Chapter IV Bisimplicial sets

This chapter is a basic exposition of the homotopy theory of bisimplicial sets and bisimplicial abelian groups.

A bisimplicial set can be viewed either as a simplicial object in the category of simplicial sets or a contravariant functor on the product $\mathbf{\Delta} \times \mathbf{\Delta}$ of two copies of ordinal number category $\mathbf{\Delta}$: both points of view are constantly exploited. Similar considerations apply to bisimplicial objects in any category, and to bisimplicial abelian groups in particular.

Categories of bisimplicial objects are ubiquitous sources of spectral sequence constructions. In many contexts, bisimplicial sets and bisimplicial abelian groups function as analogs of projective resolutions for homotopy theoretic objects. The Serre spectral sequence is one of the original examples: pullbacks over simplices of the base of a map $p : E \to B$ form a bisimplicial resolution of the total space. Then every bisimplicial set has canonically associated bisimplicial abelian groups and hence bicomplexes, and so a spectral sequence (5.3) drops out. If the map p happens to be a fibration, the resulting spectral sequence is the Serre spectral sequence (5.5). It is not much of a conceptual leap from this construction to the notion of a homology fibre sequence, which is the basis for the group completion theorem (Theorem 5.15), but then these ideas are essentially non-abelian, so that the theory can be pushed to give the basic detection principle for homotopy cartesian diagrams (Lemma 5.7) that is the basis of proof for Quillen's Theorem B (Theorem 5.6). Group completion and Theorem B are fundamental tools for algebraic K-theory. This collection of results appears in Section 5.

We begin in Section 1 with some definitions and examples of bisimplicial sets and abelian groups, which examples include all homotopy colimits. Section 2 contains a discussion of the basic features of bisimplicial abelian groups, including homotopy colimit objects and the generalized Eilenberg-Zilber theorem (Theorem 2.5). This theorem asserts that the two standard ways of extracting a chain complex from a bisimplicial abelian group, namely the chain complex associated to the diagonal simplicial abelian group and the total complex, are naturally chain homotopy equivalent.

A description of the formal homotopy theory of bisimplicial sets is given in Section 3. This homotopy theory is a little complicated, because there are closed model structures associated to multiple definitions of weak equivalence for these objects. The diagonal functor creates an external notion — one says that a map of bisimplicial sets is a diagonal weak equivalence if it induces a weak equivalence of associated diagonal simplicial sets. There are also two internal descriptions of weak equivalence, corresponding to viewing a bisimplicial set as a diagram in its vertical or horizontal simplicial sets. In particular, we say that

a bisimplicial set map is a pointwise (or vertical) weak equivalence if each of bisimplicial set
the induced maps of vertical simplicial sets is a weak equivalence. We discuss exponential law
closed model structures associated to all of these definitions. Diagonal weak bisimplex
equivalences are the objects of study in the Moerdijk structure, whereas pointwise weak equivalences figure into two different structures, namely the Bousfield-Kan structure in which the fibrations are also defined pointwise, and the Reedy structure where the cofibrations are defined pointwise. All of these theories are useful, and they are used jointly in the applications that follow, but this is by no means the end of the story: there is a further notion of E_2-weak equivalence due to Dwyer, Kan and Stover [28], [29] and a corresponding closed model structure that is not discussed here.

We confine ourselves here to applications of the homotopy theory of bisimplicial sets that involve detection of cartesian squares of bisimplicial set morphisms that become homotopy cartesian after applying the diagonal functor. There are two extant non-trivial techniques. One of these is the circle of ideas related to the Serre spectral sequence in Section 5, which has already been discussed. The other is dealt with in Section 4, and arises from the Reedy closed model structure, in the presence of the π_*-Kan condition (Theorem 4.9 of Bousfield and Friedlander). The π_*-Kan condition is satisfied widely in nature, in particular for all pointwise connected bisimplicial sets; it is best expressed by saying that the canonical maps from the homotopy group objects fibred over the simplicial set of vertical vertices of a bisimplicial set to the vertices are Kan fibrations. The Bousfield-Friedlander theorem leads to a spectral sequence (see (4.14)) for the homotopy groups of the diagonal of a pointwise connected bisimplicial set. This spectral sequence is the origin of the definition of E_2-weak equivalence that is referred to above.

1. Bisimplicial sets: first properties.

A *bisimplicial set* X is a simplicial object in the category of simplicial sets, or equivalently a functor $X : \mathbf{\Delta}^{op} \to \boldsymbol{S}$ where $\mathbf{\Delta}$ is the ordinal number category and $\boldsymbol{S}$ denotes the category of simplicial sets as before. Write $\boldsymbol{S}^2$ for the category of bisimplicial sets.

A bisimplicial set X can also be viewed as a functor

$$X : \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \to \boldsymbol{S},$$

(or as a contravariant functor on the category $\mathbf{\Delta} \times \mathbf{\Delta}$), by the categorical exponential law. From this point of view, the data for X consists of sets $X(m,n)$ with appropriately defined functions between them. The set $X(m,n)$ will often be called the set of *bisimplices of* X *of bidegree* (m,n), or the (m,n)-*bisimplices* of X. We shall also say that a bisimplex $x \in X(m,n)$ has horizontal degree m and vertical degree n.

EXAMPLE 1.1. Any simplicial set-valued functor $Z : I \to \boldsymbol{S}$ gives rise to a bisimplicial set with (m,n)-bisimplices

homotopy colimit
category, translation
classifying bisimplex

$$\bigsqcup_{i_0 \to i_1 \to \cdots \to i_m} Z(i_0)_n. \tag{1.2}$$

Note that the indexing is over simplices of degree m in the nerve BI of the category I, or equivalently over strings of arrows of length m in I.

This bisimplicial set is often called the *homotopy colimit* of the functor Z, and is denoted by $\underrightarrow{\mathrm{holim}}_I\, Z$.

The easiest way to see that $\underrightarrow{\mathrm{holim}}_I\, Z$ is in fact a bisimplicial set is to observe that each of the n-simplex functors Z_n gives rise to a *translation category* EZ_n having objects (i,x) with i an object of I and $x \in Z_n(i)$, and with morphisms $\alpha : (i,x) \to (j,y)$ where $\alpha : i \to j$ is a morphism of I such that $Z_n(\alpha)(x) = y$. Then the set of m-simplices of the nerve BEZ_n is the set displayed in (1.2). Furthermore, the data is simplicial in n, so the simplicial object $BEZ = \underrightarrow{\mathrm{holim}}_I\, Z$ is a bisimplicial set.

EXAMPLE 1.3. The bisimplicial set $\Delta^{k,l}$ is the simplicial object in $\boldsymbol{S}$ which is composed of the simplicial sets

$$\bigsqcup_{\mathbf{n} \xrightarrow{\theta} k} \Delta^l,$$

where Δ^l is the standard l-simplex in $\mathbf{S}$ and the disjoint union is indexed over morphisms $\theta : \mathbf{n} \to \mathbf{k}$ in the ordinal number category $\boldsymbol{\Delta}$. The bisimplicial set $\Delta^{k,l}$ classifies bisimplices of bidegree (k,l) in the sense that there is a simplex

$$\iota_{k,l} \in \bigsqcup_{\mathbf{k} \xrightarrow{\tau} \mathbf{k}} \Delta^l,$$

which is a *classifying* (k,l)*-simplex* in the sense that the bisimplices $x \in X(k,l)$ in a bisimplicial set X are in one to one correspondence with maps $\iota_x : \Delta^{k,l} \to X$ such that $\iota(\iota_{k,l}) = x$. Specifically, the *classifying bisimplex* $\iota_{k,l}$ is the copy of the classifying l-simplex $\iota_l \in \Delta^l_l$ in the summand corresponding to the identity map $1 : \mathbf{k} \to \mathbf{k}$.

It follows that $\Delta^{k,l}$ is the contravariant functor on $\boldsymbol{\Delta} \times \boldsymbol{\Delta}$ which is represented by the object $(\mathbf{k}, \mathbf{l})$.

EXAMPLE 1.4. Suppose that K and L are simplicial sets. Then there is a bisimplicial set $K \tilde{\times} L$ with (m,n)-bisimplices specified by

$$K \tilde{\times} L(m,n) = K_m \times L_n.$$

The bisimplicial set $K\tilde{\times}L$ will be called the *external product* of K with L. Note that the bisimplicial set $\Delta^{k,l}$ may be alternatively described as the external product $\Delta^k\tilde{\times}\Delta^l$.

external product
diagonal simplicial set
vertical, simplicial sets

The *diagonal simplicial set* $d(X)$ of a bisimplicial set X has n-simplices given by

$$d(X)_n = X(n,n).$$

It can also be viewed as the composite functor

$$\mathbf{\Delta}^{op} \xrightarrow{\Delta} \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \xrightarrow{X} \mathbf{S},$$

where Δ is the diagonal functor.

Think of the bisimplicial set X as a simplicial object in the simplicial set category by defining the *vertical simplicial sets* $X_n = X(n,*)$. Any morphism $\theta : \mathbf{m} \to \mathbf{n}$ gives rise to a diagram

(1.5)
$$\begin{tikzcd} X_n \times \Delta^m \arrow[r, "1 \times \theta_*"] \arrow[d, "\theta^* \times 1"'] & X_n \times \Delta^n \\ X_m \times \Delta^m. \end{tikzcd}$$

The collection of all such maps θ therefore determines a pair of maps

$$\bigsqcup_{\mathbf{m}\xrightarrow{\theta}\mathbf{n}} X_n \times \Delta^m \rightrightarrows \bigsqcup_n X_n \times \Delta^n,$$

by letting the restriction of the displayed maps to the summand corresponding to θ be $1 \times \theta_*$ and $\theta^* \times 1$, respectively. There are also simplicial set maps

$$\gamma_n : X_n \times \Delta^n \to d(X)$$

defined on r-simplices by

$$\gamma_n(x, \mathbf{r} \xrightarrow{\tau} \mathbf{n}) = \tau^*(x).$$

Here, x is an r-simplex of X_n, and $\tau^* : X_n \to X_r$ is a simplicial structure map of X, so that $\tau^*(x)$ is an r-simplex of $X(r,*)$ and is therefore an r-simplex of $d(X)$. The collection of the maps γ_n together determine a simplicial set map

$$\gamma : \bigsqcup_n X_n \times \Delta^n \to d(X).$$

Exercise 1.6. *Show that the resulting diagram*

$$\bigsqcup_{\mathbf{m} \xrightarrow{\theta} \mathbf{n}} X_n \times \Delta^m \rightrightarrows \bigsqcup_n X_n \times \Delta^n \xrightarrow{\gamma} d(X)$$

is a coequalizer in the category of simplicial sets.

The exercise implies that the diagonal simplicial set $d(X)$ is a coend in the category of simplicial sets for the data given by all diagrams of the form (1.5).

The diagonal simplicial set $d(X)$ has a natural filtration by subobjects $d(X)^{(p)}$, $p \geq 0$, where

$$d(X)^{(p)} = \text{image of } \Big(\bigsqcup_{0 \leq n \leq p} X_n \times \Delta^n \Big) \text{ in } d(X).$$

The degenerate part (with respect to the horizontal simplicial structure) of X_{p+1} is filtered by subobjects

$$s_{[r]} X_p = \bigcup_{0 \leq i \leq r} s_i(X_p) \subset X_{p+1}.$$

It follows from the simplicial identities that there are pushout diagrams

$$\begin{tikzcd} s_{[r]} X_{p-1} \arrow[r, "s_{r+1}"] \arrow[d, hook] & s_{[r]} X_p \arrow[d, hook] \\ X_p \arrow[r, "s_{r+1}"'] & s_{[r+1]} X_p, \end{tikzcd} \tag{1.7}$$

and

$$\begin{tikzcd} (s_{[p]} X_p \times \Delta^{p+1}) \cup (X_{p+1} \times \partial\Delta^{p+1}) \arrow[r] \arrow[d, hook] & d(X)^{(p)} \arrow[d, hook] \\ X_{p+1} \times \Delta^{p+1} \arrow[r] & d(X)^{(p+1)}. \end{tikzcd} \tag{1.8}$$

Diagrams (1.7) and (1.8) and the glueing lemma (Lemma II.9.8) are the basis of an inductive argument leading to the proof of

PROPOSITION 1.9. *Suppose that $f : X \to Y$ is a map of bisimplicial sets which is a pointwise weak equivalence in the sense that all of the maps $f : X_n \to Y_n$ are weak equivalences of simplicial sets. Then the induced map $f_* : d(X) \to d(Y)$ of associated diagonal simplicial sets is a weak equivalence.*

2. Bisimplicial abelian groups.

We collect here the basic facts about bisimplicial abelian groups, in two subsections.

The first of these is effectively about homotopy colimit constructions in the category of simplicial abelian groups. Any functor $A : I \to s\mathbf{Ab}$ taking values in simplicial abelian groups has an associated bisimplicial abelian group which we call translation object, and is formed by analogy with the homotopy colimit of a diagram of simplicial sets. We derive the basic technical result that translation object for the diagram A is weakly equivalent to the simplicial abelian group $A(t)$ if the index category I has a terminal object t.

When we say that a bisimplicial abelian group B is weakly equivalent to a simplicial abelian group C, we mean that the diagonal simplicial abelian group $d(B)$ is weakly equivalent to C within the simplicial abelian group category. One could alternatively interpret B as a bicomplex and C as a chain complex, and then ask for a weak equivalence between the chain complexes $\mathrm{Tot}(B)$ and C. The generalized Eilenberg-Zilber theorem says that $\mathrm{Tot}(B)$ and the chain complex $d(B)$ are naturally chain homotopy equivalent, so in fact there is no distinction between the two approaches to defining such weak equivalences. This is the subject of the second subsection. We further show that the standard spectral sequence

$$E_2^{p,q} = H_p(H_q A_*) \Rightarrow H_{p+q}(d(A))$$

for a bisimplicial abelian group A can be derived by methods which are completely internal to the simplicial abelian group category.

