mathbf{s}$ has a unique factorization

$$\begin{tikzcd}
\mathbf{r} \arrow[rr, "\gamma"] \arrow[dr, "\gamma_*"'] & & \mathbf{s} \\
& \mathbf{s} - \gamma(\mathbf{0}) \arrow[ur, "{[\gamma(0), s] = (d^0)^{\gamma(0)}}"'] &
\end{tikzcd}$$

where γ_* is an ordinal number map such that $\gamma_*(0) = 0$. It follows that any relation $\tau : [k, m] \leq [j, m]$ in $\mathrm{Seg}(\mathbf{m})$ induces a commutative diagram of ordinal number maps

$$\begin{tikzcd}
\mathbf{m} - \mathbf{k} \arrow[rr, "\theta_k"] \arrow[dr, "\tau"] \arrow[dd, "{[k,m]}"'] & & \mathbf{n} - \theta(\mathbf{k}) \arrow[dr, "\tau_*"] \arrow[dd, "{[\theta(k), n]}" near end] & \\
& \mathbf{m} - \mathbf{j} \arrow[rr, "\theta_j" near start] \arrow[dl, "{[j,m]}"] & & \mathbf{n} - \theta(\mathbf{j}) \arrow[dl, "{[\theta(j), n]}"] \\
\mathbf{m} \arrow[rr, "\theta"'] & & \mathbf{n} &
\end{tikzcd} \tag{4.2}$$

where the maps θ_j and θ_k take 0 to 0. Given an n-cocyle $f : \mathrm{Seg}(\mathbf{n}) \rightsquigarrow G$, define, for each relation $\tau : [k, m] \leq [j, m]$ in $\mathrm{Seg}(\mathbf{m})$, an element $\theta^*(f)(\tau) \in G_{m-k}$ by

$$\theta^*(f)(\tau) = \theta_k^*(f(\tau_*)).$$

It's not hard to see now that the collection of all such elements $\theta^*(f)(\tau)$ defines an m-cocycle $\theta^*(f) : \mathrm{Seg}(\mathbf{m}) \rightsquigarrow G$, and that the assignment $\theta \mapsto \theta^*$ is contravariantly functorial in ordinal maps θ. We have therefore constructed a simplicial set whose n-simplices are the n-cocycles $\mathrm{Seg}(\mathbf{n}) \rightsquigarrow G$, and whose simplicial structure maps are the induced maps θ^*.

This simplicial set of G-cocycles is $\overline{W}G$. This claim is checked by chasing the definition through faces and degeneracies, while keeping in mind the observation that an n-cocycle $f : \mathrm{Seg}(\mathbf{n}) \rightsquigarrow G$ is completely determined by the string of relations

$$[n, n] \xrightarrow{\tau_0} [n-1, n] \xrightarrow{\tau_1} \dots \xrightarrow{\tau_{n-2}} [1, n] \xrightarrow{\tau_{n-1}} [0, n], \tag{4.3}$$

and the corresponding element

$$(f(\tau_{n-1}), f(\tau_{n-2}), \dots, f(\tau_0)) \in G_{n-1} \times G_{n-2} \times \cdots \times G_0.$$

Of course, each τ_i is an instance of the map d^0.

The identification of the simplicial set of G-cocycles with $\overline{W}G$ leads to a "global" description of the simplicial structure of $\overline{W}G$. Suppose that $\theta : \mathbf{m} \to \mathbf{n}$ is an ordinal number map, and let

$$\overline{g} = (g_{n-1}, g_{n-2}, \dots, g_0)$$

be an element of $G_{n-1} \times G_{n-2} \times \cdots \times G_0$. Let $F_{\overline{g}}$ be the cocycle $\mathrm{Seg}(\mathbf{n}) \rightsquigarrow G$ associated to the n-tuple $\overline{g}$. Then, subject to the notation appearing in diagram (4.2), we have the relation

$$\theta^*(g_{n-1}, g_{n-2}, \dots, g_0) = (\theta_1^* F_{\overline{g}}(\tau_{m-1*}), \theta_2^* F_{\overline{g}}(\tau_{m-2*}), \dots, \theta_m^* F_{\overline{g}}(\tau_{0*})),$$

where $\tau_{m-i*} = (d^0)^{\theta(i)-\theta(i-1)}$ is the induced relation $[\theta(i), n] \leq [\theta(i-1), n]$ of $\mathrm{Seg}(\mathbf{n})$.

A simplicial map $f : X \to \overline{W}G$, from this point of view, assigns to each n-simplex x a cocycle $f(x) : \mathrm{Seg}(\mathbf{n}) \rightsquigarrow G$, such that for each ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$ and each map $\tau : [k, m] \to [j, m]$ in $\mathrm{Seg}(\mathbf{m})$ there is a relation

$$\theta_k^* f(x)(\tau_*) = f(\theta^*(x))(\tau).$$

Any element $j \in \mathbf{n}$ determines a unique diagram

$$\begin{tikzcd} \mathbf{n} \arrow[dr, "{[0,n] = 1}"] & \\ & \mathbf{n} \\ \mathbf{n-j} \arrow[uu, "{[j,n]}"] \arrow[ur, "{[j,n]}"'] & \end{tikzcd}$$

and hence unambiguously gives rise to elements

$$f(x)([j, n]) \in G_{n-j}.$$

Observe further that if $j \leq k$ and $\tau : [k, n] \leq [j, n]$ denotes the corresponding relation in $\mathrm{Seg}(\mathbf{n})$, then the cocyle condition for the composite

$$[k, n] \xrightarrow{\tau} [j, n] \xrightarrow{[j,n]} [0, n]$$

can be rephrased as the relation

$$\tau^*(f(x)([j,n])) = f(x)([k,n])f(x)(\tau)^{-1}.$$

Now, given a map (cocycle) $f : X \to \overline{W}G$, and an ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$, there is an induced function

$$\theta^* : G_n \times X_n \to G_m \times X_m,$$

which is defined by

$$(g,x) \mapsto (\theta^*(g)\theta_0^*(f(x)([\theta(0),n])), \theta^*(x)), \tag{4.4}$$

where $\theta_0 : \mathbf{m} \to \mathbf{n} - \theta(0)$ is the unique ordinal number map such that $[\theta(0),n] \circ \theta_0 = \theta$.

LEMMA 4.5. *The maps θ^* defined in (4.4) are functorial in ordinal number maps θ.*

PROOF: Suppose given ordinal number maps

$$\mathbf{k} \xrightarrow{\gamma} \mathbf{m} \xrightarrow{\theta} \mathbf{n},$$

and form the diagram

$$\begin{tikzcd}
\mathbf{k} \arrow[d, "\gamma_0"'] \arrow[dr, "\gamma"] & & \\
\mathbf{m} - \gamma(0) \arrow[r, "{[\gamma(0),m]}"'] \arrow[d, "\theta_{\gamma(0)}"'] & \mathbf{m} \arrow[d, "\theta_0"'] \arrow[dr, "\theta"] & \\
\mathbf{n} - \theta(\gamma(0)) \arrow[r, "{[\gamma(0),m]_*}"'] & \mathbf{n} - \theta(0) \arrow[r, "{[\theta(0),n]}"'] & \mathbf{n}
\end{tikzcd}$$

in the ordinal number category. In order to show that $\gamma^*\theta^*(g,x) = (\theta\gamma)^*(g,x)$ in $G_k \times X_k$, we must show that

$$\gamma^*\theta_0^*(f(x)([\theta(0),n]))\gamma_0^*(f(\theta^*(x))([\gamma(0),m])) = \gamma_0^*\theta_{\gamma(0)}^*(f(x)([\theta(\gamma(0)),n]))$$

in G_k. But

$$\gamma^*\theta_0^* = \gamma_0^*\theta_{\gamma(0)}^*[\gamma(0),m]_*^*,$$

and

$$[\gamma(0),m]_*^*(f(x)[\theta(0),n]) = f(x)([\theta\gamma(0),n])(f(x)([\gamma(0),m]_*))^{-1}$$

by the cocycle condition. Finally,

$$\theta_{\gamma(0)}^*(f(x)([\gamma(0),m]_*)) = f(\theta^*(x))([\gamma(0),m]),$$

since f is a simplicial map. The desired result follows. ∎

The simplicial set constructed in Lemma 4.5 from the map $f : X \to \overline{W}G$ will be denoted by X_f. The projection maps $G_n \times X_n \to X_n$ define a simplicial map $\pi : X_f \to X$, and this map π obviously has the structure of a G-bundle. This is a natural construction: if $h : Y \to X$ is a simplicial set map, then the maps $G_n \times Y_n \to G_n \times X_n$ defined by $(g,y) \mapsto (g,h(y))$ define a G-equivariant simplicial set map $h_* : Y_{fh} \to X_f$ such that the diagram

$$\begin{tikzcd} Y_{fh} \arrow[r, "h_*"] \arrow[d, "\pi"'] & X_f \arrow[d, "\pi"] \\ Y \arrow[r, "h"'] & X \end{tikzcd}$$

commutes. Furthermore, this diagram is easily seen to be a pullback.

The simplicial set $\overline{W}G_1$ associated to the identity map $1 : \overline{W}G \to \overline{W}G$ is WG, and the G-bundle $\pi : WG \to \overline{W}G$ is called the *canonical G-bundle.*

Lemma 4.6. *WG is contractible.*

Proof: Suppose given an element $(g_n,(g_{n-1},\dots,g_0)) \in WG_n$. Then the $(n+2)$-tuple $(e,(g_n,g_{n-1},\dots g_0))$ defines an element of WG_{n+1}, in such a way that the following diagram of simplicial set maps commutes:

$$\begin{tikzcd} \Delta^{n+1} \arrow[drr, "{(e,(g_n,g_{n-1},\dots,g_0))}"] & & \\ \Delta^n \arrow[u, "d^0"] \arrow[rr, "{(g_n,(g_{n-1},\dots,g_0))}"'] & & WG \end{tikzcd}$$

commutes. Furthermore, if $\theta : \mathbf{m} \to \mathbf{n}$ is an ordinal number map, and $\theta_* : \mathbf{m+1} \to \mathbf{n+1}$ is the unique map such that $\theta_*(0) = 0$ and $\theta_* d^0 = d^0\theta$, then

$$\theta_*^*(e, (g_n, g_{n-1}, \dots, g_0)) = (e, \theta^*(g_n, (g_{n-1}, \dots, g_0)),$$

as is easily checked. It follows that the simplices $(e, g_n, \dots, g_0)$ define an extra degeneracy on WG in the sense of Section III.5, and so Lemma III.5.1 implies that WG is contractible. ■

REMARK 4.7. Every principal G-fibration $p : Y \to X$ is isomorphic to a principal fibration $X_f \to X$ for some map $f : X \to \overline{W}G$. In effect, let $\mathbf{\Delta}_*$ denote the subcategory of the category $\mathbf{\Delta}$ consisting of all ordinal number morphisms $\gamma : \mathbf{m} \to \mathbf{n}$ such that $\gamma(0) = 0$. Then the map p restricts to a natural transformation $p_* : Y|_{\mathbf{\Delta}_*} \to X|_{\mathbf{\Delta}_*}$, and this transformation has a section $\sigma : X|_{\mathbf{\Delta}_*} \to X|_{\mathbf{\Delta}_*}$ in the category of contravariant functors on $\mathbf{\Delta}_*$, essentially since the simplicial map p is a surjective Kan fibration. Classically, the map σ is called a *pseudo cross-section* for the bundle p. The pseudo cross-section σ defines G_n-equivariant isomorphisms

$$\phi_n : G_n \times X_n \cong Y_n$$

given by $(g, x) \mapsto g \cdot \sigma(x)$. If $\tau : \mathbf{n-k} \to \mathbf{n-j}$ is a morphism of $\mathrm{Seg}(\mathbf{n})$ then

$$\tau^*(\sigma(d_0^j x)) = f_x(\tau)\sigma(\tau^* d_0^k x)$$

for some unique element $f_x(\tau) \in G_{n-k}$. The elements $f_x(\tau)$ define a cocycle $f_x : \mathrm{Seg}(\mathbf{n}) \rightsquigarrow G$ for each simplex x of X, and the collection of cocycles f_x, $x \in X$, defines a simplicial map $f : X \to \overline{W}G$ such that Y is G-equivariantly isomorphic to X_f over X via the maps ϕ_n. The classical approach to the classification of principal G-bundles is based on this construction, albeit not in these terms.