2.1. The translation object.

This section contains technical results concerning a simplicial abelian group-valued analogue of the homotopy colimit construction for diagrams of abelian groups, called the translation object. This construction is of fundamental importance for the description of the Serre spectral sequence that appears in a subsequent section of this chapter. More generally, it appears canonically in any homology spectral sequence arising from a homotopy colimit of a diagram of simplicial sets.

Supppose that $A : I \to \mathbf{Ab}$ is an abelian group valued functor, where I is a small category. There is a simplicial abelian group EA, with translation object

$$EA_n = \bigoplus_{\gamma:\mathbf{n}\to I} A_{\gamma(0)},$$

where the direct sum is indexed by the n-simplices of the nerve BI of the index category I. The abelian group homomorphism $\theta^* : EA_n \to EA_m$ induced by an ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$ is specified by requiring that all diagrams

$$\begin{tikzcd} A_{\gamma(0)} \arrow[r] \arrow[d, "in_\gamma"'] & A_{\gamma\theta(0)} \arrow[d, "in_{\gamma\theta}"] \\ \bigoplus_{\gamma:\mathbf{n}\to I} A_{\gamma(0)} \arrow[r, "\theta^*"'] & \bigoplus_{\zeta:\mathbf{m}\to I} A_{\zeta(0)} \end{tikzcd}$$

commute, where the homomorphism $A_{\gamma(0)} \to A_{\gamma\theta(0)}$ is induced by the relation $0 \leq \theta(0)$ in the ordinal number $\mathbf{n}$. The simplicial abelian group EA is called the *translation object* associated to the functor A.

Note that EA is not the nerve of a category, even though its definition is analogous to that of the nerve of a translation category for a set-valued functor.

Suppose given functors

$$J \xrightarrow{F} I \xrightarrow{A} \mathbf{Ab},$$

where J and I are small categories. The functor F induces a simplicial abelian group homomorphism $F_* : E(AF) \to EA$, which is defined on n-simplices in such a way that the diagram

$$\begin{tikzcd} AF_{\theta(0)} \arrow[r, "1"] \arrow[d, "in_\theta"'] & A_{F\theta(0)} \arrow[d, "in_{F\cdot\theta}"] \\ \bigoplus_{\theta:\mathbf{n}\to J} AF_{\theta(0)} \arrow[r, "F_*"'] & \bigoplus_{\gamma:\mathbf{n}\to I} A_{\gamma(0)} \end{tikzcd}$$

commutes.

Any natural transformation $\omega : A \to B$ of functors $I \to \mathbf{Ab}$ determines a morphism $\omega_* : EA \to EB$ of simplicial abelian groups. This morphism ω_* is defined on n-simplices by the requirement that the following diagram commutes:

$$\begin{tikzcd} A_{\theta(0)} \arrow[r, "\omega"] \arrow[d, "in_\theta"'] & B_{\theta(0)} \arrow[d, "in_\theta"] \\ \bigoplus_{\theta:\mathbf{n}\to I} A_{\theta(0)} \arrow[r, "\omega_*"'] & \bigoplus_{\theta:\mathbf{n}\to I} B_{\theta(0)}. \end{tikzcd}$$

Now consider a functor $B : I \times \mathbf{1} \to \mathbf{Ab}$, and let $d^1 : I \to I \times \mathbf{1}$ and $d^0 : I \to I \times \mathbf{1}$ be defined, respectively, by $d^1(i) = (i, 0)$ and $d^0(i) = (i, 1)$. The maps $(i, 0) \to (i, 1)$ in $I \times \mathbf{1}$ induce a natural transformation $\eta : Bd^1 \to Bd^0$ of functors $I \to \mathbf{Ab}$, and hence induce a simplicial abelian group homomorphism $\eta_* : EBd^1 \to EBd^0$. In general, the group of n-simplices of the simplicial abelian group $EA \otimes \Delta^1$ can be identifed with the direct sum

$$\bigoplus_{(\theta,\gamma):\mathbf{n}\to I\times\mathbf{1}} A_{\theta(0)}.$$

In the case at hand, one finds a canonical map $h : EBd^1 \otimes \Delta^1 \to EB$, which is defined on n-simplices by the requirement that the diagram

$$\begin{tikzcd} B_{(\theta(0),0)} \arrow[r] \arrow[d, "in_{(\theta,\gamma)}"'] & B_{(\theta(0),\gamma(0))} \arrow[d, "in_{(\theta,\gamma)}"] \\ \bigoplus_{(\theta,\gamma):\mathbf{n}\to I\times\mathbf{1}} B_{(\theta(0),0)} \arrow[r, "h"'] & \bigoplus_{(\theta,\gamma):\mathbf{n}\to I\times\mathbf{1}} B_{(\theta(0),\gamma(0))} \end{tikzcd}$$

commutes, where the top horizontal map is induced by the morphism $(\theta(0), 0) \to (\theta(0), \gamma(0))$ in $I \times \mathbf{1}$. One can now check that there is a commutative diagram of simplicial abelian group homomorphisms

$$
\begin{tikzcd}
EBd^1 \arrow[d, "d^1"'] \arrow[drr, "d^1_*"] & & \\
EBd^1 \otimes \Delta^1 \arrow[rr, "h"'] & & EB. \\
EBd^1 \arrow[u, "d^0"] \arrow[r, "\eta_*"'] & EBd^0 \arrow[ur, "d^0_*"'] &
\end{tikzcd}
\tag{2.1}
$$

Lemma 2.2. *Let $A : I \to \mathbf{Ab}$ be an abelian group valued functor on a small category I, and suppose that I has a terminal object t. Then there is a canonical weak equivalence $EA \to K(A_t, 0)$, which is specified on n-simplices by the homomorphism*

$$\bigoplus_{\theta:\mathbf{n}\to I} A_{\theta(0)} \to A_t$$

given on the summand corresponding to $\theta : \mathbf{n} \to I$ by the map $A_{\theta(0)} \to A_t$ induced by the unique morphism $\theta(0) \to t$ of the index category I.

Proof: Let t also denote the composite functor

$$I \to \{t\} \subset I.$$

Then the discrete category $\{t\}$ is a strong deformation retract of the category I in the sense that there is a commutative diagram of functors

$$
\begin{tikzcd}
I \arrow[d, "d^1"'] \arrow[dr, "1"] & \\
I \times \mathbf{1} \arrow[r, "H"] & I. \\
I \arrow[u, "d^0"] \arrow[ur, "t"'] &
\end{tikzcd}
$$

Now, $EAHd^1 = EA$, so that (2.1) can be used to show there is a commutative bisimplicial abelian group
diagram of simplicial abelian group homomorphisms

$$\begin{tikzcd}
EA \arrow[d, "d^1"'] \arrow[drr, "1"] & & \\
EA \otimes \Delta^1 \arrow[r, "h"'] & EAH \arrow[r, "H_*"'] & EA. \\
EA \arrow[u, "d^0"] \arrow[r, "\eta_*"'] & EAt \arrow[ur, "t_*"'] &
\end{tikzcd}$$

Note that $\eta_* t_*$ is the identity map on the simplicial abelian group EAt, so that the Moore complex of EA is chain homotopy equivalent to the Moore complex of EAt. The map $EA \to K(A_t, 0)$ factors as the weak equivalence $\eta_* : EA \to EA_t$, followed by the map $EAt \to K(A_t, 0)$, which map is defined on n-simplices by the codiagonal map

$$\bigoplus_{\theta : \mathbf{n} \to I} A_t \xrightarrow{\nabla} A_t.$$

This last homomorphism is a weak equivalence, because the space BI is contractible. ■

REMARK 2.3. The argument for Lemma 2.2 fails in the case where the index category has an initial object.

2.2. The generalized Eilenberg-Zilber theorem.

A *bisimplicial abelian group* A is a simplicial object in the category of simplicial abelian groups, or equivalently a functor of the form

$$A : \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \to \mathbf{Ab},$$

where $\mathbf{Ab}$ denotes the category of abelian groups, as before. Subject to the latter description, the simplicial abelian group $A(n, *)$ will be referred as the vertical simplicial abelian group in horizontal degree n associated to A, while the object $A(*, m)$ is the horizontal simplicial abelian group in vertical degree m. The category of bisimplicial abelian groups and natural transformations between them will be denoted by $s^2\mathbf{Ab}$.

It is often convenient to write $A_n = A(n, *)$ for the vertical simplicial abelian group in horizontal degree n. The simplicial abelian group morphism $A_n \to A_m$

associated to an ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$ will sometimes be denoted by θ^{*h}: this morphism is given on k-simplices by the abelian group homomorphism $(\theta, 1)^* : A(n,k) \to A(m,k)$. We shall also occasionally write $\tau^{*v} = (1,\tau)^* : A(n,k) \to A(n,p)$ for the vertical structure maps associated to $\tau : \mathbf{p} \to \mathbf{k}$.

Moore bicomplex
weak equivalence, pointwise
generalized Eilenberg-Zilber theorem

The diagonal simplicial abelian group $d(A)$ associated to the bisimplicial abelian group A has n-simplices given by $d(A)_n = A(n,n)$. The association $A \mapsto d(A)$ defines a functor $d : s^2\mathbf{Ab} \to s\mathbf{Ab}$. The diagonal functor is plainly exact. Furthermore, if B is a simplicial abelian group and $K(B,0)$ is the associated horizontally (or vertically) constant bisimplicial abelian group, then there is a natural isomorphism $d(K(B,0)) \cong B$.

The *Moore bicomplex* for a bisimplicial abelian group A is the bicomplex having (p,q)-chains $A(p,q)$, horizontal boundary

$$\partial_h = \sum_{i=0}^{p} (-1)^i d_i : A(p,q) \to A(p-1,q),$$

and vertical boundary

$$\partial_v = \sum_{j=0}^{q} (-1)^{p+j} d_j : A(p,q) \to A(p,q-1).$$

We shall also write A for the Moore bicomplex of a bisimplicial abelian group A, and $\operatorname{Tot} A$ will denote the associated total complex. Write also $A_n = A(n,*)$ for the Moore complex in horizontal degree n. Then filtering the bicomplex A in the horizontal direction gives a spectral sequence

$$E_2^{p,q} = H_p(H_q A_*) \Rightarrow H_{p+q}(\operatorname{Tot} A). \tag{2.4}$$

It follows that, if the bisimplicial abelian group map $f : A \to B$ is a *pointwise weak equivalence* in the sense that all of the maps $f : A_n \to B_n$, $n \geq 0$, of vertical simplicial abelian group are weak equivalences, then f induces an homology isomorphism $f_* : \operatorname{Tot} A \to \operatorname{Tot} B$ of the associated total complexes. Of course, the meaning of "vertical" and "horizontal" are in the eyes of the beholder, so it follows immediately that any map of bisimplicial abelian groups which consists of weak equivalences on the horizontal simplicial abelian group level must again induce a homology isomorphism of total complexes. One can, alternatively, make an argument with the second spectral sequence for the homology of $\operatorname{Tot} A$ (constructed by filtering in the Moore bicomplex in the vertical direction).

The generalized Eilenberg-Zilber theorem of Dold and Puppe [20] asserts the following:

Theorem 2.5. *The chain complexes $d(A)$ and* $\mathrm{Tot}\, A$ *are chain homotopy equivalent. This equivalence is natural with respect to morphisms of bisimplicial abelian groups A.*

Proof: Suppose that K and L are simplicial sets. The usual Eilenberg-Zilber theorem asserts that there are natural chain maps

$$f : \mathbb{Z}(K \times L) \to \mathrm{Tot}(\mathbb{Z}K \otimes \mathbb{Z}L),$$

and

$$g : \mathrm{Tot}(\mathbb{Z}K \otimes \mathbb{Z}L) \to \mathbb{Z}(K \times L),$$

and that there are natural chain homotopies $fg \simeq 1$ and $gf \simeq 1$. Specializing K and L to the standard simplices gives bicosimplicial chain maps

$$f : \mathbb{Z}(\Delta^p \times \Delta^q) \to \mathrm{Tot}(\mathbb{Z}\Delta^p \otimes \mathbb{Z}\Delta^q),$$

and

$$g : \mathrm{Tot}(\mathbb{Z}\Delta^p \otimes \mathbb{Z}\Delta^q) \to \mathbb{Z}(\Delta^p \times \Delta^q),$$

as well as bicosimplicial chain homotopies $fg \simeq 1$ and $gf \simeq 1$.

Observe that $\Delta^p \times \Delta^q = d(\Delta^{p,q})$, where $\Delta^{p,q}$ is the bisimplicial set represented by the pair of ordinal numbers $(\mathbf{p}, \mathbf{q})$. It follows that the chain complex $\mathbb{Z}(\Delta^p \times \Delta^q)$ can be identified up to natural isomorphism with $d(\mathbb{Z}\Delta^{p,q})$. Note as well that, up to natural isomorphism, $\mathrm{Tot}(\mathbb{Z}\Delta^p \otimes \mathbb{Z}\Delta^q)$ is the total complex of the bisimplicial abelian group $\mathbb{Z}\Delta^{p,q}$. Every bisimplicial abelian group A sits in a functorial exact sequence

$$\bigoplus_{(\mathbf{r},\mathbf{s}) \to (\mathbf{p},\mathbf{q})} \mathbb{Z}\Delta^{r,s} \to \bigoplus_{(\mathbf{p},\mathbf{q})} \mathbb{Z}\Delta^{p,q} \to A \to 0.$$

The functors Tot and d are both right exact and preserve direct sums, so the chain maps

$$f : d(\mathbb{Z}\Delta^{p,q}) \to \mathrm{Tot}\, \mathbb{Z}\Delta^{p,q}$$

uniquely extend to a natural chain map

$$f : d(A) \to \mathrm{Tot}\, A.$$

which is natural in bisimplicial abelian groups A. Similarly, the chain maps

$$g : \mathrm{Tot}\, \mathbb{Z}\Delta^{p,q} \to d(\mathbb{Z}\Delta^{p,q})$$

induce a natural chain map

$$g : \mathrm{Tot}\, A \to d(A).$$

The same argument implies that the bicosimplicial chain homotopies $fg \simeq 1$ and $gf \simeq 1$ extend uniquely to chain homotopies which are natural in bisimplicial abelian groups. ∎

Remark 2.6. The maps f and g in the proof of Theorem 2.5 can be precisely specified as the unique extensions of the classical Alexander-Whitneys and shuffle maps, respectively. The definitions will not be written down here (see [20], [64, pp.241–243]). Alexander-Whitney map shuffle map

The underlying acyclic models argument for the Eilenberg-Zilber theorem is somewhat less than conceptual, so that the usual approach of using the spectral sequence (2.4) and the generalized Eilenberg-Zilber theorem to construct the standard spectral sequence

$$E_2^{p,q} = \pi_p(\pi_q A_*) \Rightarrow \pi_{p+q} d(A)$$

is rather indirect. We can now give an alternative construction. The basic trick is to arrange for some independent means of showing the following:

Lemma 2.7. *A pointwise weak equivalence $f : A \to B$ of bisimplicial abelian groups induces a weak equivalence $f_* : d(A) \to d(B)$ of the associated diagonal complexes.*

Proof: There is a bisimplicial abelian group given in vertical degree m by the simplicial abelian group

$$\bigoplus_{\mathbf{m}\to\mathbf{n}_0\to\cdots\to\mathbf{n}_k} A(n_k, m).$$

This simplicial abelian group is the translation object associated to the functor $A(*, m) : (\mathbf{m} \downarrow \mathbf{\Delta})^{op} \to \mathbf{Ab}$ which is defined by associating to the object $\mathbf{m} \to \mathbf{n}$ the abelian group $A(n, m)$. The category $(\mathbf{m} \downarrow \mathbf{\Delta})^{op}$ has a terminal object, namely the identity map $1 : \mathbf{m} \to \mathbf{m}$, so Lemma 2.2 implies that the canonical simplicial abelian group map $EA(*, m) \to K(A(m, m), 0)$ is a weak equivalence. It follows that the Moore complex for $EA(*, m)$ is canonically weakly equivalent to the chain complex $A(m, m)[0]$ consisting of the group $A(m, m)$ concentrated in degree 0. The morphism of bicomplexes which is achieved by letting m vary therefore induces a natural weak equivalence of chain complexes $\operatorname{Tot} EA(*, *) \to d(A)$. On the other hand, the vertical simplicial abelian group of $EA(*, *)$ in horizontal degree k has the form

$$\bigoplus_{\mathbf{n}_0\to\cdots\to\mathbf{n}_k} \Delta^{n_0} \otimes A(n_k, *).$$

It follows that any pointwise equivalence $f : A \to B$ induces a homology isomorphism $f_* : \operatorname{Tot} EA(*, *) \to \operatorname{Tot} EB(*, *)$, and hence a weak equivalence $f_* : d(A) \to d(B)$. ∎

Define the *horizontal normalization* N_hA of a bisimplicial abelian group A horizontal normalization to be the simplicial chain complex whose "n-simplices" are given by the chain complex $N_hA_n = NA(*,n)$. The bisimplicial abelian group A can be recovered from the simplicial chain complex N_hA by applying the functor Γ in all vertical degrees. The simplicial chain complex N_hA can be filtered: one defines a simplicial chain complex F_pN_hA as a chain complex object by specifying

$$F_pN_hA_i = \begin{cases} N_hA_i & \text{if } i \leq p, \text{ and} \\ 0 & \text{if } i > p. \end{cases}$$

Now, $F_{p-1}N_hA \subset F_pN_hA$, with quotient $N_hA_p[p]$, which can be thought of as the simplicial chain complex which is the simplicial abelian group N_hA_p in chain degree p and is 0 in other chain degrees. Applying the functor Γ to the corresponding short exact sequence gives a short exact sequence of bisimplicial abelian groups

$$0 \to \Gamma F_{p-1}N_hA \to \Gamma F_pN_hA \to \Gamma N_hA_p[p] \to 0.$$

It follows that the bisimplicial abelian groups ΓF_pN_hA filter A. In vertical degree n, we have an identification

$$\Gamma N_hA_p[p](*,n) = K(NA(*,n)_p, p),$$

since, in general, the Eilenberg-Mac Lane space $K(B,n)$ can be identified with the simplicial abelian group $\Gamma B[n]$ which arises by applying the functor Γ to the chain complex $B[n]$ which consists of B concentrated in degree n.

LEMMA 2.8. *There is an isomorphism*

$$\pi_nN_hA_p \cong N(\pi_nA_*)_p.$$

PROOF: Write

$$N_h^jA_p = \cap_{i=0}^j \ker(d_i^h) \subset A_p$$

for $0 \leq j \leq p-1$. We show that the canonical map

$$\pi_nN_h^jA_p \to N^j(\pi_nA_*)_p \tag{2.9}$$

is an isomorphism for all j.

The degree 0 case is shown by observing that there is a short exact sequence

$$0 \to N_h^0A_p \to A_p \xrightarrow{d_0^h} A_{p-1} \to 0$$

which is split by $s_0^h : A_{p-1} \to A_p$, so that the map $\pi_n N_h^0 A_p \to N^0(\pi_n A_*)_p$ is an isomorphism. Furthermore, the induced map $\pi_n N_h^0 A_p \to \pi_n A_p$ is a monomorphism.

Assume that the map (2.9) is an isomorphism, and that the induced map

$$\pi_n N_j^h A_p \to \pi_n A_p$$

is monic. Consider the pullback diagram of simplicial abelian groups

$$\begin{tikzcd} N_h^{j+1} A_p \arrow[r] \arrow[d, hook] & \ker(d_{j+1}^h) \arrow[d] \\ N_h^j A_p \arrow[r] & A_p, \end{tikzcd}$$

and form the pushout

$$\begin{tikzcd} N_h^{j+1} A_p \arrow[r] \arrow[d, hook] & \ker(d_{j+1}^h) \arrow[d] \\ N_h^j A_p \arrow[r] & N_h^j A_p + \ker(d_{j+1}^h). \end{tikzcd}$$

The inclusion map $N_h^{j+1} A_p \subset N_h^j A_p$ is split by the map $x \mapsto x - s_{j+1}^h d_{j+1}^h(x)$, and so a comparison of long exact sequences shows that the induced diagram of abelian group homomorphisms

$$\begin{tikzcd} \pi_n N_h^{j+1} A_p \arrow[r] \arrow[d, hook] & \pi_n \ker(d_{j+1}^h) \arrow[d] \\ \pi_n N_h^j A_p \arrow[r] & \pi_n(N_h^j A_p + \ker(d_{j+1}^h)) \end{tikzcd}$$

is a pushout. But the inclusion $\ker(d_{j+1}^h) \subset A_p$ is a split monomorphism, so that the induced map $\pi_n \ker(d_{j+1}^h) \to \pi_n A_p$ is monic, and the map $\pi_n N_h^j A_p \to \pi_n A_p$ is monic by the inductive assumption. It follows that the induced map

$$\pi_n(N_h^j A_p + \ker(d_{j+1}^h)) \to \pi_n A_p$$

is a monomorphism, and that the canonical sequence

$$0 \to \pi_n N_h^{j+1} A_p \to \pi_n N_h^j A_p \oplus \pi_n \ker(d_{j+1}^h) \to \pi_n A_p$$

is exact. ■

The alternative method for constructing a spectral sequence

$$E_2^{p,q} = \pi_p(\pi_q A_*) \Rightarrow \pi_{p+q} d(A) \tag{2.10}$$

for a bisimplicial abelian group A is now clear. The filtration $F_p N_h A$ for the bisimplicial abelian group A gives rise to short exact sequences

$$0 \to \Gamma F_{p-1} N_h A \to \Gamma F_p N_h A \to \Gamma N_h A_p[p] \to 0$$

Let $N_h A_p\langle p\rangle$ be simplicial chain complex

$$\cdots \to 0 \to N_h \overset{p}{A_p}[p] \xrightarrow{1} N_h \overset{p-1}{A_p}[p] \to 0 \to \cdots$$

which is non-trivial in horizontal degrees p and $p-1$ only. Then $N_h A_p\langle p\rangle$ is horizontally acyclic, and there is a short exact sequence of simplicial chain complexes

$$0 \to N_h A_p[p] \to N_h A_p\langle p\rangle \to N_h A_p[p-1] \to 0.$$

It therefore follows from Lemma 2.8 that there are natural isomorphisms

$$\pi_{p+q} \Gamma N_h A_p[p] \cong \pi_q N_h A_p \cong N(\pi_q A_*)_p. \tag{2.11}$$

Furthermore, if we define a spectral sequence by letting

$$E_1^{p,q} = \pi_{p+q} \Gamma N_h A_p[p],$$

then the differential $d_1 : E_1^{p,q} \to E_1^{p-1,q}$ is identified by the isomorphisms (2.11) with the standard differential

$$\partial : N(\pi_q A_*)_p \to N(\pi_q A_*)_{p-1}$$

in the normalized complex for the simplicial abelian group $\pi_q A_*$. It follows that there are canonical isomorphisms

$$E_2^{p,q} \cong \pi_p(\pi_q A_*),$$

and the construction of the spectral sequence (2.10) is complete.

3. Closed model structures for bisimplicial sets.

closed model structure, Bousfield-Kan
weak equivalence, pointwise
fibration, pointwise

There are three closed model structures on the category $\mathbf{S}^2$ of bisimplicial sets that will be discussed in this section, namely

(1) the *Bousfield-Kan structure*, in which a fibration is a pointwise fibration and a weak equivalence is a pointwise weak equivalence,
(2) the *Reedy structure*, in which a cofibration is a pointwise cofibration (aka. inclusion) of bisimplicial sets, and a weak equivalence is a pointwise weak equivalence, and
(3) the *Moerdijk structure*, in which a fibration (respectively weak equivalence) is a map f which induces a fibration (respectively weak equivalence) $d(f)$ of associated diagonal simplicial sets.

In all of the above, a map $f : X \to Y$ is said to be a *pointwise* weak equivalence (respectively fibration, cofibration) if each of the simplicial set maps $f : X_n \to Y_n$ is a weak equivalence (respectively fibration, cofibration). This idea was partially introduced at the end of Section 1.

3.1. The Bousfield-Kan structure.

The Bousfield-Kan closed model structure on the category $\mathbf{S}^2$ is a special case of a closed model structure introduced by Bousfield and Kan for the category $\mathbf{S}^I$ of I-diagrams of simplicial sets arising from a small category I and natural transformations between such. The special case in question corresponds to letting I be the opposite category $\mathbf{\Delta}^{op}$ of the ordinal number category.

A map $f : X \to Y$ is defined to be a weak equivalence (respectively fibration) in the Bousfield-Kan structure on $\mathbf{S}^I$ if each induced simplicial set map $f : X(i) \to Y(i)$ ("in sections") is a weak equivalence (respectively fibration). One says that such maps are *pointwise weak equivalences* (respectively *pointwise fibrations*). Cofibrations are defined by the left lifting property with respect to pointwise trivial fibrations, suitably defined.

The closed model axioms for the Bousfield-Kan structure on $\mathbf{S}^I$ are verified in Example II.7.11. They can also be seen directly by using a small object argument based on the observation that a map $f : X \to Y$ of $\mathbf{S}^I$ is a pointwise fibration if it has the right lifting property with respect to all maps

$$F_i\Lambda^n_k \to F_i\Delta^n, \quad i \in I,$$

and f is a pointwise trivial fibration if it has the right lifting property with respect to all induced maps

$$F_i\partial\Delta^n \to F_i\Delta^n, \quad i \in I.$$

Here, we need to know that $K \mapsto F_i K$ is the left adjoint of the i-sections functor $X \mapsto X(i)$, and explicitly

closed model structure, Reedy
n-truncated simplicial set *a{t}

$$F_i K(j) = \bigsqcup_{i \to j} K$$

defines the functor $F_i K$ at $j \in I$.

3.2. The Reedy structure.

First, a word or two about skeleta and coskeleta of simplicial sets — the skeleton construction for more general simplicial objects will be discussed in Section V.1.

Write $\mathbf{\Delta}_n$ for the full subcategory of the ordinal numbers on the objects $\mathbf{0}, \dots, \mathbf{n}$. An *$n$-truncated simplicial set* Y is a functor $Y : \mathbf{\Delta}_n^{op} \to \mathbf{Sets}$. Let $\mathbf{S}_n$ denote the category of n-truncated simplicial sets.

Every simplicial set $X : \mathbf{\Delta}^{op} \to \mathbf{Sets}$ gives rise to an n-truncated simplicial set $i_{n*}X$ by composition with the inclusion functor $i_n : \mathbf{\Delta}_n \subset \mathbf{\Delta}$. The n-truncation functor $X \mapsto i_{n*}X$ has a left adjoint $Y \mapsto i_n^* Y$ and a right adjoint $Z \mapsto i_n^! Z$, and these adjoints are defined by left and right Kan extension respectively. Explicitly, the set $i_n^* Y_m$ of m-simplices of $i_n^* Y$ is defined by

$$i_n^* Y_m = \varinjlim_{\mathbf{m} \to \mathbf{k} \leq \mathbf{n}} Y_k,$$

while

$$i_n^! Z_m = \varprojlim_{\mathbf{n} \geq \mathbf{k} \to \mathbf{m}} Z_k,$$

where the indicated morphisms in both cases are in the ordinal number category $\mathbf{\Delta}$.

EXERCISE 3.1. *Show that the canonical maps*

$$\eta : Y \to i_{n*} i_n^* Y$$

and

$$\epsilon : i_{n*} i_n^! Z \to Z$$

are isomorphisms.

There is a canonical map of simplicial sets $\phi : i_n^* Z \to i_n^! Z$ and a commutative diagram

$$\begin{tikzcd}
Z_m \arrow[r, "(\theta\gamma)^*"] \arrow[d, "\theta^*"'] & Z_k \\
i_n^* Z_{n+1} \arrow[r, "\phi"'] & i_n^! Z_{n+1} \arrow[u, "\gamma^*"']
\end{tikzcd}$$

matching object
fibration, Reedy
n-coskeleton *a{c}
n-skeleton *a{s}

for each composite

$$\mathbf{k} \xrightarrow{\gamma} \mathbf{n+1} \xrightarrow{\theta} \mathbf{m}$$

in the ordinal number category with $k, m \leq n$. It follows that an extension of the n-truncated simplicial set Z to an $(n+1)$-truncated simplicial set consists precisely of a factorization

(3.2)
$$\begin{tikzcd} i_n^* Z_{n+1} \arrow[r, hook] \arrow[dr, "\phi"'] & Z_{n+1} \arrow[d] \\ & i_n^! Z_{n+1} \end{tikzcd}$$

of the function $\phi : i_n^* Z_{n+1} \to i_n^! Z_{n+1}$. The indicated map is a monomorphism, because $i_n^* Z_{n+1}$ must be the degenerate part of Z_{n+1} on account of the universal property implicit in diagram (1.7).