5. The loop group construction.

Suppose that $f : X \to \overline{W}G$ is a simplicial set map, and let $x \in X_n$ be an n-simplex of X. Recall that the associated cocycle $f(x) : \mathrm{Seg}(\mathbf{n}) \rightsquigarrow G$ is completely determined by the group elements

$$f(x)(d^0 : (d^0)^{k+1} \to (d^0)^k).$$

On the other hand,

$$f(x)(d^0 : (d^0)^{k+1} \to (d^0)^k) = f(d_0^k(x))(d^0 : d^0 \to 1_{\mathbf{n-k}}).$$

It follows that the simplicial map $f : X \to \overline{W}G$ is determined by the elements

$$f(x)(d^0) = f(x)(d^0 \to 1_{\mathbf{n}}) \in G_{n-1},$$

for $x \in X_n$, $n \geq 1$. Note in particular that $f(s_0 x)(d^0) = e \in G_{n-1}$.

Turning this around, suppose that $x \in X_{n+1}$, and the ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$ has the factorization

$$\begin{tikzcd} & \mathbf{n} - \theta(\mathbf{0}) \arrow[d, "\tau"] \\ \mathbf{m} \arrow[ur, "\theta_0"] \arrow[r, "\theta"'] & \mathbf{n}, \end{tikzcd}$$

where $\theta_0(0) = 0$ and τ is a segment map, and suppose that $d^0 : d^0 \to 1_{\mathbf{n+1}}$ is the inclusion in $\mathrm{Seg}(\mathbf{n+1})$. Then

$$\tau^* f(x)(d^0) = f(x)(d^0\tau)(f(d_0(x))(\tau))^{-1}.$$

by the cocycle condition for $f(x)$, and so

$$\begin{aligned} \theta^* f(x)(d^0) &= \theta_0^* \tau^*(f(x)(d^0)) \\ &= \theta_0^*(f(x)(d^0\tau))\theta_0^*(f(d_0(x))(\tau))^{-1} \\ &= f(\tilde{\theta}^*(x))(d^0)(f((c\theta)^*(d_0(x)))(d^0))^{-1}, \end{aligned}$$

where $\tilde{\theta} : \mathbf{m+1} \to \mathbf{n+1}$ is defined by

$$\tilde{\theta}(i) = \begin{cases} 0 & \text{if } i = 0, \text{ and} \\ \theta(i-1)+1 & \text{if } i \geq 1, \end{cases}$$

and $c\theta : \mathbf{m+1} \to \mathbf{n}$ is the ordinal number map defined by $(c\theta)(0) = 0$ and $(c\theta)(i) = \theta(i-1)$ for $i \geq 1$. Observe that $c\theta = s^0\tilde{\theta}$.

Define a group $GX_n = F(X_{n+1})/s_0F(X_n)$ for $n \geq 0$, where $F(Y)$ denotes the free group on a set Y. Note that GX_n may also be described as the free group on the set $X_{n+1} - s_0X_n$.

Given an ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$, define a group homomorphism $\theta^* : GX_n \to GX_m$ on generators $[x], x \in X_{n+1}$ by specifying

$$\theta^*([x]) = [\tilde{\theta}^*(x)][(c\theta)^*(d_0(x))]^{-1}. \tag{5.1}$$

If $\gamma : \mathbf{k} \to \mathbf{m}$ is an ordinal number map which is composable with θ, then the relations

$$\begin{aligned} (c\gamma)^* d_0 \tilde{\theta}^*(x) &= (c\gamma)^* \theta^* d_0(x) \\ &= (c\gamma)^* d_0 (c\theta)^* d_0(x) \end{aligned}$$

and

$$\tilde{\gamma}^*(c\theta)^* d_0(x) = (c(\theta\gamma))^* d_0(x)$$

together imply that $\gamma^*\theta^*([x]) = (\theta\gamma)^*([x])$ for all $x \in X_{n+1}$, so that we have a simplicial group, called the *loop group* of X, which will be denoted GX. This construction is plainly functorial in simplicial sets X.

Each n-simplex $x \in X$ gives rise to a string of elements

$$([x], [d_0x], [d_0^2x], \dots, [d_0^{n-1}x]) \in GX_{n-1} \times GX_{n-2} \times \dots \times GX_0,$$

which together determine a cocycle $F_x : \mathrm{Seg}(\mathbf{n}) \rightsquigarrow GX$. Suppose that $\theta : \mathbf{m} \to \mathbf{n}$ is an ordinal number map such that $\theta(0) = 0$. The game is now to obtain a recognizable formula for $[\theta^* x]$, in terms of the simplicial structure of GX.

Obviously, if $\theta(1) = \theta(0)$, then $[\theta^* x] = e \in GX_{m-1}$. Suppose that $\theta(1) > 0$. Then there is a commutative diagram of ordinal number maps

$$\begin{tikzcd}
\mathbf{m-1} \arrow[r, "\theta_1"] \arrow[d, "d^0"'] & \mathbf{n}-\theta(\mathbf{1}) \arrow[d, "\tau"] \arrow[dr, "(d^0)^{\theta(1)-1}"] & \\
& & \mathbf{n-1} \arrow[dl, "d^0"] \\
\mathbf{m} \arrow[r, "\theta"'] & \mathbf{n} &
\end{tikzcd}$$

If $\gamma = (d^0)^{\theta(1)-1}\theta_1$, then $\theta = \tilde{\gamma}$, and so

$$[\theta^*(x)] = (\theta_1^* d_0^{\theta(1)-1}[x])[f(\theta)^*(d_0x)],$$

where $f(\theta)$ is defined by $f(\theta) = c\gamma$. We have $f(\theta)(0) = 0$ by construction, and there is a commutative diagram

$$\begin{tikzcd}
\mathbf{m-1} \arrow[r, "\theta_1"] \arrow[d, "d^0"'] & \mathbf{n}-\theta(\mathbf{1}) \arrow[d, "(d^0)^{\theta(1)-1}"] \\
\mathbf{m} \arrow[r, "f(\theta)"'] & \mathbf{n-1},
\end{tikzcd}$$

so an inductive argument on the exponent $\theta(1)-1$ implies that there is a relation

$$[f(\theta)^*(d_0x)] = (\theta_1^* d_0^{\theta(1)-2}[d_0x]) \dots (\theta_1^*[d_0^{\theta(1)-1}(x)]).$$

It follows that

$$(5.2)\quad [\theta^*(x)] = (\theta_1^* d_0^{\theta(1)-1}[x])(\theta_1^* d_0^{\theta(1)-2}[d_0 x]) \dots (\theta_1^*[d_0^{\theta(1)-1}(x)]) = \theta_1^*(F_x(\tau)).$$

LEMMA 5.3.

(a) *The assignment*

$$x \mapsto ([x], [d_0 x], [d_0^2 x], \dots, [d_0^{n-1} x])$$

defines a natural simplicial map $\eta : X \to \overline{W}GX$.

(b) *The map* η *is one of the canonical homomorphisms for an adjunction*

$$\hom_{s\mathbf{Gr}}(GX, H) \cong \hom_{\mathbf{S}}(X, \overline{W}H),$$

where $s\mathbf{Gr}$ *denotes the category of simplicial groups.*

PROOF:
(a) Suppose that $\theta : \mathbf{m} \to \mathbf{n}$ is an ordinal number map, and recall the decomposition of (4.2). It will suit us to observe once again that the map $[j, m]$ is the composite $(d^0)^j$, and that $\tau_* = (d^0)^{\theta(k)-\theta(j)}$. Note in particular that $\theta = (d^0)^{\theta(0)}\theta_0$, and recall that $\theta_0(0) = 0$. It is also clear that there is a commutative diagram

$$\begin{tikzcd}
X_n \arrow[r, "\eta"] \arrow[d, "d_0^{\theta(0)}"'] & \overline{W}GX_n \arrow[d, "d_0^{\theta(0)}"] \\
X_{n-\theta(0)} \arrow[r, "\eta"'] & \overline{W}GX_{n-\theta(0)}
\end{tikzcd}$$

Let F_x be the cocycle $\mathrm{Seg}(\mathbf{n}) \rightsquigarrow GX$ associated to the element

$$([x], [d_0 x], [d_0^2 x], \dots, [d_0^{n-1} x]).$$

Then, for $x \in X_n$,

$$\begin{aligned}
\theta_0^*([d_0^{\theta(0)} x], [d_0^{\theta(0)+1} x], \dots, [d_0^{n-1} x]) &= (\theta_1^* F_x(\tau_{m-1*}), \dots, \theta_m^* F_x(\tau_{0*})) \\
&= ([\theta_0^* d_0^{\theta(0)} x], [\theta_1^* d_0^{\theta(1)} x], \dots, [\theta_{m-1}^* d_0^{\theta(m-1)} x]) \\
&= ([\theta^* x], [d_0 \theta^* x], \dots, [d_0^{m-1} \theta^* x]),
\end{aligned}$$

where $\tau_{m-i*} = (d^0)^{\theta(i)-\theta(i-1)}$ as before, and this by repeated application of the formula (5.2). In particular, η is a simplicial set map. The naturality is obvious.

(b) Suppose that $f : X \to \overline{W}H$ is a simplicial set map, where H is a simplicial group. Recall that the cocycle $f(x) : \mathrm{Seg}(\mathbf{n}) \rightsquigarrow H$ can be identified with the element

$$(f(x)(d^0), f(d_0x)(d^0), \dots, f(d_0^{n-1}x)(d^0)) \in H_{n-1} \times H_{n-2} \times \cdots \times H_0.$$

The simplicial structure for GX given by the formula (5.1) implies that $f : X \to \overline{W}H$ induces a simplicial group map $f_* : GX \to H$ which is specified on generators by $f_*([x]) = f_x(d^0)$. It follows that the function

$$\hom_{s\mathbf{Gr}}(GX, H) \to \hom_{\mathbf{S}}(X, \overline{W}H)$$

defined by $g \mapsto (\overline{W}g) \circ \eta$ is surjective. Furthermore, any map $f : X \to \overline{W}H$ is uniquely specified by the elements $f(x)(d^0)$, and hence by the simplicial group homomorphism f_*. ∎

REMARK 5.4. Any simplicial group homomorphism $f : G \to H$ induces a f-equivariant morphism of associated principal fibrations of the form

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d] & H \arrow[d] \\ WG \arrow[r, "Wf"] \arrow[d] & WH \arrow[d] \\ \overline{W}G \arrow[r, "\overline{W}f"'] & \overline{W}H, \end{tikzcd}$$

as can be seen directly from the definitions. The canonical map $\eta : X \to \overline{W}GX$ induces a morphism

$$\begin{tikzcd} GX \arrow[r, equal] \arrow[d] & GX \arrow[d] \\ X_\eta \arrow[r] \arrow[d] & WGX \arrow[d] \\ X \arrow[r, "\eta"'] & \overline{W}GX \end{tikzcd}$$

of GX-bundles. It follows that, for any simplicial group homomorphism $f : GX \to H$, the map f and its adjoint $f_* = \overline{W}f \circ \eta$ fit into a morphism of bundles

$$\begin{tikzcd}
GX \arrow[r, "f"] \arrow[d] & H \arrow[d] \\
X_\eta \arrow[r] \arrow[d] & WH \arrow[d] \\
X \arrow[r, "f_*"'] & \overline{W}H.
\end{tikzcd}$$