All of the foregoing is completely natural, and gives corresponding results for diagrams of simplicial sets. In particular, an extension of an n-truncated bisimplicial set $Z : \mathbf{\Delta}_n^{op} \to \mathbf{S}$ to an $(n+1)$-truncated bisimplicial set consists of a factorization of the canonical simplicial set map $\phi : i_n^* Z_{n+1} \to i_n^! Z_{n+1}$ of the form (3.2). Of course, the extended object consists of the simplicial sets $Z_0, \dots Z_n, Z_{n+1}$.

In the literature, the simplicial set $i_{n*} i_{n-1}^! i_{(n-1)*} X$ arising from a bisimplicial set X is denoted by $M_n X$, and is called the n^{th} *matching object* for X [28], [29].

A map $p : X \to Y$ of bisimplicial sets is said to be *Reedy fibration* if

(1) the map $p : X_0 \to Y_0$ is a Kan fibration of simplicial sets, and
(2) each of the induced maps $p_* : X_n \to Y_n \times_{M_n Y} M_n X$ is a Kan fibration for $n > 0$.

It is common to write

$$\operatorname{cosk}_n X = i_n^! i_{n*} X.$$

and

$$\operatorname{sk}_n X = i_n^* i_{n*} X.$$

From this point of view, the canonical maps in the second condition in the definition are induced by the commutative diagram of simplicial set maps

$$\begin{tikzcd} X_n \arrow[r, "\eta"] \arrow[d, "p"'] & \operatorname{cosk}_{n-1} X_n \arrow[d, "p_*"] \\ Y_n \arrow[r, "\eta"'] & \operatorname{cosk}_{n-1} Y_n \end{tikzcd}$$

which arises from the naturality of the adjunction maps. In particular, a bisimplicial set X is Reedy fibrant if the simplicial set X_0 is a Kan complex and each of the maps $X_n \to \operatorname{cosk}_{n-1} X_n$, $n > 0$, is a Kan fibration.

LEMMA 3.3.

(1) *Suppose that a map $p : X \to Y$ is a Reedy fibration. Then p has the right lifting property with respect to all maps of bisimplicial sets which are pointwise cofibrations and pointwise weak equivalences.*

(2) *Suppose that $p : X \to Y$ is a Reedy fibration such that each of the fibrations*

$$X_n \to Y_n \times_{\operatorname{cosk}_{n-1} Y_n} \operatorname{cosk}_{n-1} X_n$$

is also a weak equivalence. Then p has the right lifting property with respect to all maps which are pointwise cofibrations.

PROOF: We' ll prove the first assertion. The second is similar.

Suppose given a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ V \arrow[r] & Y \end{tikzcd}$$

of bisimplicial set maps in which the map i is a pointwise cofibration and a pointwise weak equivalence. Suppose inductively that there is a commutative diagram of n-truncated bisimplicial set maps of the form

$$\begin{tikzcd} i_{n*}U \arrow[r] \arrow[d, "i"'] & i_{n*}X \arrow[d, "p"] \\ i_{n*}V \arrow[r] \arrow[ur, "\theta"] & i_{n*}Y \end{tikzcd}$$

Then there is an induced solid arrow diagram of simplicial set maps

$$\begin{tikzcd}
\mathrm{sk}_n V_{n+1} \cup_{\mathrm{sk}_n V_{n+1}} U_{n+1} \arrow[r] \arrow[d, "i_*"'] & X_{n+1} \arrow[d, "p_*"] \\
V_{n+1} \arrow[r] \arrow[ur, dashed] & Y_{n+1} \times_{\mathrm{cosk}_n Y_{n+1}} \mathrm{cosk}_n X_{n+1}
\end{tikzcd}$$

The map i_* is a trivial cofibration, since the functor $U \mapsto \mathrm{sk}_n U_{n+1}$ takes maps which are pointwise weak equivalences to weak equivalences of simplicial sets via diagram (1.7), and the diagram of simplicial set maps

$$\begin{tikzcd}
\mathrm{sk}_n U_{n+1} \arrow[r, hook] \arrow[d, "i_*"'] & U_{n+1} \arrow[d, "i"] \\
\mathrm{sk}_n V_{n+1} \arrow[r, hook] & V_{n+1}
\end{tikzcd}$$

is a pullback. The map p_* is a Kan fibration by assumption, so the indicated dotted arrow exists. ∎

Corollary 3.4. *Suppose that $p : X \to Y$ is a Reedy fibration such that each of the fibrations*

$$X_n \to Y_n \times_{\mathrm{cosk}_{n-1} Y_n} \mathrm{cosk}_{n-1} X_n$$

is also a weak equivalence. Then p is a pointwise weak equivalence.

Proof: The map p has the right lifting property with respect to the cofibrations $F_{\mathbf{n}}\partial\Delta^n \to F_{\mathbf{n}}\Delta^n$, so that p is a pointwise trivial fibration. ∎

Lemma 3.5. *Suppose that $f : X \to Y$ is a map of bisimplicial sets. Then f has factorizations*

$$\begin{tikzcd}
& Z \arrow[dr, "p"] & \\
X \arrow[ur, "i"] \arrow[rr, "f"] \arrow[dr, "j"'] & & Y \\
& W, \arrow[ur, "q"'] &
\end{tikzcd}$$

where

(1) *the map i is a pointwise cofibration and a pointwise weak equivalence and p is a Reedy fibration, and*
(2) *the map j is a pointwise cofibration and q is a Reedy fibration and a pointwise weak equivalence.*

Proof: We'll prove the second claim. The first has a similar argument.

It suffices to find a factorization $f = q \cdot j$, where j is a pointwise cofibration and q meets the conditions of Corollary 3.4.

Suppose, inductively, that we've found a factorization

$$\begin{tikzcd} i_{n*}X \arrow[r, "j"] \arrow[dr, "f"'] & W \arrow[d, "q"] \\ & i_{n*}Y \end{tikzcd}$$

in the category of n-truncated bisimplicial sets, such that j is a pointwise cofibration, and such that the maps $q : W_0 \to Y_0$ and

$$q_* : W_m \to Y_m \times_{i^!_{m-1}Y_m} i^!_{m-1}W_m, \qquad 0 < m \leq n$$

are trivial fibrations of simplicial sets. The commutative diagram

$$\begin{tikzcd} i_n^*X_{n+1} \arrow[rr] \arrow[d, "j_*"'] \arrow[ddr, "f_*"] & & X_{n+1} \arrow[rr] \arrow[ddr, "f_*"] & & i_n^!X_{n+1} \arrow[d, "j_*"'] \arrow[ddr, "f_*"] & \\ i_n^*W_{n+1} \arrow[rrrr] \arrow[dr, "q_*"'] & & & & i_n^!W_{n+1} \arrow[dr, "q_*"'] & \\ & i_n^*Y_{n+1} \arrow[rr] & & Y_{n+1} \arrow[rr] & & i_n^!Y_{n+1} \end{tikzcd}$$

induces a diagram

$$\begin{tikzcd} i_n^*W_{n+1} \arrow[r] \arrow[d] & i_n^*W_{n+1} \cup_{i_n^*X_{n+1}} X_{n+1} \arrow[d] \arrow[dl, "\gamma"'] \\ Y_{n+1} \times_{i_n^!Y_{n+1}} i_n^!W_{n+1} \arrow[r] & i_n^!W_{n+1}. \end{tikzcd}$$

Choose a factorization

$$\begin{tikzcd}
i_n^* W_{n+1} \cup_{i_n^* X_{n+1}} X_{n+1} \arrow[r, "j'"] \arrow[dr, "\gamma"'] & W_{n+1} \arrow[d, "q'"] \\
& Y_{n+1} \times_{i_n^! Y_{n+1}} i_n^! W_{n+1}
\end{tikzcd}$$

such that j' is a cofibration and q' is a trivial fibration of simplicial sets. Then the desired factorization of f at level $n+1$ is given by the maps

$$X_{n+1} \xrightarrow{j' \cdot j_*} W_{n+1} \xrightarrow{q_* \cdot q'} Y_{n+1}.$$

Note that the map

$$j_* : X_{n+1} \to i_n^* W_{n+1} \cup_{i_n^* X_{n+1}} X_{n+1}$$

is a cofibration. ∎

The matching space $M_n X = \operatorname{cosk}_{n-1} X_n$ is a special case of a construction which associates a simplicial set $M_K X$ to each pair consisting of a simplicial set K and a bisimplicial set X. Explicitly, the p-simplices of $M_K X$ are defined to be a set of simplicial set morphisms by setting

$$M_K X_p = \hom_{\mathbf{S}}(K, X(*, p)).$$

The simplicial set $M_K X$ is a *matching space* for K in the bisimplicial set X.

Subject to the tacit indentification

$$X_n = X(n, *),$$

the bisimplicial set $\operatorname{cosk}_{n-1} X$ has (m,p)-bisimplices specified by

$$\operatorname{cosk}_{n-1} X(m,p) = (\operatorname{cosk}_{n-1} X(*,p))_m.$$

It follows that the p-simplices of the simplicial set $\operatorname{cosk}_{n-1} X_n$ have the form

$$\begin{aligned}
\operatorname{cosk}_{n-1} X(n,p) &= (\operatorname{cosk}_{n-1} X(*,p))_n \\
&= \hom_{\mathbf{S}}(\operatorname{sk}_{n-1} \Delta^n, X(*,p)) \\
&= \hom_{\mathbf{S}}(\partial\Delta^n, X(*,p)) \\
&= M_{\partial\Delta^n} X_p,
\end{aligned}$$

so that $M_{\partial\Delta^n}X$ is naturally isomorphic to $M_nX = \operatorname{cosk}_{n-1} X_n$. The functor $K \mapsto M_KX$ is right adjoint to the functor $\mathbf{S} \to \mathbf{S}^2$ which is defined by $L \mapsto K\tilde{\times}L$.

Suppose given integers $s_0, \dots, s_r$ such that $0 \le s_0 < s_1 < \dots < s_r \le n$, and let $\Delta^n\langle s_0, \dots, s_r\rangle$ be the subcomplex of $\partial\Delta^n$ which is generated by the simplices $d_{s_j}\iota_n$, $j = 0, \dots, r$. Then the simplicial identities $d_id_j = d_{j-1}d_i$ for $i < j$ imply that the complex $\Delta^n\langle s_0, \dots, s_r\rangle$ can be inductively constructed by pushout diagrams or the form

$$\begin{tikzcd} \Delta^{n-1}\langle s_0, \dots, s_{r-1}\rangle \arrow[r, "d^{s_r-1}"] \arrow[d, hook] & \Delta^n\langle s_0, \dots, s_{r-1}\rangle \arrow[d, hook] \\ \Delta^{n-1} \arrow[r, "d^{s_r}"'] & \Delta^n\langle s_0, \dots, s_r\rangle. \end{tikzcd} \tag{3.6}$$

If $k \neq s_j$ for $j = 0, \dots, r$, so that $\Delta^n\langle s_0, \dots, s_r\rangle$ is a subcomplex of Λ^n_k, then k is a vertex of all of the generating simplices of the object $\Delta^{n-1}\langle s_0, \dots, s_{r-1}\rangle$ in the diagram (3.6). It follows that this copy of $\Delta^{n-1}\langle s_0, \dots, s_{r-1}\rangle$ is a subcomplex of $\Lambda^{n-1}_q \subset \Delta^{n-1}$, where $q = k$ if $k < s_r$ and $q = k-1$ if $k > s_r$.

Following Bousfield and Friedlander, we shall write

$$M_n^{(s_0,\dots,s_r)}X = M_{\Delta^n\langle s_0,\dots,s_r\rangle}X.$$

Then, in particular, $M_n^{(0,1,\dots,n)}X$ is yet another notation for $\operatorname{cosk}_{n-1} X_n$.

Lemma 3.7. *Suppose that the map $p : X \to Y$ is a Reedy fibration and a pointwise weak equivalence. Then p has the right lifting property with respect to all pointwise cofibrations.*

Proof: We show that each of the fibrations

$$X_m \to Y_m \times_{\operatorname{cosk}_{m-1} Y_m} \operatorname{cosk}_{m-1} X_m$$

is a weak equivalence, and then apply Lemma 3.3.

There are canoncial simplicial set morphisms

$$X_{n+1} \to Y_{n+1} \times_{M_{n+1}^{(0,\dots,k)}Y} M_{n+1}^{(0,\dots,k)}X$$

which generalize the map

$$X_{n+1} \to Y_{n+1} \times_{\operatorname{cosk}_n Y_{n+1}} \operatorname{cosk}_n X_{n+1}.$$

The idea of proof is to show that each of the maps

$$X_{n+1} \to Y_{n+1} \times_{M_{n+1}^{(0,\dots,k)}Y} M_{n+1}^{(0,\dots,k)}X$$

is a trivial fibration of simplicial sets, by induction on n.

There is a pullback diagram

$$\begin{tikzcd} Y_{n+1} \times_{M_{n+1}^{(0,\dots,k+1)}Y} M_{n+1}^{(0,\dots,k+1)}X \arrow[r] \arrow[d] & X_n \arrow[d] \\ Y_{n+1} \times_{M_{n+1}^{(0,\dots,k)}Y} M_{n+1}^{(0,\dots,k)}X \arrow[r] & Y_n \times_{M_n^{(0,\dots,k)}Y} M_n^{(0,\dots,k)}X, \end{tikzcd}$$

and so each of the vertical maps in the diagram

$$\begin{tikzcd} X_{n+1} \arrow[r] \arrow[dddddd, "p"'] & Y_{n+1} \times_{M_{n+1}^{(0,\dots,n+1)}Y} M_{n+1}^{(0,\dots,n+1)}X \arrow[d] \\ & Y_{n+1} \times_{M_{n+1}^{(0,\dots,n)}Y} M_{n+1}^{(0,\dots,n)}X \arrow[d] \\ & \vdots \arrow[d] \\ & Y_{n+1} \times_{M_{n+1}^{(0)}Y} M_{n+1}^{(0)}X \arrow[d] \\ & Y_{n+1} \arrow[from=uuuul, ""] \end{tikzcd}$$

is a trivial fibration by the inductive hypothesis. The map p is a weak equivalence, so that all of the intermediate maps

$$X_{n+1} \to Y_{n+1} \times_{M_{n+1}^{(0,\dots,k)}Y} M_{n+1}^{(0,\dots,k)}X$$

are weak equivalences as well. ∎

We've now done all the work that goes into the proof of

THEOREM 3.8. *The category $\mathbf{S}^2$ of bisimplicial sets, together with the classes of pointwise weak equivalences, pointwise cofibrations and Reedy fibrations, satisfies the axioms for a closed model category.*

simplicial presheaves
closed model structure, Moerdijk
fibration, diagonal
weak equivalence, diagonal
cofibration, Moerdijk

PROOF: The factorization axiom **CM5** is Lemma 3.5, and the lifting axiom **CM4** is Lemma 3.3 together with Lemma 3.7. ■

REMARK 3.9. There is a completely different approach to proving Theorem 3.8, which involves a closed model structure for a certain category of presheaves of simplicial sets [46], [51] (ie. for the "chaotic topology" on the ordinal number category $\mathbf{\Delta}$). The global fibrations for that theory coincide with the Reedy fibrations: seeing this requires having both the Reedy structure and the closed model structure for the simplicial presheaf category in hand. The applications of the Reedy structure given in the next section depend on the matching space description of Reedy fibrations, so there's nothing to be gained by introducing simplicial presheaf homotopy theory here.

3.3. The Moerdijk structure.

A map $f : X \to Y$ is said to be a *diagonal fibration* (respectively *diagonal weak equivalence*) if the induced map $f_* : d(X) \to d(Y)$ of associated diagonal simplicial sets is a Kan fibration (respectively weak equivalence). A *Moerdijk cofibration* is a map which has the left lifting property with respect to all maps which are diagonal fibrations and diagonal weak equivalences.

The diagonal functor $d : \mathbf{S}^2 \to \mathbf{S}$ has a left adjoint $d^* : \mathbf{S} \to \mathbf{S}^2$, which is completely determined by the requirements

(1) $d^*(\Delta^n) = \Delta^{n,n}$, and

(2) d^* preserves colimits.

This follows from the fact that every simplicial set is a colimit of its simplices. It implies that the bisimplicial set $d^*(\Lambda^n_k)$ is the subcomplex of $\Delta^{n,n}$ which is generated by the bisimplices $(d_i\iota_n, d_i\iota_n)$ for $i \neq k$, whereas, $d^*(\partial\Delta^n)$ is the subcomplex of $\Delta^{n,n}$ which is generated by all $(d_i\iota_n, d_i\iota_n)$.

Alternatively, the set of bisimplices $d^*(\Lambda^n_k)(r,s)$ can be characterized as the set of all pairs of ordinal number maps $(\alpha : \mathbf{r} \to \mathbf{n}, \beta : \mathbf{s} \to \mathbf{n})$ such that the images of the functions α and β miss some common element i, where $i \neq k$. Put a different way, the simplicial set of bisimplices $d^*(\Lambda^n_k)(*,\beta)$ can be identified with the subcomplex C_β of Λ^n_k which is generated by the faces $d_i\iota_n$ which contain the s-simplex β. This observation is the heart of the proof of

LEMMA 3.10. *The inclusion map $d^*(\Lambda^n_k) \subset d^*(\Delta^n) = \Delta^{n,n}$ is a diagonal weak equivalence of bisimplicial sets.*

Proof: The projection map

$$(\alpha : \mathbf{r} \to \mathbf{n}, \beta : \mathbf{s} \to \mathbf{n}) \mapsto \beta$$

defines a map of bisimplicial sets

$$\bigsqcup_{\beta \in \Lambda^n_k} C_\beta \to \bigsqcup_{\beta \in \Lambda^n_k} *,$$

which in turn induces a map of simplicial sets

$$pr : d(d^*(\Lambda^n_k)) \to \Lambda^n_k$$

after applying the diagonal functor. The complex C_β of Δ^n is covered by subcomplexes isomorphic to Δ^{n-1}, each of which contains the vertex k. It follows that the contracting homotopy

$$(x, t) \mapsto tx + (1-t)v_k$$

of the affine simplex $|\Delta^n|$ onto the vertex v_k corresponding to the element $k \in \mathbf{n}$ restricts to a contracting homotopy

$$|C_\beta| \times I \to |C_\beta|.$$

The map pr is therefore a weak equivalence of simplicial sets, by Proposition 1.9, so $d(d^*(\Lambda^n_k))$ is contractible. ∎

Remark 3.11. In the proof of Lemma 3.10, the complex C_β is a subcomplex of Λ^n_k of the form

$$C_\beta = \Delta^n\langle s_0, \dots, s_r\rangle.$$

One can alternatively give a combinatorial argument for the "contractibility" of this complex by making an inductive argument based on the existence of the pushout diagram (3.6).

The diagonal functor d also has a right adjoint $d_* : \mathbf{S} \to \mathbf{S}^2$: the bisimplicial set d_*K which is associated to the simplicial set K by this functor is defined by

$$d_*K_{p,q} = \hom_{\mathbf{S}}(\Delta^p \times \Delta^q, K).$$

Lemma 3.12. *The functor $d_* : \mathbf{S} \to \mathbf{S}^2$ takes fibrations to diagonal fibrations.*

Proof: Suppose that $p : X \to Y$ is a Kan fibration, and suppose that there is a commutative diagram of simplicial set maps

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, "j"'] & d(d_*X) \arrow[d, "d(d_*p)"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & d(d_*Y) \end{tikzcd}$$

Then the indicated lifting exists if and only if the lifting exists in the diagram

$$\begin{tikzcd} d(d^*\Lambda^n_k) \arrow[r] \arrow[d, "d(d^*j)"'] & X \arrow[d, "p"] \\ d(d^*\Delta^n) \arrow[r] \arrow[ur, dashed] & Y, \end{tikzcd}$$

and Lemma 3.10 says that the inclusion $d(d^*j)$ is a trivial cofibration. ∎

Now suppose that the map $i : U \to V$ of bisimplicial sets has the left lifting property with respect to all diagonal fibrations. Then the lifting exists in all diagrams

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & d_*X \arrow[d, "d_*p"] \\ V \arrow[r] \arrow[ur, dashed] & d_*Y \end{tikzcd}$$

where $p : X \to Y$ is a Kan fibration, so that the induced simplicial set map $i_* : d(U) \to d(V)$ has the left lifting property with respect to all Kan fibrations. In particular, the map i is a diagonal weak equivalence. In view of Theorem II.5.1, this implies the following:

Theorem 3.13. *The category $\mathbf{S}^2$ of bisimplicial sets, together with the classes of Moerdijk cofibrations, diagonal fibrations and diagonal weak equivalences, satisfies the axioms for a closed model category.*

An alternative proof of this result can be given by using a small object argument group object (as was done originally in [73]), based on the observation that a map $p : X \to Y$ of bisimplicial sets is a diagonal fibration (respectively a diagonal fibration and a diagonal weak equivalence) if and only if f has the right lifting property with respect to all maps $d^*(\Lambda^n_k) \subset \Delta^{n,n}$ (respectively with respect to all maps $d^*(\partial\Delta^n) \subset \Delta^{n,n}$). One of the outcomes of the small object argument is the assertion that every Moerdijk cofibration is a monomorphism of bisimplicial sets.

4. The Bousfield-Friedlander theorem.

Suppose that X is a pointwise fibrant bisimplicial set. The simplicial set

$$\bigsqcup_{x \in X(m,0)} \pi_n(X_m, x)$$

and the obvious simplicial map

$$\pi_m X : \bigsqcup_{x \in X(m,0)} \pi_n(X_m, x) \to \bigsqcup_{x \in X(m,0)} * = X(m,0)$$

together form a group object in the category $\mathbf{S} \downarrow X(*,0)$ of simplicial sets over the vertex simplicial set $X(*,0)$. This group object is abelian if $m > 1$.

Recall that a vertex $v \in M_K X$ is a simplicial set morphism $v : K \to X(*,0)$. The set of morphisms

$$M_K(\pi_m X, v) = \hom_{\mathbf{S}\downarrow X(*,0)}(v, \pi_m X)$$

therefore has a group structure for $m \geq 1$, which is abelian for $m \geq 2$.

Suppose that $i \mapsto K(i)$ defines an I-diagram $K : I \to \mathbf{S}$ in the category of simplicial sets, and let

$$v : \varinjlim_{i \in I} K(i) \to X(*,0)$$

be a map of simplicial sets. Let $v(i) : K(i) \to X(*,0)$ be the composite

$$K(i) \xrightarrow{in_i} \varinjlim_{i \in I} K(i) \xrightarrow{v} X(*,0)$$

of v with the canonical map in_i of the colimiting cone. Then

$$v = \varinjlim_{i \in I} v_i$$

in the category $\mathbf{S} \downarrow X(*, 0)$ of simplicial sets over $X(*, 0)$, and so there is an isomorphism π_*-Kan condition *a{p}
vertical, path components

$$\hom_{\mathbf{S}\downarrow X(*,0)}(v, \pi_m X) \cong \varprojlim_{i \in I} \hom_{\mathbf{S}\downarrow X(*,0)}(v_i, \pi_m X).$$

It follows that there is an isomorphism

$$M_{\varinjlim_i K(i)}(\pi_m X, v) \cong \varprojlim_{i \in I} M_{K(i)}(\pi_m X, v_i).$$

In particular, the group $M_{\Delta^n}(\pi_m X, v)$ is canonically isomorphic to the group $\pi_m(X_n, v)$, where the map $v : \Delta^n \to X(*, 0)$ is identified with a vertex $v \in X(n, 0)$. Any such vertex $v : \Delta^n \to X(*, 0)$ restricts to a composite map

$$\Lambda^n_k \subset \Delta^n \xrightarrow{v} X(*, 0),$$

which will also be denoted by dv. It follows in particular that the corresponding group $M_{\Lambda^n_k}(\pi_m X, dv)$ fits into an equalizer diagram

$$M_{\Lambda^n_k}(\pi_m X, dv) \to \prod_{i \neq k} \pi_m(X_{n-1}, d_i v) \rightrightarrows \prod_{i<j;i,j\neq k} \pi_m(X_{n-2}, d_i d_j v), \tag{4.1}$$

where the parallel pair of arrows is defined by the simplicial identities $d_i d_j = d_{j-1} d_i$.

A pointwise fibrant bisimplicial set X is said to satisfy the *π_*-Kan condition* if the map

$$d : \pi_m(X_n, v) \to M_{\Lambda^n_k}(\pi_m X, dv)$$

induced by restriction along the inclusion $\Lambda^n_k \subset \Delta^n$ is a surjective group homomorphism for all $m \geq 1$ and all n, k that make sense. The π_*-Kan condition for X is equivalent to the requirement that all of the structure maps

$$\pi_m X \to X(*, 0)$$

for the homotopy group objects $\pi_m X$, $m \geq 1$, are Kan fibrations, on account of the description of $M_{\Lambda^n_k}(\pi_m X, dv)$ given in (4.1).

Suppose that Y is an arbitrary bisimplicial set, and write $\pi_0 Y$ for the simplicial set having n-simplices $\pi_0 Y_n = \pi_0 Y(n, *)$. This is the simplicial set of *vertical path components* of the bisimplicial set Y. There is a canonical simplicial set map $Y(*, 0) \to \pi_0 Y$.

Lemma 4.2.

(1) *A pointwise fibrant bisimplicial set X satisfies the π_*-Kan condition if all of the vertical simplicial sets $X_n = X(n,*)$ are path connected.*

(2) *Suppose that $f : X \to Y$ is a pointwise weak equivalence of pointwise fibrant bisimplicial sets. Then X satisfies the π_*-Kan condition if and only if Y satisfies the π_*-Kan condition.*

Proof: In the case of statement (1), there is a path ω from a given vertex $x \in X(n,*)$ to a horizontally degenerate vertex $s(y)$, where $y \in X(0,*)$ and so the action the corresponding morphism $[\omega]$ of the fundamental groupoid for X_n induces an isomorphism of maps

$$\begin{tikzcd}
\pi_m(X_n, x) \arrow[r, "d"] \arrow[d, "{[\omega]_*}"', "\cong"] & M_{\Lambda^n_k}(\pi_m X, x) \arrow[d, "\cong"', "{[\omega]_*}"] \\
\pi_m(X_n, s(y)) \arrow[r, "d"'] & M_{\Lambda^n_k}(\pi_m X, s(y)),
\end{tikzcd}$$

and the simplicial group $n \mapsto \pi_m(X_n, s(y))$ is a Kan complex.

To prove statement (2), note first of all that the map $f : X \to Y$ is a pointwise weak equivalence of pointwise fibrant bisimplicial sets if and only if

(a) the induced map $f_* : \pi_0 X \to \pi_0 Y$ is an isomorphism of simplicial sets, and

(b) the induced simplicial set diagrams

$$\begin{tikzcd}
\pi_m X \arrow[r, "f_*"] \arrow[d] & \pi_m Y \arrow[d] \\
X(*,0) \arrow[r, "f_*"'] & Y(*,0)
\end{tikzcd} \tag{4.3}$$

are pullbacks for $m \geq 1$.

Kan fibrations are stable under pullback, so if f is a pointwise weak equivalence and Y satisfies the π_*-Kan condition, then X satisfies the π_*-Kan condition.