Suppose now that the simplicial set X is *reduced* in the sense that it has only one vertex. A *closed n-loop of length 2k* in X is defined to be a string

$$(x_{2k}, x_{2k-1}, \dots, x_2, x_1)$$

of $(n+1)$-simplices x_j of X such that $d_0 x_{2i-1} = d_0 x_{2i}$ for $0 \leq i \leq k$. Define an equivalence relation on loops by requiring that

$$(x_{2k}, \dots, x_1) \sim (x_{2k}, \dots, x_{i+2}, x_{i-1}, \dots, x_1)$$

if $x_i = x_{i+1}$. Let

$$\langle x_{2k}, \dots, x_1 \rangle$$

denote the equivalence class of the loop $(x_{2k}, \dots, x_1)$. Write $G'X_n$ for the set of equivalence classes of n-loops under the relation $\sim$. Loops may be concatenated, giving $G'X_n$ the structure of a group having identity represented by the empty n-loop. Any ordinal number morphism $\theta : \mathbf{m} \to \mathbf{n}$ induces a group homomorphism

$$\theta^* : G'X_n \to G'X_m,$$

which is defined by the assignment

$$\langle x_{2k}, \dots, x_2, x_1 \rangle \mapsto \langle \tilde{\theta}^* x_{2k}, \dots \tilde{\theta}^* x_2, \tilde{\theta}^* x_1 \rangle.$$

The corresponding simplicial group will be denote by $G'X$. This construction is clearly functorial with respect to morphisms of reduced simplicial sets.

There is a homomorphism

$$\phi_n : G'X_n \to GX_n$$

which is defined by

$$\phi_n\langle x_{2k}, x_{2k-1}, \dots, x_2, x_1\rangle = [x_{2k}][x_{2k-1}]^{-1} \dots [x_2][x_1]^{-1}.$$

Observe that

$$\begin{aligned}\theta^*([x_{2i}][x_{2i-1}]^{-1}) &= [\tilde{\theta}^*(x_{2i})][(c\theta)^* d_0(x_{2i})]^{-1}[(c\theta)^* d_0(x_{2i-1})][\tilde{\theta}^*(x_{2i-1}]^{-1} \\ &= [\tilde{\theta}^*(x_{2i})][\tilde{\theta}^*(x_{2i-1}]^{-1},\end{aligned}$$

so that

$$\theta^*([x_{2k}][x_{2k-1}]^{-1} \dots [x_2][x_1]^{-1}) = [\tilde{\theta}^* x_{2k}][\tilde{\theta}^* x_{2k-1}]^{-1} \dots [\tilde{\theta}^* x_2][\tilde{\theta}^* x_1]^{-1}.$$

The homomorphisms $\phi_n : G'X_n \to GX_n$, taken together, therefore define a simplicial group homomorphism $\phi : G'X \to GX$.

LEMMA 5.5. *The homomorphism $\phi : G'X \to GX$ is an isomorphism of simplicial groups which is natural with respect to morphisms of reduced simplicial sets X.*

PROOF: The homomorphism $\phi_n : G'X_n \to GX_n$ has a section, which is defined on generators by

$$[x] \mapsto \langle x, s_0 d_0 x\rangle,$$

and elements of the form $\langle x, s_0 d_0 x\rangle$ generate $G'X_n$. ■

Again, let X be a reduced simplicial set. The set $E'X_n$ consists of equivalence classes of strings of $(n+1)$-simplices

$$(x_{2k}, \dots, x_1, x_0)$$

with $d_0 x_{2i} = d_0 x_{2i-1}$, $i \geq 1$, subject an equivalence relation generated by relations if the form

$$(x_{2k}, \dots, x_0) \sim (x_{2k}, \dots, x_{i+2}, x_{i-1}, \dots, x_0)$$

if $x_i = x_{i+1}$. We shall write $\langle x_{2k}, \dots, x_0\rangle$ for the equivalence class containing the element $(x_{2k}, \dots, x_0)$. Any ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$ determines a function $\theta^* : E'X_n \to E'X_m$, which is defined by

$$\theta^*\langle x_{2k}, \dots, x_0\rangle = \langle\tilde{\theta}^*(x_{2k}), \dots, \tilde{\theta}^*(x_0)\rangle,$$

and so we obtain a simplicial set $E'X$. Concatenation induces a left action $G'X \times E'X \to E'X$ of the simplicial group $G'X$ on $E'X$.

There is a function

$$\phi'_n : E'X_n \to GX_n \times X_n$$

which is defined by

$$\phi'_n\langle x_{2k}, \dots, x_1, x_0\rangle = ([x_{2k}][x_{2k-1}]^{-1} \dots [x_2][x_1]^{-1}[x_0], d_0x_0).$$

The function ϕ'_n is ϕ_n-equivariant, and so

$$\phi'_n(\phi_n^{-1}(g)\langle s_0x\rangle) = (g, x)$$

for any $(g, x) \in GX_n \times X_n$, and ϕ'_n is surjective. There is an equation

$$\langle x_{2k}, \dots, x_0\rangle = \langle x_{2k}, \dots, x_0, s_0d_0(x_0)\rangle\langle s_0d_0(x_0)\rangle$$

for every element of $E'X_n$, so that $E'X_n$ consists of $G'X_n$-orbits of elements $\langle s_0x\rangle$. The function ϕ'_n preserves orbits and ϕ_n is a bijection, so that ϕ'_n is injective as well.

The set $GX_n \times X_n$ is the set of n-simplices of the GX-bundle X_η which is associated to the natural map $\eta : X \to \overline{W}GX$. If $\theta : \mathbf{m} \to \mathbf{n}$ is an ordinal number map, then the associated simplicial structure map θ^* in X_η has the form

$$\begin{aligned}&\theta^*([x_{2k}] \dots [x_1]^{-1}[x_0], d_0x_0)\\ &= ([\tilde{\theta}^*(x_{2k})] \dots [\tilde{\theta}^*(x_1)]^{-1}[\tilde{\theta}^*(x_0)][(c\theta)^*(d_0x_0)]^{-1}\theta_0^*(\eta(x)([\theta(0), n])), d_0\tilde{\theta}^*(x_0))\end{aligned}$$

since $d_0\tilde{\theta}^*(x_0) = \theta^*(d_0x_0)$. But

$$[(c\theta)^*(d_0x_0))] = \theta_0^*(\eta(x)([\theta(0), n]),$$

by equation (5.2). The bijections ϕ'_n therefore define a ϕ-equivariant simplicial map, and so we have proved

LEMMA 5.6. *There is a ϕ-equivariant isomorphism*

$$\phi' : E'X \to X_\eta.$$

This isomorphism is natural with respect to maps of reduced simplicial sets.

There is a simplicial set $E''X$ whose n-simplices consist of the strings of $(n+1)$-simplices $(x_{2k}, \dots, x_0)$ of X as above, and with simplicial structure maps defined by

$$\theta^*(x_{2k}, \dots, x_0) = (\tilde{\theta}^* x_{2k}, \dots, \tilde{\theta}^* x_0)$$

for $\theta : \mathbf{m} \to \mathbf{n}$. Observe that $E'X = E''X/\sim$.

Given this description of the simplicial structure maps in $E''X$, the best way to think of the members of an n-simplex is as a string $(x_{2k}, \dots, x_0)$ of cones on their 0^{th} faces, with the obvious incidence relations. A homotopy $\Delta^n \times \Delta^1 \to E''X$ can therefore be identified with a string

$$(h_{2k}, \dots, h_1, h_0),$$

where

(1) $h_i : C(\Delta^n \times \Delta^1) \to X$ is a map defined on the cone $C(\Delta^n \times \Delta^1)$ on the simplicial set $\Delta^n \times \Delta^1$, and

(2)
$$h_{2i}|_{\Delta^n \times \Delta^1} = h_{2i-1}|_{\Delta^n \times \Delta^1}$$
for $1 \leq i \leq k$.

We shall say that maps of the form $C(\Delta^n \times \Delta^1) \to Y$ are *cone homotopies*. Examples of such include the following:

(1) The canonical contracting homotopy

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow \cdots & & \longrightarrow & 0 \\ \downarrow & & \downarrow & & & & \downarrow \\ 0 & \longrightarrow & 1 & \longrightarrow \cdots & & \longrightarrow & n+1 \end{array}$$

of Δ^{n+1} onto the vertex 0 induces a map $C(\Delta^n \times \Delta^1) \to \Delta^{n+1}$ which is jointly specified by the vertex 0 and the restricted homotopy

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow \cdots & & \longrightarrow & 0 \\ \downarrow & & \downarrow & & & & \downarrow \\ 1 & \longrightarrow & 2 & \longrightarrow \cdots & & \longrightarrow & n+1. \end{array}$$

This map is a "contracting" cone homotopy.

(2) The vertex 0 and the constant homotopy

$$\begin{array}{ccccccc} 1 & \longrightarrow & 2 & \longrightarrow \cdots & \longrightarrow & n+1 \\ \downarrow & & \downarrow & & & \downarrow \\ 1 & \longrightarrow & 2 & \longrightarrow \cdots & \longrightarrow & n+1. \end{array}$$

jointly specify a "constant" cone homotopy $C(\Delta^n \times \Delta^1) \to \Delta^{n+1}$.

In both of these cases, it's helpful to know that the cone CBP on the nerve BP of a poset P can be identified with the nerve of the cone poset CP which is obtained from P by adjoining a disjoint initial object. Furthermore, a poset map $\gamma : P \to Q$ can be extended to a map $CP \to Q$ by mapping the initial object of CP to some common lower bound of the objects in the image of γ, if such a lower bound exists.

LEMMA 5.7. *$E'X$ is acyclic in the sense that $\tilde{H}_*(E'X, \mathbb{Z}) = 0$.*

PROOF: Both the contracting and constant cone homotopies defined above are natural in Δ^n in the sense that the diagram

$$\begin{tikzcd} C(\Delta^m \times \Delta^1) \arrow[r, "h"] \arrow[d, "C(\theta \times 1)"'] & \Delta^{m+1} \arrow[d, "\tilde{\theta}_*"] \\ C(\Delta^n \times \Delta^1) \arrow[r, "h"'] & \Delta^{n+1} \end{tikzcd}$$

commutes for each ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$, where h denotes one of the two types. It follows that there is a homotopy from the identity map on $E''X$ to the map $E''X \to E''X$ defined by

$$(x_{2k}, \ldots, x_1, x_0) \mapsto (x_{2k}, \ldots, x_1, *),$$

and that this homotopy can be defined on the level of simplices by strings of cone homotopies

$$(h(x_{2k}), \ldots, h(x_1), h(x_0)),$$

where $h(x_0)$ is contracting on $d_0 x_0$, and all other $h(x_i)$ are constant. This homotopy, when composed with the canonical map $E''X \to E'X$, determines a chain homotopy $\mathbf{S}$ from the induced map $\mathbb{Z}E''X \to \mathbb{Z}E'X$ to the map $\mathbb{Z}E''X \to \mathbb{Z}E'X$ which is induced by the simplicial set map defined by

$$(x_{2k}, \dots, x_1, x_0) \mapsto \langle x_{2k}, \dots, x_1, *\rangle.$$

For each element $(x_{2k}, \dots, x_1, x_0)$, the chain $S(x_{2k}, \dots, x_1, x_0)$ is an alternating sum of the simplices comprising the homotopy $(h(x_{2k}), \dots, h(x_1), h(x_0))$. It follows in particular, that if $x_i = x_{i+1}$ for some $i \geq 1$, then the corresponding adjacent simplices of the components of $S(x_{2k}, \dots, x_1, x_0)$ are also equal.

It also follows that there is a chain homotopy defined by

$$(x_{2k}, \dots, x_1, x_0) \mapsto S(x_{2k}, \dots, x_1, x_0) - S(x_{2k}, \dots, x_1, x_1),$$

and that this is a chain homotopy from the chain map induced by the canonical map $E''X \to E'X$ to the chain map induced by the simplicial set map

$$(x_{2k}, \dots, x_1, x_0) \mapsto \langle x_{2k}, \dots, x_3, x_2\rangle$$

This construction can be iterated, to produce a chain homotopy H defined by

$$(x_{2k}, \dots, x_0) \mapsto (\sum_{i=0}^{k-1}(S(x_{2k}, \dots, x_{2i+1}, x_{2i}) - S(x_{2k}, \dots, x_{2i+1}, x_{2i+1}))) + S(x_{2k})$$

from the chain map $\mathbb{Z}E''X \to \mathbb{Z}E'X$ to the chain map induced by the simplicial set map $E''X \to E'X$ which takes all simplices to the base point $*$. One can show that

$$H(x_{2k}, \dots, x_0) = H(x_{2k}, \dots x_{i+2}, x_{i-1}, \dots, x_0)$$

if $x_i = x_{i+1}$. It follows that H induces a contracting chain homotopy on the complex $\mathbb{Z}E'X$. ■

LEMMA 5.8. *$E'X$ is simply connected.*

PROOF: Following Lemma 5.6, we shall do a fundamental groupoid calculation in $X_\eta \cong E'X$.

The boundary of the 1-simplex $(s_0 g, x)$ in X_η has the form

$$\partial(s_0 g, x) = ((g[x], *), (g, *)).$$

There is an oriented graph $T(X)$ (hence a simplicial set) having vertices coinciding with the elements of GX_0 and with edges $x : g \to gx$ for $x \in X_1 - \{*\}$. There is plainly a simplicial set map $T(X) \to X_\eta$ which is the identity on vertices and sends each edge $x : g \to gx$ to the 1-simplex $(s_0 g, x)$. This map induces a map of fundamental groupoids

$$\pi T(X) \to \pi X_\eta$$

which is surjective on objects. A reduced word argument shows that $T(X)$ is contractible, hence has trivial fundamental groupoid, so we end up showing that X_η is simply connected if we show that the 1-simplices $(s_0 g, x)$ generate the fundamental groupoid πX_η.

There are boundary relations

$$\begin{aligned}\partial(s_1 g, s_0 x) &= (d_0(s_1 g, s_0 x), d_1(s_1 g, s_0 x), d_2(s_1 g, s_0 x)) \\ &= (s_0 d_0 g, x), (g, x), (g, *))\end{aligned}$$

and in the same notation,

$$\partial(s_0 g, y) = ((g[y], d_0 y), (g, d_1 y), (s_0 d_1 g, d_2 y)).$$

The upshot is that there are commuting diagrams in πX_η of the form

$$\begin{tikzcd}
* \arrow[rr, "{(s_0 d_1 g, d_2 y)}"] \arrow[dr, swap, "{(g, d_1 y)}"] & & * \arrow[rr, "{(g[y], x)}"] \arrow[dl, swap, "{(g[y], d_0 y)}"] \arrow[dr, "{(g[y], *)}"] & & * \\
& * & & * \arrow[ll, "{(s_0 d_0(g[y]), d_0 y)}"] \arrow[ur, swap, "{(s_0 d_0(g[y]), x)}"] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
* \arrow[dr, swap, "{(g, d_0 y)}"] & & * \arrow[ll, swap, "{(s_0 d_1(g[y]^{-1}), d_2 y)}"] \arrow[rr, "{(g[y]^{-1}, x)}"] \arrow[dl, swap, "{(g[y]^{-1}, d_1 y)}"] \arrow[dr, "{(g[y]^{-1}, *)}"] & & * \\
& * & & * \arrow[ll, "{(s_0 d_0(g[y]^{-1}), d_1 y)}"] \arrow[ur, swap, "{(s_0 d_0(g[y]^{-1}), x)}"] &
\end{tikzcd}$$

It follows that any generator $(g[y], x)$ (respectively $(g[y]^{-1}, x)$ of πX_η can be replaced by a generator $(g, d_1 y)$ (respectively $(g, d_0 y)$) of πX_η up to multiplication by elements of $\pi T(X)$. In particular, any generator (h, x) of πX_η can be replaced up to multiplication by elements of $\pi T(X)$ by a generator (h', x') such that h' has strictly smaller word length as an element of the free group GX_1. An induction on word length therefore shows that the groupoid πX_η is generated by the image of $T(X)$. ∎

REMARK 5.9. The object $T(X)$ in the proof of Lemma 5.8 is the Serre tree associated to the generating set $X_1 - \{*\}$ of the free group GX_0. See p.16 and p.26 of [84].

We have proved

THEOREM 5.10. *Suppose that X is a reduced simplicial set. Then the total space X_η of the principal GX-fibration $X_\eta \to X$ is contractible.*

COROLLARY 5.11. *There are weak equivalences*

$$GX \xrightarrow{\simeq} X_\eta \times_X PX \xleftarrow{\simeq} \Omega X,$$

which are natural with respect to morphisms of reduced Kan complexes X.

6. Reduced simplicial sets, Milnor's FK-construction.

The proof of Theorem 5.10 depends on an explicit geometric model for the space X_η, and the construction of this model uses the assumption that the simplicial set X is reduced. There is no such restriction on the loop group functor: GY is defined for all simplicial sets Y. The geometric model for X_η can be expanded to more general simplicial sets (see Kan's paper), but Theorem 5.10 fails badly in the non-reduced case: the loop group $G(\Delta^1)$ on the simplex Δ^1 is easily seen to be the constant simplicial group on the free group $\mathbb{Z}$ on one letter, which is manifestly not contractible. This sort of example forces us (for the time being — see Section 8) to restrict our attention to spaces with one vertex.

We now turn to the model category aspects of the loop group and $\overline{W}$ functors.

LEMMA 6.1. *Let $f : X \to Y$ be a cofibration of simplicial sets. Then $Gf : GX \to GY$ is a cofibration of simplicial groups. In particular, for all simplicial sets X, GX is a cofibrant simplicial group.*

PROOF: This result is a consequence of Corollary 1.10.

Note that since $s_0 X_n \subseteq X_{n+1}$ there is an isomorphism of groups

$$GX_n \cong F(X_{n+1} - s_0 X_n).$$

Furthermore, for all $i \geq 0$, the map $s_{i+1} : X_n \to X_{n+1}$ restricts to a map

$$s_{i+1} : X_n - s_0 X_{n-1} \to X_{n+1} - s_0 X_n$$

since $s_{i+1} s_0 X = s_0 s_i X$. Hence there is a diagram

$$\begin{tikzcd} GX_{n-1} \arrow[r, "\cong"] \arrow[d, "s_i"'] & F(X_n - s_0 X_n) \arrow[d, "Fs_{i+1}"] \\ GX_n \arrow[r, "\cong"] & F(X_{n+1} - s_0 X_n) \end{tikzcd}$$

and GX is almost free, hence cofibrant. For the general case, if $X \to Y$ is a level-wise inclusion

$$Y_{n+1} - s_0Y_n = (X_{n+1} - s_0X_n) \cup Z_{n+1}$$

where $Z_{n+1} = Y_{n+1} - (X_{n+1} \cup s_0Y_n)$. Thus

$$GY_n \cong GX_n * FZ_{n+1}$$

where the $*$ denotes the free product. Now $s_{i+1} : Y_n \to Y_{n+1}$ restricts to a map $s_{i+1} : Z_n \to Z_{n+1}$ and, hence, the inclusion $GX \to GY$ is almost-free and a cofibration. ∎

As a result of Theorem 5.10 and Lemma 6.1 one sees that G preserves cofibrations and weak equivalences between spaces with one vertex. This suggests that the proper domain category for G — at least from a model category point of view — is the category $\mathbf{S}_0$ of simplicial sets with one vertex. Our next project then is to give that category a closed model structure.

Proposition 6.2. *The category $\mathbf{S}_0$ has a closed model category structure where a morphism $f : X \to Y$ is a*

1) *a weak equivalence if it is a weak equivalence as simplicial sets;*
2) *a cofibration if it is a cofibration as simplicial sets; and*
3) *a fibration if it has the right lifting property with respect to all trivial cofibrations.*

The proof is at the end of the section, after we explore some consequences.

Proposition 6.3.

1) *The functor $G : \mathbf{S}_0 \to s\mathbf{Gr}$ preserves cofibrations and weak equivalences.*
2) *The functor $\overline{W} : s\mathbf{Gr} \to \mathbf{S}_0$ preserves fibrations and weak equivalences.*
3) *Let $X \in \mathbf{S}_0$ and $G \in s\mathbf{Gr}$. Then a morphism $f : GX \to G$ is a weak equivalence in $s\mathbf{Gr}$ if only if the adjoint $f_* : X \to \overline{W}G$ is a weak equivalence in $\mathbf{S}_0$.*

Proof: Part 1) follows from Lemma 6.1 and Theorem 5.10. For part 2) notice that since $\overline{W}$ is right adjoint to a functor which preserves trivial cofibrations, it preserves fibrations. The clause about weak equivalences follows from Lemma 4.6 Finally, part 3), follows from Remark 5.4, Lemma 4.6 and Theorem 5.10.

COROLLARY 6.4. *Let* $\mathrm{Ho}(\mathbf{S}_0)$ *and* $\mathrm{Ho}(s\mathbf{Gr})$ *denote the homotopy categories. Then the functors* G *and* $\overline{W}$ *induce an equivalence of categories*

$$\mathrm{Ho}(\mathbf{S}_0) \cong \mathrm{Ho}(s\mathbf{Gr}).$$

PROOF: Proposition 6.3 implies that the natural maps $\epsilon : G\overline{W}H \to H$ and $\eta : X \to \overline{W}GX$ are weak equivalences for all simplicial groups H and reduced simplicial sets X. ∎

REMARK 6.5. If $\mathrm{Ho}(\mathbf{S})_c \subseteq \mathrm{Ho}(\mathbf{S})$ is the full sub-category of the usual homotopy category with objects the connected spaces, then the inclusion $H_0(\mathbf{S}_0) \to H_0(\mathbf{S})_c$ is an equivalence of categories. To see this, it is sufficient to prove if X is connected there is a Y weakly equivalent to X with a single vertex. One way is to choose a weak equivalence $X \to Z$ with Z fibrant and then let $Y \subseteq Z$ be a minimal subcomplex weakly equivalent to Z.

We next relate the fibrations in $\mathbf{S}_0$ to the fibrations in $\mathbf{S}$.