Suppose that X satisfies the π_*-Kan condition, and that there is a diagram

fibrant model, pointwise
π_*-Kan condition *a{p}

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\alpha"] \arrow[d, hook] & \pi_m Y \arrow[d] \\ \Delta^n \arrow[r, "v"'] & Y(*,0) \end{tikzcd}$$

The simplicial set map $f_* : \pi_0 X \to \pi_0 Y$ is an isomorphism, so that the vertex $v \in Y(n,*)$ is homotopic to a vertex $f(w)$ for some vertex $w \in X(n,*)$. The argument for statement (1) implies that the π_*-Kan condition for the vertex v is equivalent to the π_*-Kan condition for the vertex $f(w)$, so we can replace v by $f(w)$. But now the diagram (4.3) is a pullback, by assumption, and so there is a commutative diagram

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\alpha_*"] \arrow[d, hook] & \pi_m X \arrow[r, "f_*"] \arrow[d] & \pi_m Y \arrow[d] \\ \Delta^n \arrow[r, "w"'] \arrow[ur, dashed] & X(*,0) \arrow[r, "f_*"'] & Y(*,0) \end{tikzcd}$$

where $f_* \circ \alpha_* = \alpha$. Then X satisfies the π_*-Kan condition, by assumption, so that the indicated lifting exists, and the π_*-Kan condition for the vertex $f(w)$ is verified. ∎

A *pointwise fibrant model* of a bisimplicial set X consists of a pointwise weak equivalence $j : X \to Z$, where Z is a pointwise fibrant bisimplicial set.

We shall say that an arbitrary bisimplicial set X satisfies the *π_*-Kan condition* if for any pointwise fibrant model $j : X \to Z$ of X, the pointwise fibrant object Z satisfies the π_*-Kan condition in the sense described above. Lemma 4.2 says that the π_*-Kan condition for pointwise fibrant bisimplicial sets is an invariant of pointwise weak equivalence, so (by appropriate manipulation of the Bousfield-Kan closed model structure for bisimplicial sets) it suffices to find only one pointwise fibrant model $j : X \to Z$ which satisfies the π_*-Kan condition. The π_*-Kan condition for arbitrary bisimplicial sets is also an invariant of weak equivalence.

Suppose that $f : X \to Y$ is a Reedy fibration. Let $i : K \subset L$ be an inclusion of simplicial sets, and observe that all induced bisimplicial set maps

$$(K \tilde{\times} \Delta^n) \cup_{(K \tilde{\times} \Lambda^n_k)} (L \tilde{\times} \Lambda^n_k) \hookrightarrow L \tilde{\times} \Delta^n$$

are pointwise trivial cofibrations of bisimplicial sets. The functor $X \mapsto M_K X$ is right adjoint to the functor $Y \mapsto K \tilde{\times} Y$, so it follows that the simplicial set map

$$M_L X \xrightarrow{(f_*, i^*)} M_L Y \times_{M_L X} M_K X \tag{4.4}$$

which is jointly induced by the Reedy fibration f and the inclusion i is a Kan fibration.

Lemma 4.5. *Suppose X is a Reedy fibrant bisimplicial set X that satisfies the π_*-Kan condition. Take a vertex $x \in X(n,*)$. Then there is a canonical isomorphism*

$$\pi_m(M_{\Lambda^n_k} X, dx) \cong M_{\Lambda^n_k}(\pi_m X, dx).$$

There is also an isomorphism

$$\pi_0(M_{\Lambda^n_k} X) \cong M_{\Lambda^n_k}(\pi_0 X).$$

Proof: Choose integers $0 \leq s_0 < s_1 < \cdots < s_r \leq n$ with $s_i \neq k$, and recall that

$$M_n^{(s_0,\dots,s_r)} X = M_{\Delta^n\langle s_0,\dots,s_r\rangle} X$$

is a subcomplex of $M_{\Lambda^n_k} X$. Write

$$M_n^{(s_0,\dots,s_r)}(\pi_m X, dx) = M_{\Delta^n\langle s_0,\dots,s_r\rangle}(\pi_m X, dx),$$

and let $M_n^{(s_0,\dots,s_r)}\pi_0 X$ denote the set

$$M_{\Delta^n\langle s_0,\dots,s_r\rangle}(\pi_0 X) = \hom_{\mathbf{S}}(\Delta^n\langle s_0,\dots,s_r\rangle, \pi_0 X).$$

The pushout diagram (3.6) induces a pullback diagram

$$\begin{tikzcd} M_n^{(s_0,\dots,s_r)} X \arrow[r] \arrow[d] & X_{n-1} \arrow[d, "d"] \\ M_n^{(s_0,\dots,s_{r-1})} X \arrow[r] & M_{n-1}^{(s_0,\dots,s_{r-1})} X \end{tikzcd} \tag{4.6}$$

Then the map d is an instance of the fibration (4.4), and inductively the canonical map

$$\pi_m(M_n^{(s_0,\dots,s_{r-1})} X, dx) \to M_n^{(s_0,\dots,s_{r-1})}(\pi_m X, dx)$$

is an isomorphism. The map

$$d : \pi_m(X_{n-1}, d_{s_r}x) \to M_{n-1}^{(s_0,\dots,s_{r-1})}(\pi_m X, dx)$$

is surjective for all $m \geq 1$, since X satisfies the π_*-Kan condition. It follows that the induced map

$$d_* : \pi_m(X_{n-1}, x) \to \pi_m(M_{n-1}^{(s_0,\dots,s_{r-1})}X, dx)$$

is surjective for all $m \geq 1$, and so the commutative square

$$\begin{tikzcd}
\pi_m(M_n^{(s_0,\dots,s_r)}X, dx) \arrow[r] \arrow[d] & \pi_m(X_{n-1}, d_{s_r}x) \arrow[d, "d_*"] \\
\pi_m(M_n^{(s_0,\dots,s_{r-1})}X, dx) \arrow[r] & \pi_m(M_{n-1}^{(s_0,\dots,s_{r-1})}X, dd_{s_r}x)
\end{tikzcd}$$

of abelian group homomorphisms is a pullback. The group $\pi_m(M_n^{(s_0,\dots,s_r)}X, dx)$ therefore has the required form.

The map

$$d_* : \pi_1(X_{n-1}, x) \to \pi_1(M_{n-1}^{(s_0,\dots,s_{r-1})}X, dx)$$

is surjective for all choices of base point $x \in X_{n-1}$, and so all inclusions $F_x \hookrightarrow X_{n-1}$ of fibres over dx induce injections $\pi_0 F_x \to \pi_0 X_{n-1}$. It follows that applying the path component functor π_0 to all diagrams of the from (4.6) gives pullback diagrams of sets. This is what is required to show inductively that the canonical maps

$$\pi_0(M_n^{(s_0,\dots s_r)}X) \to M_n^{(s_0,\dots,s_r)}\pi_0 X$$

are bijections. ■

LEMMA 4.7. *Suppose that X and Y are Reedy fibrant bisimplicial sets which satisfy the π_*-Kan condition, and that the bisimplicial set map $f : X \to Y$ is a Reedy fibration. Suppose further that the induced simplicial set map $f_* : \pi_0 X \to \pi_0 Y$ of vertical path components is a Kan fibration. Then the map f is a horizontal pointwise Kan fibration.*

To understand the meaning of the word "horizontal" in the statement of Lemma 4.7, note that every Reedy fibration is a pointwise Kan fibration, since the maps $F_{\mathbf{n}}\Lambda^m_k \to F_{\mathbf{n}}\Delta^m$ are trivial cofibrations. The assumptions of the lemma therefore imply that the simplicial sets $X(n,*)$ and $Y(n,*)$ are Kan complexes, and that the simplicial set maps $f : X(n,*) \to Y(n,*)$ are Kan fibrations. The lemma asserts that under the stated conditions the simplicial set maps $f : X(*,m) \to Y(*,m)$ are Kan fibrations as well.

Proof: We shall prove the lemma by showing that each canonical map

$$d : X_m \to Y_m \times_{M_{\Lambda^m_k} Y} M_{\Lambda^m_k} X$$

is a surjective simplicial set map. The map d is an instance of (4.4), hence a Kan fibration, so it suffices to show that d induces a surjective function

$$d_* : \pi_0 X_m \to \pi_0(Y_m \times_{M_{\Lambda^m_k} Y} M_{\Lambda^m_k} X)$$

in path components.

From the previous result, the induced map

$$\pi_0 X_m \to \pi_0(Y_m \times_{M_{\Lambda^n_k} Y} M_{\Lambda^n_k} X)$$

can be identified up to isomorphism with the map

$$\pi_0 X_n \to \pi_0 Y_n \times_{M_{\Lambda^n_k} \pi_0 Y} M_{\Lambda^n_k} \pi_0 X,$$

and the latter is surjective on account of the assumption that the induced map $\pi_0 X \to \pi_0 Y$ in vertical path components is a Kan fibration. ∎

Recall that the bisimplicial set $F_{\mathbf{n}} K$ is canonically isomorphic to the object $\Delta^n \tilde{\times} K$, and that the classifying bisimplex $\Delta^{m,n}$ is a copy of $\Delta^m \tilde{\times} \Delta^n$.

Lemma 4.8. *Suppose that $f : X \to Y$ is a Reedy fibration and a horizontal pointwise Kan fibration in the sense that each of the maps $f : X(*,n) \to Y(*,n)$ is a Kan fibration. Then f is a diagonal fibration.*

Proof: The cofibration

$$d^*\Lambda^n_k \to d^*\Delta^n = \Delta^{n,n}$$

factors as a composite of two maps

$$d^*\Lambda^n_k \subset \Lambda^n_k \tilde{\times} \Delta^n \subset \Delta^n \tilde{\times} \Delta^n.$$

The first map is a pointwise trivial cofibration, by the proof of Lemma 3.10. The second has the left lifting property with respect to all horizontal pointwise Kan fibrations. ∎

homotopy cartesian, pointwise

A commutative square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd}$$

of bisimplicial set maps is said to be *pointwise homotopy cartesian* if each of the induced squares of simplicial set maps

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & Y_n \arrow[d] \\ Z_n \arrow[r] & W_n \end{tikzcd}$$

is homotopy cartesian, for $n \geq 0$.

THEOREM 4.9 (BOUSFIELD-FRIEDLANDER). *Suppose given a pointwise homotopy cartesian square*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ Z \arrow[r] & W \end{tikzcd}$$

in the category of bisimplicial sets such that Y and W satisfy the π_*-Kan condition. Suppose further that the induced map $p : \pi_0 Y \to \pi_0 W$ of vertical path components is a Kan fibration. Then the associated commutative square

$$\begin{tikzcd} d(X) \arrow[r] \arrow[d] & d(Y) \arrow[d, "d(p)"] \\ d(Z) \arrow[r] & d(W) \end{tikzcd}$$

of diagonal simplicial sets is homotopy cartesian.

Proof: Construct a diagram

$$\begin{tikzcd} Y \arrow[r, "j_Y"] \arrow[d, "p"'] & Y' \arrow[d, "p'"] \\ W \arrow[r, "j_W"'] & W' \end{tikzcd}$$

in which j_Y and j_W are pointwise trivial cofibrations, W' is Reedy fibrant, and p' is a Reedy fibration. Then the composite square

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathbf{I}"] & Y \arrow[r, "j_Y"] \arrow[d, "p"] \arrow[dr, phantom, "\mathbf{II}"] & Y' \arrow[d, "p'"] \\ Z \arrow[r] & W \arrow[r, "j_W"'] & W' \end{tikzcd}$$

is pointwise homotopy cartesian by Lemma II.9.20, and the bisimplicial sets Y' and W' satisfy the π_*-Kan condition by Lemma 4.2. The Reedy fibrant objects Y' and W' are both pointwise fibrant.

Applying the diagonal functor gives a composite diagram in the simplicial set category

$$\begin{tikzcd} d(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathbf{I}"] & d(Y) \arrow[r, "j_{Y*}"] \arrow[d, "p_*"] \arrow[dr, phantom, "\mathbf{II}"] & d(Y') \arrow[d, "p'_*"] \\ d(Z) \arrow[r] & d(W) \arrow[r, "j_{W*}"'] & d(W'), \end{tikzcd} \tag{4.10}$$

in which the induced maps j_{Y*} and j_{W*} are weak equivalences by Proposition 1.9. Thus, in order to demonstrate that the square

$$\begin{tikzcd} d(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathbf{I}"] & d(Y) \arrow[d, "p_*"] \\ d(Z) \arrow[r] & d(W) \end{tikzcd}$$

is homotopy cartesian, it suffices, by Lemma II.9.20, to show that the composite pointwise k-connected
square (4.10) is homotopy cartesian.

The map p' is a diagonal fibration, by Lemma 4.7 and Lemma 4.8, and the induced bisimplicial set map

$$X \to Z \times_{W'} Y'$$

is a pointwise weak equivalence, since the Reedy fibration p' is a pointwise fibration. It follows that the induced map

$$d(X) \to d(Z) \times_{d(W')} d(Y')$$

of diagonal simplicial sets is a weak equivalence. ■

Corollary 4.11. *Suppose that a pointed bisimplicial set X is pointwise fibrant and pointwise connected. Then there is a weak equivalence*

$$d(\Omega X) \simeq \Omega d(X).$$

Proof: We have tacitly chosen a base point $x \in X(0,0)$ for all of the vertical simplicial sets $X(n,*)$ in our assumption that X is pointed. The loop object ΩX can then be characterized as the pointed bisimplicial set having vertical simplicial sets $\Omega X(n,*)$. There is a corresponding path space PX, which is the pointed bisimplicial set with vertical simplicial sets $PX(n,*)$, and a pointwise fibre sequence

$$\Omega X \to PX \xrightarrow{p} X.$$

The map p induces an isomorphism $\pi_0 PX \cong \pi_0 X$ of vertical path component simplicial sets. The bisimplicial sets PX and X both satisfy the π_*-Kan condition, and so applying the diagonal functor gives a homotopy fibre sequence

$$d(\Omega X) \to d(PX) \xrightarrow{d(p)} d(X)$$

in the category of pointed simplicial sets, by Theorem 4.9. The simplicial set $d(PX)$ is contractible, by Proposition 1.9. ■

A bisimplicial set X is said to be *pointwise k-connected* if each of the associated simplicial sets X_n is k-connected, for $n \geq 0$. A pointwise connected (or 0-connected) bisimplicial set can also be characterized as a bisimplicial set X such that the associated vertical path component simplicial set $\pi_0 X$ is a copy of the terminal object $*$.