LEMMA 6.6. *Let* $f : X \to Y$ *be a fibration in* $\mathbf{S}_0$. *Then* f *is a fibration in* $\mathbf{S}$ *if and only if* f *has the right lifting property with respect to*

$$* \to S^1 = \Delta^1/\partial\Delta^1.$$

PROOF: First suppose f is a fibration in $\mathbf{S}$. Consider a lifting problem

$$\begin{tikzcd} \Delta^0 \arrow[r] \arrow[d, hook, "d^i"'] & * \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \Delta^1 \arrow[r] & S^1 \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd} \tag{6.7}$$

Since f is a fibration in $\mathbf{S}$, there is a map $g : \Delta^1 \to X$ solving the lifting problem for the outer rectangle. Since X has one vertex g factors through the quotient map,

$$\Delta^1 \to \Delta^1/\operatorname{sk}_0 \Delta^1 = S^1 \xrightarrow{\overline{g}} X$$

and $\overline{g}$ solves the original lifting problem. Here sk_0 denotes the zero skeleton.

Now suppose f has the stipulated lifting property. Then one must solve all lifting problems

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd}$$

If $n = 0$, this is trivial since $X_0 \cong Y_0$. If $n > 1$, this diagram can be expanded to

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & \Lambda^n_k / \operatorname{sk}_0 \Lambda^n_k \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] & \Delta^n / \operatorname{sk}_0 \Delta^n \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd}$$

The map

$$\Lambda^n_k / \operatorname{sk}_0 \Lambda^n_k \to \Delta^n / \operatorname{sk}_0 \Delta^n$$

is still a trivial cofibration, now in $\mathbf{S}_0$. So the lift exists. If $n = 1$, the expanded diagram is an instance of diagram (6.7), and the lift exists by hypothesis. ∎

Corollary 6.8. *Let $X \in \mathbf{S}_0$ be fibrant in $\mathbf{S}_0$, then X is fibrant in $\mathbf{S}$. In particular, if $G \in s\mathbf{Gr}$, then $\overline{W}G$ is fibrant in $\mathbf{S}$.*

Proof: The first clause follows from the previous lemma. For the second, note that every object of $s\mathbf{Gr}$ is fibrant. Since $\overline{W} : s\mathbf{Gr} \to \mathbf{S}_0$ preserves fibrations, $\overline{W}G$ is fibrant in $\mathbf{S}_0$. ∎

Corollary 6.9. *Let $f : X \to Y$ be a fibration in $\mathbf{S}_0$ between fibrant spaces. Then f is a fibration in $\mathbf{S}$ if and only if*

$$f_* : \pi_1 X \to \pi_1 Y$$

is onto. In particular, if $G \to H$ is a fibration of simplicial groups, $\overline{W}G \to \overline{W}H$ is a fibration of simplicial sets if and only if $\pi_0 G \to \pi_0 H$ is onto.

Proof: Consider a lifting problem

$$\begin{tikzcd} * \arrow[r] \arrow[d] & X \arrow[d] \\ S^1 \arrow[r, "\alpha"'] \arrow[ur, dashed] & Y. \end{tikzcd}$$

This can be solved up to homotopy; that is there is a diagram

$$\begin{tikzcd} & S^1 \arrow[r] \arrow[d, "d^1"] & X \arrow[d] \\ S^1 \arrow[r, "d^0"] & S^1 \wedge \Delta^1_+ \arrow[r, "h"] & Y. \end{tikzcd}$$

where $h \circ d^0 = \alpha$. But $d^1 : S^1 \to S^1 \wedge \Delta^1_+$ is a trivial cofibration in $\mathbf{S}_0$ so the homotopy h can be lifted to $\widetilde{h} : S^1 \wedge \Delta^1_+ \to X$ and $\widetilde{h} \circ d^0$ solves the original lifting problem.

For the second part of the corollary, note that Proposition 3.5 implies $\pi_1 \overline{W}G \cong \pi_0 G$. ■

We now produce the model category structure promised for $\mathbf{S}_0$. The following lemma sets the stage. If X is a simplicial set, let $\#X$ denote the cardinality of the non-degenerate simplices in X. Let ω be the first infinite cardinal.

LEMMA 6.10.

1) *Let $A \to B$ be a cofibration in $\mathbf{S}$ and $x \in B_k$ a k-simplex. Then there is a subspace $C \subseteq B$ so that $\#C < \omega$ and $x \in C$.*
2) *Let $A \to B$ be a trivial cofibration in $\mathbf{S}$ and $x \in B_k$ a k-simplex. Then there is a subspace $D \subseteq B$ so that $\#D \leq \omega$, $x \in D$ and $A \cap D \to D$ is a trivial cofibration.*

PROOF: Part 1) is a reformulation of the statement that every simplicial set is the filtered colimit of its finite subspaces. For part 2) we will construct an expanding sequence of subspaces

$$D_1 \subseteq D_2 \subseteq \cdots \subseteq B$$

so that $x \in D_1$, $\#D_n \leq \omega$ and

$$\pi_p(|D_n|, |D_n \cap A|) \to \pi_p(|D_{n+1}|, |D_{n+1} \cap A|)$$

is the zero map. Then we can set $D = \bigcup_n D_n$.

To get D_1, simply choose a finite subspace $D_1 \subseteq B$ with $x \in D_1$. Now suppose $D_q, q \leq n$, have been constructed and satisfy the above properties. Let

$$\alpha \in \pi_*(|D_n|, |D_n \cap A|).$$

Since α maps to zero under

$$\pi_*(|D_n|, |D_n \cap A|) \to \pi_*(|B|, |A|)$$

there must be a subspace $D_\alpha \subseteq B$, such that $\#D < \omega$ and so that α maps to zero under

$$\pi_*(|D_n|, |D_n \cap A|) \to \pi_*(|D_n \cup D_\alpha|, (D_n \cup D_\alpha) \cap A|).$$

Set $D_{n+1} = D_n \cup (\bigcup_\alpha D_\alpha)$. ■

Remark 6.11. The relative homotopy groups $\pi_*(|B|, |A|)$ for a cofibration $i : A \to B$ of simplicial sets are defined to be the homotopy groups of the homotopy fibre of the realized map $i_* : |A| \hookrightarrow |B|$, up to a dimension shift. The realization of a Kan fibration is a Serre fibration (Theorem I.10.10), so it follows that these groups coincide up to isomorphism with the simplicial homotopy groups $\pi_* F_i$ of any choice of homotopy fibre F_i in the simplicial set category. One can use Kan's Ex^∞ functor along with an analog of the classical method of replacing a continuous map by a fibration to give a rigid construction of the Kan complex F_i which satisfies the property that the assignment $i \mapsto F_i$ preserves filtered colimits in the maps i. The argument for part 2) of Lemma 6.10 can therefore be made completely combinatorial. This observation becomes quite important in contexts where preservation of functoriality is vital — see [38].

Lemma 6.12. *A morphism $f : X \to Y$ in $\mathbf{S}_0$ is a fibration if and only if it has the right lifting property with respect to all trivial cofibrations $C \to D$ in $\mathbf{S}_0$ with $\#D \leq \omega$.*

Proof: Consider a lifting problem

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "j"'] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where j is a trivial cofibration. We solve this by a Zorn's Lemma argument. Consider the set Λ of pairs (Z, g) where $A \subseteq Z \subseteq B$, $A \to Z$ is a weak equivalence and g is a solution to the restricted lifting problem

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d] & X \arrow[d, "f"] \\ Z \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd}$$

Partially order Λ by setting $(Z, g) < (Z', g')$ if $Z \subseteq Z'$ and g' extends g. Since $(A, a) \in \Lambda$, Λ is not empty and any chain

$$(Z_1, g_1) < (Z_2, g_2) < \cdots$$

in Λ has an upper bound, namely $(\cup Z_n, \cup g_n)$. Thus Λ satisfies the hypotheses of Zorn's lemma and has a maximal element (B_0, g_0). Suppose $B_0 \neq B$. Consider the diagram

$$\begin{tikzcd} B_0 \arrow[r, "g_0"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r] & Y. \end{tikzcd}$$

Then i is a trivial cofibration. Choose $x \in B$ with $x \notin B_0$. By Lemma 6.10.2 there is a subspace $D \subseteq B$ with $x \in D$, $\#D \leq \omega$ and $B_0 \cap D \to D$ a trivial cofibration. The restricted lifting problem

$$\begin{tikzcd} B_0 \cap B \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ D \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

has a solution, by hypotheses. Thus g_0 can be extended over $B_0 \cup D$. This contradicts the maximality of (B_0, g_0). Hence $B_0 = B$. ■

REMARK 6.13. The proofs of Lemma 6.10 and Lemma 6.12 are actually standard moves. The same circle of ideas appears in the arguments for the closed model structures underlying both the Bousfield homology localization theories [8], [9] and the homotopy theory of simplicial presheaves [46], [51], [38]. We shall return to this topic in Chapter IX.

THE PROOF OF PROPOSITION 6.2: Axioms **CM1–CM3** for a closed model category are easy in this case. Also, the "trivial cofibration-fibration" part of **CM4** is the definition of fibration. We next prove the factorization axiom **CM5** holds, then return to finish **CM4**.

Let $f : X \to Y$ be a morphism in $\mathbf{S}_0$. To factor f as a cofibration followed by a trivial fibration, use the usual small object argument with pushout along cofibrations $A \to B$ in $\mathbf{S}_0$ with $\#B < \omega$ to factor f as $X \xrightarrow{j} Z \xrightarrow{q} Y$ where j is cofibration and q is a map with the right lifting property with respect to all cofibrations $A \to B$ with $\#B < \omega$. The evident variant on the Zorn's lemma argument given in the proof of Lemma 6.12 using 6.10.1 implies q has the right lifting property with respect to all cofibrations in $\mathbf{S}_0$. Hence q is a fibration. We claim it is a weak equivalence and, in fact, a trivial fibration in $\mathbf{S}$. To see this consider a lifting problem

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, hook] & Z \arrow[d, "q"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd}$$

If $n = 0$ this has a solution, since $Z_0 \cong Y_0$. If $n > 0$, this extends to a diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, hook] & \partial\Delta^n / \operatorname{sk}_0(\partial\Delta^n) \arrow[r] \arrow[d, "j"] & Z \arrow[d, "q"] \\ \Delta^n \arrow[r] & \Delta^n / \operatorname{sk}_0 \Delta^n \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd}$$

Since $n \geq 1$, $sk(\partial\Delta^n) = \operatorname{sk}_0(\Delta^n)$, so j is a cofibration between finite complexes in $\mathbf{S}_0$ and the lift exists.

Return to $f : X \to Y$ in $\mathbf{S}_0$. To factor f as a trivial cofibration followed by a fibration, we use a transfinite small object argument.

We follow the convention that a cardinal number is the smallest ordinal number within a given bijection class; we further interpret a cardinal number β as a poset consisting of strictly smaller ordinal numbers, and hence as a category. Choose a cardinal number β such that $\beta > 2^\omega$.

Take the map $f : X \to Y$, and define a functor $X : \beta \to \mathbf{S}_0$ and a natural transformation $f_s : X(s) \to Y$ such that

(1) $X(0) = X$,
(2) $X(t) = \varinjlim_{s<t} X(s)$ for all limit ordinals $t < \beta$, and
(3) the map $X(s) \to X(s+1)$ is defined by the pushout diagram

$$\begin{tikzcd} \bigsqcup_D A_D \arrow[r, "(\alpha_D)"] \arrow[d, "\sqcup i_D"'] & X(s) \arrow[d] \\ \bigsqcup_D B_D \arrow[r] & X(s+1) \end{tikzcd}$$

where the index D refers to a set of representatives for all diagrams

$$\begin{tikzcd} A_D \arrow[r, "\alpha_D"] \arrow[d, "i_D"'] & X(s) \arrow[d, "f_s"] \\ B_D \arrow[r] & Y \end{tikzcd}$$

such that $i_D : A_D \to B_D$ is a trivial cofibration in $\mathbf{S}_0$ with $\#B_D \leq \omega$. Then there is a factorization

$$\begin{tikzcd} X \arrow[r, "i_0"] \arrow[dr, "f"'] & X_\beta \arrow[d, "f_\beta"] \\ & Y \end{tikzcd}$$

for the map f, where $X_\beta = \varinjlim_s X(s)$, and $i_0 : X = X(0) \to X_\beta$ is the canonical map into the colimit. Lemma 6.12. A pushout along a trivial cofibration in $\mathbf{S}_0$ is a trivial cofibration in $\mathbf{S}_0$ because the same is true in $\mathbf{S}$, so i_0 is a trivial cofibration. Also, any map $A \to X_\beta$ must factor through one of the canonical maps $i_s : X(s) \to X_\beta$ if $\#A \leq \omega$, for otherwise A would have too many subobjects on account of the size of β. It follows that the map $f_\beta : X_\beta \to Y$ is a fibration of $\mathbf{S}_0$. This finishes **CM5**.

To prove **CM4** we must show any trivial fibration $f : X \to Y$ in $\mathbf{S}_0$ has the right lifting property with respect to all cofibrations. However, we factored f as a composite

$$X \xrightarrow{j} Z \xrightarrow{q} Y$$

where j is a cofibration and q is a trivial fibration with the right lifting property with respect to all cofibrations. Now j is a trivial cofibration, since f is a weak equivalence. Thus there is a lifting in

$$\begin{tikzcd} X \arrow[r, "="] \arrow[d, "j"'] & X \arrow[d, "f"] \\ Z \arrow[r, "q"'] \arrow[ur, dashed] & Y \end{tikzcd}$$

since f is a fibration. This shows f is a retract of q and has the requisite lifting property, since q does. ∎

As an artifact of the proof we have:

LEMMA 6.14. *A morphism $f : X \to Y$ in $\mathbf{S}_0$ is a trivial fibration in $\mathbf{S}_0$ if and only if it is a trivial fibration in $\mathbf{S}$.*

The *Milnor FK construction* associates to a pointed simplicial set K the simplicial group FK, which is given in degree n by

$$FK_n = F(K_n - \{*\}),$$

so that FK_n is the free group on the set $K_n - \{*\}$. This construction obviously gives a functor from pointed simplicial sets to simplicial groups. The group FK is also a loop group:

THEOREM 6.15. *There is a natural isomorphism*

$$G(\Sigma K) \cong FK,$$

for pointed simplicial sets K.

PROOF: Recall that ΣK denotes the Kan suspension of K. The group of n-simplices of $G(\Sigma K)$ is defined to be the quotient

$$G(\Sigma K)_n = F(\Sigma K_{n+1})/F(s_0 \Sigma K_n).$$

The map $s_0 : \Sigma K_n \to \Sigma K_{n+1}$ can be identified with the wedge summand inclusion

$$K_{n-1} \vee \cdots \vee K_0 \hookrightarrow K_n \vee K_{n-1} \vee \cdots \vee K_0,$$

so that the composite group homomorphism

$$F(K_n) \xrightarrow{\eta_{n*}} F(\Sigma K_{n+1}) \to F(\Sigma K_{n+1})/F(s_0 \Sigma K_n)$$

can be identified via an isomorphism

$$F(\Sigma K_{n+1})/F(s_0 \Sigma K_n) \cong FK_n \tag{6.16}$$

with the quotient map

$$F(K_n) \to F(K_n)/F(*) \cong FK_n.$$

Recall that for $\theta : \mathbf{m} \to \mathbf{n}$, the map $\theta^* : G(\Sigma K)_n \to G(\Sigma K)_m$ is specified on generators $[x]$ by

$$\theta^*([x]) = [\tilde{\theta}^*(x)][(c\theta)^*(d_0(x))]^{-1}.$$

But then

$$\begin{aligned} \theta^*([\eta_n(x)]) &= [\tilde{\theta}^*(\eta_n(x))][(c\theta)^*(d_0(\eta_n(x)))]^{-1} \\ &= [\tilde{\theta}^*(\eta_n(x))] \\ &= [\eta_m(\theta^*(x))], \end{aligned}$$

since $d_0(\eta_n(x)) = *$. It follows that the isomorphisms (6.16) respect the simplicial structure maps. ■

The proof of Theorem 6.15 is easy enough, but this result has important consequences:

Corollary 6.17.

(1) *The Milnor FK construction takes weak equivalences of pointed simplicial sets to weak equivalences of simplicial groups.*

(2) *The simplicial group FK is a natural fibrant model for $\Omega\Sigma K$, in the category of pointed simplicial sets.*

Proof: The first assertion is proved by observing that the Kan suspension functor preserves weak equivalences; the loop group construction has the same property by Theorem 5.10 (see also Section II.8).

Let $\Sigma K \to Y$ be a fibrant model for ΣK in the category of reduced simplicial sets. Then Y is a Kan complex which is weakly equivalent to ΣK, so that ΩY is a model for $\Omega\Sigma K$. The loop group functor preserves weak equivalences, so that the induced map $G(\Sigma K) \to GY$ is a weak equivalence of simplicial groups. Finally, we know that GY is weakly equivalent to ΩY, so that $G(\Sigma K)$ and hence FK is a model for $\Omega\Sigma K$. ■

7. Simplicial groupoids.

A *simplicial groupoid* G, for our purposes, is a simplicial object in the category of groupoids whose simplicial set of objects is discrete. In other words, G consists of small groupoids G_n, $n \geq 0$ with a functor $\theta^* : G_m \to G_n$ for each ordinal number map $\theta : \mathbf{n} \to \mathbf{m}$, such that all sets of objects $\mathrm{Ob}(G_n)$ coincide with a fixed set $\mathrm{Ob}(G)$, and all functors θ^* induce the identity function on $\mathrm{Ob}(G)$. Of course, $\theta \mapsto \theta^*$ is also contravariantly functorial in ordinal number maps θ. The set of morphisms from x to y in G_n will be denoted by $G_n(x, y)$, and there is a simplicial set $G(x,y)$ whose n-simplices are the morphism set $G_n(x,y)$ in the groupoid G_n. We shall denote the category of simplicial groupoids by $s\mathbf{Gd}$.

The free groupoid $G(X)$ on a graph X has the same set of objects as X, and has morphisms consisting of reduced words in arrows of X and their inverses. There is a canonical graph morphism $\eta : X \to G(X)$ which is the identity on objects, and takes an arrow α to the reduced word represented by the string consisting of α alone. Any graph morphism $f : X \to H$ taking values in a groupoid H extends uniquely to a functor $f_* : G(X) \to H$, in the sense that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\eta"'] & H \\ G(X) \arrow[ur, "f_*"'] & \end{tikzcd}$$

There is a similar construction of a free groupoid $G\mathcal{C}$ on a category $\mathcal{C}$, which has been used without comment until now. The groupoid $G\mathcal{C}$ is obtained by the free groupoid on the graph underlying the category $\mathcal{C}$ by killing the normal subgroupoid generated by the composition relations of $\mathcal{C}$ and the strings associated to the identity morphisms of $\mathcal{C}$ (see also Sections I.8 and III.1). The category of groupoids obviously has all small coproducts, given by disjoint unions. This category also has pushouts, which are actually pushouts in the category of small categories, so the category of groupoids is cocomplete. Note that filtered colimits are formed in the category of groupoids just as filtered colimits on the object and morphism levels. The initial object in the category of groupoids has an empty set of morphisms and an empty set of objects and is denoted by $\emptyset$.

It is also completely straightforward to show that the category of simplicial groupoids has all inverse limits.

Dwyer and Kan define [25], for every simplicial set X, a groupoid $F'X$ having object set $\{0,1\}$, such that the set of n-simplices X_n is identified with a set of arrows from 0 to 1, and such that $F'X_n$ is the free groupoid on the resulting graph.

The groupoid $F'K$ is morally the same thing as the Milnor construction, for pointed simplicial sets K. If x denotes the base point of K, then there is a homomorphism of simplicial groups

$$g : FK \to F'K(0,0)$$

which is defined on generators $y \in K_n - \{x\}$ by $y \mapsto x^{-1}y$. Also, regarding FK as a simplicial groupoid with one object, we see that there is a map of simplicial groupoids

$$f : F'K \to FK$$

defined by sending x to e in all degrees and such that $y \in K_n - \{x\}$ maps to the arrow y. The collection of all products $y^{-1}z$, $y, z \in K_n$, generates $F'K(0,0)$ in degree n, and so it follows that the composite simplicial group homomorphism

$$F'K(0,0) \xrightarrow{f} FK \xrightarrow{g} F'K(0,0)$$

is the identity. The composite

$$FK \xrightarrow{g} F'K(0,0) \xrightarrow{f} FK$$

sends $y \in K_n$ to $x^{-1}y = y \in FK_n$, so the homomorphism g is an isomorphism.

LEMMA 7.1. *Suppose that K is a pointed simplicial set. Then the simplicial sets $F'K(a,b)$, $a, b \in \{0,1\}$, are all isomorphic to the Milnor FK construction.*

PROOF: The base point x of K determines an isomorphism $x : 0 \to 1$ in the groupoid $F'K_n$ for all $n \geq 0$. Composition and precomposition with x therefore determines a commutative diagram of simplicial set isomorphisms

(7.2)
$$\begin{tikzcd} F'K(0,0) \arrow[r, "x_*", "\cong"'] & F'K(0,1) \\ F'K(1,0) \arrow[u, "x^*", "\cong"'] \arrow[r, "\cong", "x_*"'] & F'K(1,1), \arrow[u, "\cong", "x^*"'] \end{tikzcd}$$

and of course we've seen that $F'K(0,0) \cong FK$. ■

COROLLARY 7.3. *A weak equivalence $f : X \to Y$ of simplicial sets induces weak equivalences $f_* : F'X(a,b) \to F'Y(a,b)$ for all objects $a, b \in \{0,1\}$.*

PROOF: We can suppose that X is non-empty. Pick a base point x in X, and observe that the diagram (7.2) is natural in pointed simplicial set maps, as is the isomorphism $F'X(0,0) \cong FX$. We've already seen that the Milnor FX construction preserves weak equivalences. ■

For an ordinary groupoid H, it's standard to write $\pi_0 H$ for the set of *path components* of H. By this, one means that

$$\pi_0 H = \mathrm{Ob}(H)/\sim,$$

where there is a relations $x \sim y$ between two objects of H if and only if there is a morphism $x \to y$ in H. This is plainly an equivalence relation since H is a groupoid, but more generally $\pi_0 H$ is the specialization of a notion of the set of path components $\pi_0 \mathcal{C}$ for a small category $\mathcal{C}$.

If now G is a simplicial groupoid, it's easily seen that all of the simplicial structure functors $\theta^* : G_n \to G_m$ induce isomorphisms $\pi_0 G_n \cong \pi_0 G_m$. We shall therefore refer to $\pi_0 G_0$ as the set of *path components of the simplicial groupoid* G, and denote it by $\pi_0 G$.

A map $f : G \to H$ of simplicial groupoids is said to be a *weak equivalence of* $s\mathbf{Gd}$ if

(1) the morphism f induces an isomorphism $\pi_0 G \cong \pi_0 H$, and
(2) each induced map $f : G(x,x) \to H(f(x), f(x))$, $x \in \mathrm{Ob}(G)$ is a weak equivalence of simplicial groups (or of simplicial sets).

Corollary 7.3 says that the functor $F' : \mathbf{S} \to s\mathbf{Gd}$ takes weak equivalences of simplicial sets to weak equivalences of simplicial groupoids.