Lemma 4.12. *Suppose that X is a bisimplicial set which is pointwise connected. Then the diagonal simplicial set $d(X)$ is connected.*

Proof: There is a coequalizer diagram

$$\pi_0 X(1,*) \underset{d_{1*}}{\overset{d_{0*}}{\rightrightarrows}} \pi_0 X(1,*) \longrightarrow \pi_0 d(X),$$

where d_{0*} and d_{1*} are induced by the horizontal face maps $d_0, d_1 : X(1,*) \to X(0,*)$. ∎

Proposition 4.13. *Suppose that X is a pointed bisimplicial set which is pointwise k-connected. The the diagonal $d(X)$ is a k-connected simplicial set.*

Proof: Choose a pointwise weak equivalence $i : X \to \tilde{X}$, where $\tilde{X}$ is pointwise fibrant. Then $d(i) : d(X) \to d(\tilde{X})$ is a weak equivalence. We may therefore presume that X is pointwise fibrant.

The space $d(X)$ is connected, by Lemma 4.12. Also, the bisimplical set ΩX is pointwise $(k-1)$-connected, and there are isomorphisms

$$\pi_j d(X) \cong \pi_{j-1} d(\Omega X),$$

for $j \geq 1$, by Corollary 4.11. This does it, by induction on k. ∎

Proposition 4.13 admits a vast generalization, in the sense that it is a consequence of a very general spectral sequence calculation. The existence of the spectral sequence in question is an easy consequence of Theorem 4.9, modulo a few technical observations.

First of all, if G is a simplicial group, then there is a bisimplicial set BG whose vertical simplicial in horizontal degree n is the classifying space BG_n of the group G_n of n-simplices of G. Similarly, the translation categories of the various groups G_n can be collected together to form a bisimplicial set EG and a canonical map $\pi : EG \to BG$. These bisimplicial sets are pointwise fibrant and connected in each horizontal degree, and the map π is a pointwise fibration. It therefore follows from Theorem 4.9 that there is an induced fibre sequence

$$G \to d(EG) \xrightarrow{\pi_*} d(BG).$$

Furthermore, the bisimplicial set EG consists of contractible simplicial sets EG_n, so that the associated diagonal $d(EG)$ is contractible, by Proposition 1.9. It follows that there are natural isomorphisms

$$\pi_n d(BG) \cong \pi_{n-1} G$$

for $n \geq 1$. The space $d(BG)$ is connected, by Lemma 4.12.

Bousfield-Kan spectral sequence

Suppose that X is a pointwise fibrant and pointed bisimplicial set such that each of the vertical simplicial sets X_n is an Eilenberg-Mac Lane space of the form $K(\pi_m, m)$, for some fixed number $m \geq 2$. Then Corollary III.3.8 implies that there is a pointwise weak equivalence of bisimplicial sets of the form

$$X \to \Gamma(\pi_m X[m]),$$

where $\pi_m X[m]$ denotes the chain complex of simplicial groups, concentrated in chain degree m. Then, by Theorem 4.9 or via chain complex arguments, one sees that there are natural isomorphisms

$$\pi_j d(X) \cong \pi_{j-m}(\pi_m X)$$

for $j \geq 0$.

Suppose finally that X is a pointwise fibrant bisimplicial set which is pointed and pointwise connected. Then the Postnikov tower construction applied to each of the vertical simplicial sets X_n induces a tower of pointwise fibrations

$$\cdots \to X(n) \to X(n-1) \to \cdots \to X(1) \to X(0),$$

such that the the fibre F_n of the map $X(n) \to X(n-1)$ is a diagram of Eilenberg-Mac Lane spaces of the form $K(\pi_n X, n)$. Each of the pointwise fibre sequences

$$F_n \to X(n) \to X(n-1)$$

induces a fibre sequence of associated diagonal simplicial sets, by Theorem 4.9. The resulting long exact sequences

$$\cdots \to \pi_{j+1} d(X(n-1)) \xrightarrow{\partial} \pi_j d(F_n) \to \pi_j d(X(n)) \to \pi_j d(X(n-1)) \to \dots$$

determine an exact couple which gives rise to a convergent spectral sequence with

$$E_2^{s,t} = \pi_{s+t} F_t \cong \pi_s(\pi_t X) \Rightarrow \pi_{s+t} d(X), \quad t+s \geq 0. \tag{4.14}$$

This spectral sequence is due to Bousfield and Friedlander [12, p.122]. It is a reindexed example of the Bousfield-Kan spectral sequence for a tower of fibrations, which will be discussed at more length in Section VII.3. Convergence for such spectral sequences is usually an issue, but it follows in this case from Proposition 4.13, which implies that the map $X \to X(n)$ induces isomorphisms

$$\pi_j d(X) \cong \pi_j d(X(n))$$

for $j \leq n$.

5. Theorem B and group completion.

translation category

The stream of ideas leading to Quillen's Theorem B begins with the most general formulation of the Serre spectral sequence.

5.1. The Serre spectral sequence.

Suppose that $p : E \to B$ is an arbitrary map of simplicial sets, and recall that $\mathbf{\Delta} \downarrow B$ denotes the simplex category for B. There is a functor $p^{-1} : \mathbf{\Delta} \downarrow B \to \mathbf{S}$ taking values in the simplicial set category which is defined by associating to the simplex $\sigma : \Delta^n \to B$ the simplicial set $p^{-1}(\sigma)$, where $p^{-1}(\sigma)$ is defined by the pullback diagram

$$\begin{tikzcd} p^{-1}(\sigma) \arrow[r, "\sigma_*"] \arrow[d] & E \arrow[d, "p"] \\ \Delta^n \arrow[r, "\sigma"'] & B. \end{tikzcd}$$

The homotopy colimit $\underrightarrow{\mathrm{holim}}\, p^{-1}$ arising from the functor $p^{-1} : \mathbf{\Delta} \downarrow B \to \mathbf{S}$ has vertical simplicial set in horizontal degree n given by

$$\bigsqcup_{\sigma_0 \to \cdots \to \sigma_n} p^{-1}(\sigma_0).$$

Note that this disjoint union is indexed by strings of arrows of length n in the simplex category $\mathbf{\Delta} \downarrow B$, and that these strings form the set of n-simplices of the nerve $B(\mathbf{\Delta} \downarrow B)$.

The simplicial set B is a colimit of its simplices in the simplicial set category, and pulling back along $p : E \to B$ is right exact, so that the maps $\sigma_* : p^{-1}(\sigma) \to E$ induce an isomorphism of simplicial sets

$$\varinjlim_{\sigma \in \mathbf{\Delta} \downarrow B} p^{-1}(\sigma) \cong E.$$

In particular, the set E_m of m-simplices of E may be identified with the set of path components of the translation category Ep_m^{-1} arising from the functor

$$\sigma \mapsto p^{-1}(\sigma)_m.$$

The objects of this category are pairs (σ, x), where $\sigma : \Delta^n \to B$ is a simplex of B and $x \in p^{-1}(\sigma)_m$, and a morphism $\theta : (\sigma, x) \to (\tau, y)$ consists of a morphism $\tau : \sigma \to \tau$ in $\mathbf{\Delta} \downarrow B$ such that $\tau_*(x) = y$. The nerve BEp_m^{-1} for this translation category coincides with the horizontal simplicial set in $\underrightarrow{\text{holim}}\, p^{-1}$ appearing in vertical degree m.

Write $Ep^{-1}_{m,x}$ for the path component of the translation category Ep^{-1}_m corresponding to a simplex x of E_m. This component $Ep^{-1}_{m,x}$ is the full subcategory of Ep^{-1}_m on objects of the form (σ, y), where $\sigma_*(y) = x$. In particular, y must have the form $y = (\theta, x)$ in

$$p^{-1}(\sigma) = \Delta^n \times_B E.$$

One sees, now, that the object $(p(x), (\iota_m, x))$ is initial in the category $Ep^{-1}_{m,x}$, so that $BEp^{-1}_{m,x}$ is contractible, and the simplicial set map

$$BEp^{-1}_m \to E_m \tag{5.1}$$

is a weak equivalence. We have taken the liberty of identifying the set E_m with the corresponding constant simplicial set $K(E_m, 0)$; by further abuse, the maps (5.1) are the horizontal components of a bisimplicial set map

$$\underrightarrow{\text{holim}}\, p^{-1} \to E,$$

which is a weak equivalence in each vertical degree. Proposition 1.9 therefore implies the following:

LEMMA 5.2. *Suppose that $p : E \to B$ is a map of simplicial sets. Then the canonical bisimplicial set map*

$$\underrightarrow{\text{holim}}\, p^{-1} \to E$$

induces a weak equivalence

$$d(\underrightarrow{\text{holim}}\, p^{-1}) \simeq E$$

of simplicial sets.

Now let A be an abelian group, and consider the bisimplicial abelian group $\mathbb{Z}(\underrightarrow{\text{holim}}\, p^{-1}) \otimes A$. The diagonal of this object is the simplicial abelian group $\mathbb{Z}(d(\underrightarrow{\text{holim}}\, p^{-1})) \otimes A$, and the weak equivalence $d(\underrightarrow{\text{holim}}\, p^{-1}) \to E$ of Lemma 5.2 induces a weak equivalence of simplicial groups

$$\mathbb{Z}(\underrightarrow{\text{holim}}\, p^{-1}) \otimes A \to \mathbb{Z}(E) \otimes A,$$

by universal coefficients and the fact that the free abelian group functor preserves weak equivalences (see Lemma III.2.14). Note as well that the bisimplicial abelian group $\mathbb{Z}(\underrightarrow{\operatorname{holim}}\, p^{-1}) \otimes A$ can be identified with the translation object associated to the functor defined on the simplex category $\mathbf{\Delta} \downarrow B$ by $\sigma \mapsto \mathbb{Z}(p^{-1}(\sigma)) \otimes A$ (Section II.4). It follows, by the results of Section 2, that there is a spectral sequence

Grothendieck spectral sequence

Serre spectral sequence

$$E_2^{p,q} = \pi_p E H_q(\mathbb{Z}(p^{-1}) \otimes A) \Rightarrow H_{p+q}(E, A). \tag{5.3}$$

In other words, $E_2^{p,q}$ is the p^{th} homotopy group of the translation object for the abelian group valued functor $H_q(\mathbb{Z}(p^{-1}) \otimes A) : \mathbf{\Delta} \downarrow B \to \mathbf{Ab}$ defined by

$$\sigma \mapsto H_q(\mathbb{Z}(p^{-1}(\sigma)) \otimes A) = H_q(p^{-1}(\sigma), A). \tag{5.4}$$

The spectral sequence (5.3) is sometimes called the *Grothendieck spectral sequence*, and is defined for any simplicial set map $p : E \to B$. This spectral sequence specializes to the *Serre spectral sequence* in the case where the map $p : E \to B$ is a fibration. When p is a fibration, any map $\theta : \sigma \to \tau$ in the simplex category $\mathbf{\Delta} \downarrow B$ induces a weak equivalence $\theta_* : p^{-1}(\sigma) \to p^{-1}(\tau)$, and hence induces isomorphisms

$$H_q(p^{-1}(\sigma), A) \xrightarrow[\cong]{\theta_*} H_q(p^{-1}(\tau), A)$$

for all $q \geq 0$. It follows (Theorem III.1.1) that the functors (5.4) factor through functors

$$H_q(\mathbb{Z}(p^{-1}) \otimes A) : \pi(B) = G(\mathbf{\Delta} \downarrow B) \to \mathbf{Ab},$$

which are defined on the fundamental groupoid $\pi(B)$ of the space B (see Theorem III.1.1).

If in addition B is simply connected, and F is the fibre over a choice of base point for B, then the functor $H_q(\mathbb{Z}(p^{-1}) \otimes A)$ is naturally isomorphic to the constant functor $\sigma \mapsto H_q(F, A)$ on the simplex category for B, and so there is a natural isomorphism

$$E_2^{p,q} = \pi_p E H_q(\mathbb{Z}(p^{-1}) \otimes A) \cong H_p(B(\mathbf{\Delta} \downarrow B), H_q(F, A)).$$

The assertion that there is a natural weak equivalence $d(\underrightarrow{\operatorname{holim}}\, p^{-1}) \to E$ can be specialized to the case of the identity map $B \to B$, implying that the bisimplicial set map

$$\bigsqcup_{\sigma_0 \to \cdots \to \sigma_n} \Delta^{m_0} \to B$$

(where $\sigma_j : \Delta^{m_j} \to B$ are simplices of B) induces a weak equivalence of diagonal simplicial sets. One also knows that the canonical bisimplicial set map Theorem B

$$\bigsqcup_{\sigma_0 \to \dots \to \sigma_n} \Delta^{m_0} \to \bigsqcup_{\sigma_0 \to \dots \to \sigma_n} *$$

is a pointwise weak equivalence, and hence induces a weak equivalence

$$d(\bigsqcup_{\sigma_0 \to \dots \to \sigma_n} \Delta^{m_0}) \to B(\mathbf{\Delta} \downarrow B).$$

It follows that there are isomorphisms

$$H_p(B(\mathbf{\Delta} \downarrow B), H_q(F, A)) \cong H_p(B, H_q(F, A)),$$

$p \geq 0$, and we obtain the standard form of the Serre spectral sequence

$$E_2^{p,q} = H_p(B, H_q(F, A)) \Rightarrow H_{p+q}(E, A). \tag{5.5}$$

5.2. Theorem B.

Quillen's Theorem B is the following:

THEOREM 5.6. *Suppose that $F : C \to D$ is a functor between small categories such that for every morphism $\alpha : y \to y'$ of D the induced simplicial set map $\alpha^* : B(y' \downarrow F) \to B(y \downarrow F)$ is a weak equivalence. Then, for every object y of D, the commutative diagram*

$$\begin{tikzcd}
B(y \downarrow F) \arrow[r] \arrow[d] & BC \arrow[d, "F_*"] \\
B(y \downarrow D) \arrow[r] & BD
\end{tikzcd}$$

of simplicial set maps is homotopy cartesian.