A map $g : H \to K$ of simplicial groupoids is said to be a *fibration* if

(1) the morphism g has the *path lifting property* in the sense for every object x of H and morphism $\omega : g(x) \to y$ of the groupoid K_0, there is a morphism $\hat{\omega} : x \to z$ of H_0 such that $g(\hat{\omega}) = \omega$, and
(2) each induced map $g : H(x,x) \to K(g(x), g(x))$, $x \in \mathrm{Ob}(H)$, is a fibration of simplicial groups (or of simplicial sets).

According to this definiton, every simplicial groupoid G is fibrant, since the map $G \to *$ which takes values in the terminal simplicial groupoid $*$ is a fibration. A *cofibration of simplicial groupoids* is defined to be a map which has the left lifting property with respect to all morphisms of $s\mathbf{Gd}$ which are both fibrations and weak equivalences.

Several comments are in order. First of all, picking a representative $x \in [x]$ for each $[x] \in \pi_0 G$ determines a map of simplicial groupoids

$$i : \bigsqcup_{[x] \in \pi_0 G} G(x,x) \to G$$

which is plainly a weak equivalence. But more is true, in that the simplicial groupoid $\bigsqcup_{[x] \in \pi_0 G} G(x,x)$ is a deformation retract of G in the usual groupoid-theoretic sense. To see this, pick morphisms $\omega_y : y \to x$ in G_0 for each $y \in [x]$ and for each $[x] \in \pi_0 G$, such that $\omega_x = 1_x$ for all the fixed choices of representatives x of the various path components x. Then there is a simplicial groupoid morphism

$$r : G \to \bigsqcup_{[x]\in\pi_0 G} G(x,x),$$

which is defined by conjugation by the paths ω_y, in that $r(y) = x$ if and only if $y \in [x]$ for all objects y of G, and $r : G(y,z) \to G(x,x)$ is the map sending $\alpha : y \to z$ to the composite $\omega_z \alpha \omega_y^{-1} \in G(x,x)$ for all $y, z \in [x]$, and for each $[x] \in \pi_0 G$. The morphisms ω_y also determine a groupoid homotopy

$$h : G \times I \to G$$

where I denotes the free groupoid on the ordinal number (category) $\mathbf{1}$. This homotopy is from the identity on G to the composite ir, and is given by the obvious conjugation picture.

The choices of the paths which define the retraction map r are non-canonical, and are certainly not natural with respect to morphisms of simplicial groupoids, except in certain useful isolated cases (see the proof of Lemma 7.5 below). The choices of the representatives x for path components $[x]$ are non-canonical as well, but are more nearly natural, in that for every weak equivalence $f : G \to H$ of simplicial groupoids, choices of representatives can be made (in G) such that there is a commutative diagram

$$\text{(7.4)} \qquad \begin{tikzcd} \bigsqcup_{[x]\in\pi_0 G} G(x,x) \arrow[r] \arrow[d, "i"'] & \bigsqcup_{[x]\in\pi_0 G} H(f(x),f(x)) \arrow[d, "i"] \\ G \arrow[r, "f"] & H. \end{tikzcd}$$

of weak equivalences of simplicial groupoids. We can also prove the following:

LEMMA 7.5. *Suppose that the morphism $j : A \to B$ of simplicial groupoids has the left lifting property with respect to all fibrations. Then the map j is a weak equivalence.*

PROOF: The initial object in the category of simplicial groupoids is the category $\emptyset$ having no objects and no arrows. The path lifting property for the canonical map $\emptyset \to G$ is vacuously satisfied, and, similarly, all simplicial set maps $\emptyset \to X$ are Kan fibrations, so that each map $\emptyset \to G$ is a fibration of simplicial groupoids.

If $A = \emptyset$, then the lifting exists in the diagram

$$\begin{tikzcd} \emptyset \arrow[r, "="] \arrow[d] & \emptyset \arrow[d] \\ B \arrow[r, "="'] \arrow[ur, dashed] & B, \end{tikzcd}$$

forcing $B = \emptyset$, and so j is a weak equivalence in this case.

In general, there is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "="] \arrow[d, "j"'] & A \arrow[d] \\ B \arrow[r] \arrow[ur, "r"'] & * \end{tikzcd}$$

since A is fibrant. In particular, j induces an injection $i : \mathrm{Ob}(A) \hookrightarrow \mathrm{Ob}(B)$, as well as an injection $i_* : \pi_0 A \to \pi_0 B$. Let C be the subobject of B, given by the union of path components containing objects of A. If $C \neq B$, then $B = C \sqcup C'$ and the map $j : A \to B$ is a disjoint union of maps of the form $A \to C$ and $\emptyset \to C'$, each of which has the left lifting property with respect to all fibrations. It follows that $C' = \emptyset$, and so j induces an isomorphism on π_0.

Recall that j induces a monomorphism on objects. It follows that, by choosing paths $\omega : y \to x$ for a fixed list of representatives x for elements $[x] \in \pi_0 A$, first in A and then for the remaining objects in B, we can construct a commutative diagram of simplicial groupoid maps

$$\begin{tikzcd} \bigsqcup_{[x]\in\pi_0 A} A(x,x) \arrow[r, "i"] \arrow[d, "j"'] & A \arrow[r, "r"] \arrow[d, "j"] & \bigsqcup_{[x]\in\pi_0 A} A(x,x) \arrow[d, "j"] \\ \bigsqcup_{[x]\in\pi_0 A} B(j(x),j(x)) \arrow[r, "i"'] & B \arrow[r, "r"'] & \bigsqcup_{[x]\in\pi_0 A} B(j(x),j(x)) \end{tikzcd}$$

such that the top and bottom horizontal composites are the identity. The restricted map

$$j : \bigsqcup_{[x]\in\pi_0 A} A(x,x) \to \bigsqcup_{[x]\in\pi_0 A} B(j(x),j(x))$$

therefore has the right lifting property with respect to all fibrations, and so each of the maps $j : A(x,x) \to B(j(x),j(x))$ has the same lifting property, and is therefore a trivial cofibration of simplicial groups. ■

THEOREM 7.6. *With these definitions, the category $s\mathbf{Gd}$ of simplicial groupoids satisfies the axioms for a closed model category.*

PROOF: Only the factorization axiom has an interesting proof. Write $F'\partial\Delta^0$ to denote the discrete simplicial groupoid on the object set $\{0,1\}$, and write $F'\Lambda^0_0$ to denote the terminal groupoid $*$. Then a map of simplicial groupoids $f : G \to H$ is a fibration if and only if it has the right lifting property with respect to all morphisms $F'\Lambda^n_k \to F'\Delta^n$, $0 \leq k \leq n$, and f is a trivial fibration (aka. fibration and weak equivalence) if and only if it has the right lifting property with respect to all morphisms $F'\partial\Delta^n \to F'\Delta^n$, $n \geq 0$, and the morphism $\emptyset \to *$ (compare [25]). We can therefore use a small object argument to show that every simplicial groupoid morphism $g : K \to L$ has factorizations

$$\begin{tikzcd} & N \arrow[dr, "p"] & \\ K \arrow[ur, "i"] \arrow[rr, "g"] \arrow[dr, "j"'] & & L \\ & M \arrow[ur, "q"'] & \end{tikzcd}$$

where p is a fibration and i has the left lifting property with respect to all fibrations, and q is a trivial fibration and j is a cofibration. Lemma 7.5 implies that i is a weak equivalence.

The proof of the lifting axiom **CM4** is a standard consequence of the proof of the factorization axiom: any map which is both a cofibration and a weak equivalence (ie. a trivial cofibration) is a retract of a map which has the left lifting property with respect to all fibrations, and therefore has that same lifting property. ■

There is a simplicial set $\overline{W}G$ for a simplicial groupoid G that is defined by analogy with and extends the corresponding object for a simplicial group. Explicitly, suppose that G is a simplicial groupoid. An *n-cocycle* $X : \mathrm{Seg}(\mathbf{n}) \rightsquigarrow G$ associates to each object $[k,n]$ an object X_k of G, and assigns to each relation $\tau : [j,n] \leq [k,n]$ in $\mathrm{Seg}(\mathbf{n})$ a morphism $X(\tau) : X_j \to X_k$ in G_{n-j}, such that the following conditions hold:

(1) $X(1_j) = 1_{X_j} \in G_{n-j}$, where 1_j is the identity relation $[j,n] \leq [j,n]$,

(2) for any composeable pair of relations $[l,n] \xrightarrow{\zeta} [k,n] \xrightarrow{\tau} [j,n]$, there is a commutative diagram

$$\begin{tikzcd} X_l \arrow[r, "X(\zeta)"] \arrow[dr, "X(\tau\zeta)"'] & X_k \arrow[d, "\zeta^* X(\tau)"] \\ & X_j \end{tikzcd}$$

in the groupoid G_{n-j}.

Suppose that $\theta : \mathbf{m} \to \mathbf{n}$ is an ordinal number map. As before, θ induces a functor $\theta_* : \mathrm{Seg}(\mathbf{m}) \to \mathrm{Seg}(\mathbf{n})$, which is defined by sending the morphism $\tau : [k,m] \to [j,m]$ to the morphism $\tau_* : [\theta(k),n] \to [\theta(j),n]$. "Composing" the n-cocycle $X : \mathrm{Seg}(\mathbf{n}) \rightsquigarrow G$ with θ_* gives a cocycle $\theta^* X : \mathrm{Seg}(\mathbf{m}) \rightsquigarrow G$, defined for each relation $\tau : [k,m] \leq [j,m]$ in $\mathrm{Seg}(\mathbf{m})$, (and in the notation of (4.2)) by the morphism

$$\theta^* X(\tau) = \theta_k^*(X(\tau_*)) : X_{\theta(k)} \to X_{\theta(j)}.$$

of G_{m-k} The assignment $\theta \mapsto \theta^*$ is contravariantly functorial in ordinal maps θ.

We have therefore constructed a simplicial set whose n-simplices are the n-cocycles $\mathrm{Seg}(\mathbf{n}) \rightsquigarrow G$, and whose simplicial structure maps are the induced maps θ^*. This simplicial set of G-cocycles is $\overline{W}G$. In particular, an n-cocycle $X : \mathrm{Seg}(\mathbf{n}) \rightsquigarrow G$ is completely determined by the string of relations

$$[n,n] \xrightarrow{\tau_0} [n-1,n] \xrightarrow{\tau_1} \dots \xrightarrow{\tau_{n-2}} [1,n] \xrightarrow{\tau_{n-1}} [0,n],$$

and the corresponding maps

$$X_n \xrightarrow{X(\tau_0)} X_{n-1} \xrightarrow{X(\tau_1)} X_{n-2} \to \cdots \to X_1 \xrightarrow{X(\tau_{n-1})} X_0.$$

Each τ_i is an instance of the map d^0, and $X(\tau_i)$ is a morphism of the groupoid G_i. Note, in particular, that the i^{th} vertex of the cocycle $X : \mathrm{Seg}(\mathbf{n}) \rightsquigarrow G$ is the object X_i of G: this means that X_i can be identified with the "cocycle" i^*X, where $i : \mathbf{0} \to \mathbf{n}$.

Suppose that $\theta : \mathbf{m} \to \mathbf{n}$ is an ordinal number map, and let $\overline{g}$ denote the string of morphisms

$$X_n \xrightarrow{g_0} X_{n-1} \xrightarrow{g_1} X_{n-2} \to \cdots \to X_1 \xrightarrow{g_{n-1}} X_0$$

in G, with g_i a morphism of G_i. Let $X_{\overline{g}}$ be the cocycle $\mathrm{Seg}(\mathbf{n}) \rightsquigarrow G$ associated to the n-tuple $\overline{g}$. Then, subject to the notation appearing in diagram (4.2), $\theta^* X_{\overline{g}}$ is the string

$$X_{\theta(m)} \xrightarrow{\theta_m^* X_{\overline{g}}(\tau_{0*})} X_{\theta(m-1)} \xrightarrow{\theta_{m-1}^* X_{\overline{g}}(\tau_{1*})} X_{\theta(m-2)} \to \cdots \to X_{\theta(1)} \xrightarrow{\theta_1^* X_{\overline{g}}(\tau_{m-1*})} X_{\theta(0)}.$$

It's easily seen that this definition specializes to the cocycle definition of $\overline{W}G$ in the case where G is a simplicial group.

A simplicial map $f : X \to \overline{W}G$ assigns to each n-simplex x a cocycle $f(x) : \mathrm{Seg}(\mathbf{n}) \rightsquigarrow G$, such that for each ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$ and each map $\tau : [k, m] \to [j, m]$ in $\mathrm{Seg}(\mathbf{m})$ there is a relation

$$\theta_k^* f(x)(\tau_*) = f(\theta^*(x))(\tau).$$

Furthermore, $f(x)$ is determined by the string of maps

$$f(x_n) \xrightarrow{f(x)(\tau_0)} f(x_{n-1}) \xrightarrow{f(x)(\tau_1)} f(x_{n-2}) \to \cdots \to f(x_1) \xrightarrow{f(x)(\tau_{n-1})} f(x_0),$$

in G, where x_i is the i^{th} vertex of x, and τ_{n-i} is the map $\tau_{n-i} = d^0 : [i, n] \to [i-1, n]$ of $\mathrm{Seg}(\mathbf{n})$. By the simplicial relations, $f(x)(\tau_{n-i}) = f(d_0^{i-1}(x))(\tau_{n-i})$, so that the simplicial map $f : X \to \overline{W}G$ is completely determined by the morphisms

$$f(x)(d^0 = \tau_{n-1} : [1, n] \to \mathbf{n}) : f(x_1) \to f(x_0)$$

in G_{n-1}, $x \in X$. In alternate notation then, the cocycle $f(x)$ is completely determined by the string of morphisms

$$f(x_n) \xrightarrow{f(d_0^{n-1}x)(d^0)} f(x_{n-1}) \to \ldots \xrightarrow{f(d_0 x)(d^0)} f(x_1) \xrightarrow{f(x)(d^0)} f(x_0)$$

in G.

The morphism $f(s_0x)(d^0)$ is the identity on $f(x_0)$. We now can define a groupoid GX_n to be the free groupoid on generators $x : x_1 \to x_0$, where $x \in X_{n+1}$, subject to the relations $s_0 x = 1_{x_0}$, $x \in X_n$. The objects of this groupoid are just the vertices of X. Following the description of the loop group from a previous section, we can define a functor $\theta^* : GX_n \to GX_m$ for each ordinal number morphism $\theta : \mathbf{m} \to \mathbf{n}$ by specifying that θ^* is the identity on objects, and is defined on generators $[x]$, $x \in X_{n+1}$, by requiring that the following diagram commutes:

$$\begin{tikzcd} x_{\theta(0)+1} \arrow[r, "{[(c\theta)^* d_0 x]}"] \arrow[dr, "{[\tilde{\theta}^*(x)]}"'] & x_1 \arrow[d, "{\theta^*[x]}"] \\ & x_0, \end{tikzcd}$$

or rather that

$$\theta^*[x] = [\tilde{\theta}^*(x)][(c\theta)^* d_0 x]^{-1}.$$

One checks, as before, that this assignment is functorial in ordinal number morphisms θ, so that the groupoids GX_n, $n \geq 0$, and the functors θ^* form a simplicial groupoid GX, which we call the *loop groupoid* for X.

Any n-simplex x of the simplicial set X determines a string of morphisms

$$x_n \xrightarrow{[d_0^{n-1}x]} x_{n-1} \to \dots \xrightarrow{[d_0 x]} x_1 \xrightarrow{[x]} x_0$$

in GX, which together determine a cocycle $\eta(x) : \mathrm{Seg}(\mathbf{n}) \rightsquigarrow GX$ in the simplicial groupoid GX. The calculations leading to Lemma 5.3 also imply the following:

Lemma 7.7.

(a) *The assignment $x \mapsto \eta(x)$ defines a natural simplicial map $\eta : X \to \overline{W}GX$.*

(b) *The map η is one of the canonical homomorphisms for an adjunction*

$$\hom_{s\mathbf{Gd}}(GX, H) \cong \hom_{\mathbf{S}}(X, \overline{W}H),$$

where $s\mathbf{Gd}$ denotes the category of simplicial groupoids.

Here's the homotopy theoretic content of these functors:

Theorem 7.8.

(1) *The functor $G : \mathbf{S} \to s\mathbf{Gd}$ preserves cofibrations and weak equivalences.*

(2) *The functor $\overline{W} : s\mathbf{Gd} \to \mathbf{S}$ preserves fibrations and weak equivalences.*

(3) *A map $K \to \overline{W}X \in \mathbf{S}$ is a weak equivalence if and only if its adjoint $GK \to X \in s\mathbf{Gd}$ is a weak equivalence.*

Proof: The heart of the matter for this proof is statement (2). We begin by showing that $\overline{W}$ preserves weak equivalences.

Suppose that A is a simplicial groupoid, and choose a representative x for each $[x] \in \pi_0 A$. Recall that the inclusion

$$i : \bigsqcup_{[x] \in \pi_0 A} A(x,x) \to A$$

is a homotopy equivalence of simplicial groupoids in the sense that there is a groupoid map

$$r : A \to \bigsqcup_{[x] \in \pi_0 A} A(x,x)$$

which is determined by paths, such that ri is the identity and such that the paths defining r determine a groupoid homotopy

$$h : A \times I \to A$$

from the identity on A to the composite morphism ir. The object I is the constant simplicial groupoid associated to the groupoid having two objects $0, 1$ and exactly one morphism $a \to b$ for any $a, b \in \{0, 1\}$. One sees that $\overline{W}I = BI$ and that $\overline{W}$ preserves products. It follows that the groupoid homotopy h induces a homotopy of simplicial sets from the identity on $\overline{W}A$ to the composite map $\overline{W}i \circ \overline{W}r$, and so $\overline{W}i$ is a weak equivalence. If $f : A \to B$ is a weak equivalence of simplicial groupoids, then f induces an isomorphism $\pi_0 A \cong \pi_0 B$, and there is a commutative diagram of simplicial groupoid maps

$$\begin{tikzcd}
\displaystyle\bigsqcup_{x \in \pi_0 A} A(x,x) \arrow[r] \arrow[d, "i"'] & \displaystyle\bigsqcup_{x \in \pi_0 A} B(f(x), f(x)) \arrow[d, "i"] \\
A \arrow[r] & B
\end{tikzcd}$$

in which the vertical maps are homotopy equivalences. To see that $\overline{W}f$ is a weak equivalence, it therefore suffices to show that $\overline{W}$ takes the top horizontal map to a weak equivalence. But $\overline{W}$ preserves disjoint unions, and then one uses the corresponding result for simplicial groups (ie. Proposition 6.3).

To show that $\overline{W}$ preserves fibrations, we have to show that a lifting exists for all diagrams

(7.9)
$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\alpha"] \arrow[d, hook] & \overline{W}A \arrow[d, "\overline{W}f"] \\ \Delta^n \arrow[r, "\beta"'] & \overline{W}B, \end{tikzcd}$$

given that $f : A \to B$ is a fibration of $s\mathbf{Gd}$. We can assume that A and B are connected. The lifting problem is solved by the path lifting property for f if $n = 1$.

Otherwise, take a fixed $x \in A_0$ and choose paths $\eta_i : y_i \to x$ in A_0, where y_i is the image of the i^{th} vertex in Λ^n_k. Note that the vertices of Λ^n_k coincide with those of Δ^n, since $n \geq 2$. These paths, along with their images in the groupoid B_0 determine "cocycle homotopies" from the diagram (7.9) to a diagram

(7.10)
$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\alpha'"] \arrow[d, hook] & \overline{W}A(x,x) \arrow[d, "\overline{W}f"] \\ \Delta^n \arrow[r, "\beta'"'] \arrow[ur, dashed, "\gamma"] & \overline{W}B(f(x),f(x)). \end{tikzcd}$$

More explicitly, if the simplicial set map β is determined by the string of morphisms

$$f(y_n) \xrightarrow{g_{n-1}} f(y_{n-1}) \xrightarrow{g_{n-2}} f(y_{n-2}) \to \dots \to f(y_1) \xrightarrow{g_0} f(y_0)$$

in B, then the cocycle homotopy from β to β' is the diagram

$$\begin{tikzcd} f(y_n) \arrow[r, "g_{n-1}"] \arrow[d, "f(\eta_n)"'] & f(y_{n-1}) \arrow[r, "g_{n-2}"] \arrow[d, "f(\eta_{n-1})"'] & \dots \arrow[r, "g_1"] & f(y_1) \arrow[r, "g_0"] \arrow[d, "f(\eta_1)"'] & f(y_0) \arrow[d, "f(\eta_0)"] \\ f(x) \arrow[r, "h_{n-1}"'] & f(x) \arrow[r, "h_{n-2}"'] & \dots \arrow[r, "h_1"'] & f(x) \arrow[r, "h_0"'] & f(x) \end{tikzcd}$$

where $h_i = f(\eta_i)g_i f(\eta_{i+1})^{-1}$, and β' is defined by the string of morphisms h_i. The cocycle β' is a cocycle conjugate of β, in an obvious sense.

The indicated lift exists in the diagram (7.10), because the simplicial set map $\overline{W}f : A(x,x) \to B(f(x), f(x))$ satisfies the lifting property for $n \geq 2$ (see the proof of Lemma 6.6). The required lift for the diagram (7.9) is cocycle conjugate to γ.

We have therefore proved statement (2) of the theorem. An adjointness argument now implies that the functor G preserves cofibrations and trivial cofibrations. Every weak equivalence $K \to L$ of simplicial sets can be factored as a trivial cofibration, followed by a trivial fibration, and every trivial fibration in $\mathbf{S}$ is left inverse to a trivial cofibration. It follows that G preserves weak equivalences, giving statement (1).

Statement (3) is proved by showing that the unit and counit of the adjunction are both weak equivalences. Let A be a simplicial groupoid. To show that the counit $\epsilon : G\overline{W}A \to A$ is a weak equivalence, we form the diagram

$$\begin{tikzcd}
G\overline{W}(\bigsqcup_{x\in\pi_0 A} A(x,x)) \arrow[r, "\epsilon"] \arrow[d, "G\overline{W}i"'] & \bigsqcup_{x\in\pi_0 A} A(x,x) \arrow[d, "i", "\simeq"'] \\
G\overline{W}A \arrow[r, "\epsilon"'] & A,
\end{tikzcd}$$

where we note that $G\overline{W}i$ is a weak equivalence by statements (1) and (2). The functors G and $\overline{W}$ both preserve disjoint unions, so it's enough to show that the simplicial group map $\epsilon : G\overline{W}A(x,x) \to A(x,x)$ is a weak equivalence, but this is the traditional result for simplicial groups (Proposition 6.3; see also Corollary 6.4).

Let K be a simplicial set. To show that the unit $\eta : K \to \overline{W}GK$ is a weak equivalence, it suffices to assume that K is a reduced Kan complex, by statements (1) and (2). Now apply Proposition 6.3. ■

Corollary 7.11. *The functors G and $\overline{W}$ induce an equivalence of homotopy categories*

$$\mathrm{Ho}(s\mathbf{Gd}) \simeq \mathrm{Ho}(\mathbf{S}).$$