Here, one should recall that the objects of the comma category $y \downarrow F$ are pairs (τ, x), where x is an object of C and $\tau : y \to F(x)$ is a morphism of D. A morphism $\alpha : (\tau_0, x_0) \to (\tau_1, x_1)$ of $y \downarrow F$ is a morphism $\alpha : x_0 \to x_1$ of C such that the diagram

$$\begin{tikzcd} & F(x_0) \arrow[dd, "F(\alpha)"] \\ y \arrow[ur, "\tau_0"] \arrow[dr, "\tau_1"'] & \\ & F(x_1) \end{tikzcd}$$

commutes in the category D.

Theorem B has important applications in algebraic K-theory. In some sense, however, one of the steps in its proof is even more important, this being the following result:

Lemma 5.7. *Suppose that $X : I \to \mathbf{S}$ is a simplicial set valued functor which is defined on a small category I. Suppose further that the induced simplicial set map $X(\alpha) : X(i) \to X(j)$ is a weak equivalence for each morphism $\alpha : i \to j$ of the index category I. Then, for each object j of I the pullback diagram of simplicial sets*

$$\begin{tikzcd} X(j) \arrow[r] \arrow[d] & d(\underrightarrow{\operatorname{holim}}\, X) \arrow[d, "\pi"] \\ * \arrow[r, "j"'] & BI \end{tikzcd} \tag{5.8}$$

is homotopy cartesian.

Proof: The diagram (5.8) is obtained by applying the diagonal functor to the following pullback diagram of bisimplicial sets:

$$\begin{tikzcd} X(j) \arrow[r] \arrow[d] & \displaystyle\bigsqcup_{i_0 \to \dots \to i_n} X(i_0) \arrow[d] \\ * \arrow[r] & \displaystyle\bigsqcup_{i_0 \to \dots \to i_n} * \end{tikzcd}$$

In particular, the canonical map $\pi : d(\underrightarrow{\operatorname{holim}}\, X) \to BI$ is the map which is associated to the natural transformation of functors on I which is given by the simplicial set maps $X(i) \to *$.

The strategy of proof is to find a factorization

$$\begin{tikzcd} * \arrow[rr, "j"] \arrow[dr, "i"'] & & BI \\ & U \arrow[ur, "p"'] & \end{tikzcd}$$

of the inclusion of the vertex j in BI such that j is a trivial cofibration, p is a fibration, and such that the induced map $X(j) \to U \times_{BI} d(\underrightarrow{\operatorname{holim}}\, X)$ is a weak equivalence.

Pulling back along the map $\pi : d(\underrightarrow{\operatorname{holim}}\, X) \to BI$ preserves colimits in the category $\mathbf{S} \downarrow BI$ of simplicial set maps $K \to BI$. The small object argument therefore implies that it suffices to show that any diagram

(5.9)
$$\begin{tikzcd} & & d(\underrightarrow{\operatorname{holim}}\, X) \arrow[d, "\pi"] \\ \Lambda^n_k \arrow[r, hook, "i"'] & \Delta^n \arrow[r, "\sigma"'] & BI \end{tikzcd}$$

induces a weak equivalence

(5.10)
$$i_* : \Lambda^n_k \times_{BI} d(\underrightarrow{\operatorname{holim}}\, X) \to \Delta^n \times_{BI} d(\underrightarrow{\operatorname{holim}}\, X).$$

The simplex σ in the diagram (5.9) is a functor $\sigma : \mathbf{n} \to I$, and the space $\Delta^n \times_{BI} d(\underrightarrow{\operatorname{holim}}\, X)$ coincides up to isomorphism with the diagonal of the homotopy colimit $\underrightarrow{\operatorname{holim}}\, X\sigma$ associated to the composite functor

$$\mathbf{n} \xrightarrow{\sigma} I \xrightarrow{X} \mathbf{S}.$$

Furthermore, the map i_* in (5.10) can be identified with the diagonal of the map i_* in the following pullback diagram of bisimplicial sets:

$$\begin{tikzcd}
\displaystyle\bigsqcup_{\substack{k_0\to\cdots\to k_r\\ \in\Lambda^n_k}} X\sigma(k_0) \arrow[r, "i_*"] \arrow[d] & \displaystyle\bigsqcup_{\substack{k_0\to\cdots\to k_r\\ \in\Delta^n}} X\sigma(k_0) \arrow[d] \\
\displaystyle\bigsqcup_{\substack{k_0\to\cdots\to k_r\\ \in\Lambda^n_k}} * \arrow[r, "i"'] & \displaystyle\bigsqcup_{\substack{k_0\to\cdots\to k_r\\ \in\Delta^n}} *.
\end{tikzcd}$$

The initial object $0 \in \mathbf{n}$ determines a natural transformation $\theta : X\sigma(0) \to X\sigma$, where $X\sigma(0)$ denotes the constant functor at the object of the same name, and there is an induced diagram of bisimplicial set maps

$$\begin{tikzcd}
\displaystyle\bigsqcup_{\substack{k_0\to\cdots\to k_r\\ \in\Lambda^n_k}} X\sigma(0) \arrow[r] \arrow[d, "\theta_*"'] & \displaystyle\bigsqcup_{\substack{k_0\to\cdots\to k_r\\ \in\Delta^n}} X\sigma(0) \arrow[d, "\theta_*"] \\
\displaystyle\bigsqcup_{\substack{k_0\to\cdots\to k_r\\ \in\Lambda^n_k}} X\sigma(k_0) \arrow[r, "i_*"'] & \displaystyle\bigsqcup_{\substack{k_0\to\cdots\to k_r\\ \in\Delta^n}} X\sigma(k_0).
\end{tikzcd}$$

The vertical maps θ_* induce weak equivalences of associated diagonal simplicial sets, by Proposition 1.9, and the diagonal of the top horizontal map is the weak equivalence

$$i \times 1 : \Lambda^n_k \times X\sigma(0) \to \Delta^n \times X\sigma(0).$$

It follows that the map i_* of (5.10) is a weak equivalence. ■

There is a homology version of Lemma 5.7 for every homology theory h_* which satisfies the wedge axiom. Here is a specimen statement:

Lemma 5.11. *Suppose that $X : I \to \mathbf{S}$ is a simplicial set valued functor which is defined on a small category I, and that A is an abelian group. Suppose further that the induced simplicial set map $X(\alpha) : X(i) \to X(j)$ induces an isomorphism*

$H_*(X(i),A) \cong H_*(X(j),A)$ for each morphism $\alpha : i \to j$ of the index category I. Then, for each object j of I the pullback diagram of simplicial sets

$$\begin{tikzcd} X(j) \arrow[r] \arrow[d] & d(\underrightarrow{\mathrm{holim}}\, X) \arrow[d, "\pi"] \\ * \arrow[r, "j"'] & BI \end{tikzcd}$$

is homology cartesian in the sense that the corresponding map $X(j) \to F_j$ from $X(j)$ to the homotopy fibre F_j over j induces an isomorphism $H_*(X(j),A) \cong H_*(F_j,A)$.

The proof of Lemma 5.11 is a spectral sequence argument which follows the basic outline of the proof of Lemma 5.7.

PROOF OF THEOREM 5.6: The functor $y \mapsto B(y \downarrow F)$ determines a contravariant simplicial set valued functor $D \to \mathbf{S}$, with homotopy colimit having (m,n)-bisimplices

$$\bigsqcup_{y_n \to \cdots \to y_0} B(y_0 \downarrow F)_m$$

This set of bisimplices can also be identified with the set of all strings of arrows in D of the form

$$y_n \to \cdots \to y_0 \to F(x_0) \to \cdots \to F(x_m).$$

The degenerate simplices

$$y \xrightarrow{1} y \xrightarrow{1} \ldots \xrightarrow{1} y$$

determine a commutative diagram of bisimplicial set maps

$$\begin{tikzcd} B(y \downarrow F)_m \arrow[r] \arrow[d] & \bigsqcup_{y_n \to \cdots \to y_0} B(y_0 \downarrow F)_m \arrow[r, "Q_*"] \arrow[d] & BC_m \arrow[d, "F_*"] \\ B(y \downarrow D)_m \arrow[r] \arrow[d, "\simeq"'] & \bigsqcup_{y_n \to \cdots \to y_0} B(y_0 \downarrow D)_m \arrow[r, "Q_*"'] \arrow[d, "\simeq"] & BD_m \\ * \arrow[r] & \bigsqcup_{y_n \to \cdots \to y_0} *. & \end{tikzcd} \tag{5.12}$$

The bisimplicial set map

$$\bigsqcup_{y_n \to \cdots \to y_0} B(y_0 \downarrow F)_m \xrightarrow{Q_*} BC_m \tag{5.13}$$

is an alternate way of representing the forgetful map

$$\bigsqcup_{y_n \to \dots y_0 \to F(x_0) \to \cdots \to F(x_m)} * \to \bigsqcup_{x_0 \to \cdots \to x_m} *$$

corresponding to the functor F, and can also be identified with the map

$$\bigsqcup_{x_0 \to \cdots \to x_m} B(F(x_0) \downarrow D)^{op} \to \bigsqcup_{x_0 \to \cdots \to x_m} *.$$

The category $(F(x_0) \downarrow D)^{op}$ has a terminal object, so the map Q_* in (5.13) induces a weak equivalence of associated diagonals, by Proposition 1.9. The bisimplicial set map

$$\bigsqcup_{y_n \to \cdots \to y_0} B(y_0 \downarrow D)_m \xrightarrow{Q_*} BD_m$$

is an instance of the map in (5.13), corresponding to the case where F is the identity functor on the category D, so it induces a weak equivalence of associated diagonal simplicial sets as well. The categories $y \downarrow D$ and $y_0 \downarrow D$ have initial objects, so Proposition 1.9 implies that the indicated maps in the diagram (5.12) induce weak equivalences of associated diagonals.

Thus (see Lemma II.9.20), to show that the simplicial set diagram

$$\begin{tikzcd}
B(y \downarrow F) \arrow[r] \arrow[d] & BC \arrow[d, "F_*"] \\
B(y \downarrow D) \arrow[r] & BD
\end{tikzcd}$$

is homotopy cartesian, it suffices to see that the bisimplicial set diagram

$$\begin{tikzcd}
B(y \downarrow F)_m \arrow[r] \arrow[d] & \bigsqcup_{y_n \to \cdots \to y_0} B(y \downarrow F)_m \arrow[d] \\
* \arrow[r, "y"'] & \bigsqcup_{y_n \to \cdots \to y_0} *
\end{tikzcd}$$

induces a homotopy cartesian diagram of the associated diagonal simplicial sets. group completion
This is a consequence of Lemma 5.7. ■

5.3. The group completion theorem.

Suppose that M is a simplicial monoid, and that X is a simplicial set with an M-action $M \times X \to X$. There is a Borel construction for this action, namely a bisimplicial set $EM \times_M X$ having vertical simplicial set in horizontal degree n given by $M^{\times n} \times X$. The fastest way to convince yourself that this thing actually exists is to observe that the action $M \times X \to X$ is composed of actions $M_n \times X_n \to X_n$ of monoids of n-simplices on the corresponding sets X_n. These actions admit Borel constructions $EM_n \times_{M_n} X_n$ (or nerves of translation categories), and this construction is natural in n.

The canonical maps $\pi : EM_n \times_{M_n} X_n \to BM_n$ are also natural in n, and therefore define a map

$$\pi : EM \times_M X \to BM$$

of bisimplicial sets, which is given in horizontal degree n by the projection

$$M^{\times n} \times X \to M^{\times n}.$$

There is an obvious pullback diagram of bisimplicial set maps

$$\begin{tikzcd} X \arrow[r] \arrow[d] & EM \times_M X \arrow[d] \\ * \arrow[r] & BM \end{tikzcd} \tag{5.14}$$

The group completion theorem gives a criterion for this diagram to be homology cartesian.

Theorem 5.15 (Group Completion). *Suppose that $M \times X \to X$ is an action of a simplicial monoid M on a simplicial set X, and let A be an abelian group. Suppose further that the action of each vertex v of M induces an isomorphism $v_* : H_*(X, A) \cong H_*(X, A)$. Then the diagram (5.14) is homology cartesian in the sense that the map $X \to F$ to the homotopy fibre of the simplicial set map $d(\pi)$ induces an isomorphism in homology with coefficients in A.*

Theorem 5.15 is used, in the main, to analyze the output of infinite loop space machines. It implies, for example, that each connected component of the 0^{th} space of the Ω-spectrum corresponding to the sphere spectrum is a copy of the space $B\Sigma_\infty^+$ obtained by applying Quillen's plus construction to the classifying space of the infinite symmetric group [6], [83]. Here is another typical calculation:

Example 5.16. Suppose that R is a ring with identity. Then matrix addition induces a simplicial monoid structure on the simplicial set

$$M(R) = \bigsqcup_{n \geq 0} BGl_n(R).$$

In particular, right multiplication by the vertex $e = * \in BGl_1(R)_0$ (and all of its degeneracies) induces a simplicial set map

$$? \oplus e : M(R) \to M(R),$$

which restricts, on the n^{th} summand, to the map $BGl_n(R) \to BGl_{n+1}(R)$ which is induced by the canonical inclusion $Gl_n(R) \hookrightarrow Gl_{n+1}(R)$ defined by $A \mapsto A \oplus I$, where I denotes the identity element of $Gl_1(R)$. The filtered colimit of the system

$$M(R) \xrightarrow{? \oplus e} M(R) \xrightarrow{? \oplus e} \dots$$

can be identified up to isomorphism with the simplicial set

$$X(R) = \bigsqcup_{\mathbb{Z}} BGl(R).$$

The simplicial set $X(R)$ has an obvious left $M(R)$ action, and Theorem 5.15 implies that the diagram

$$\begin{tikzcd} X(R) \arrow[r] \arrow[d] & EM(R) \times_{M(R)} X(R) \arrow[d] \\ * \arrow[r] & BM(R) \end{tikzcd}$$

is homology cartesian. In effect, left multiplication by a vertex $n \in M(R)$ just shifts the vertices of $X(R)$: the summand corresponding to $r \in \mathbb{Z}$ is taken to the summand corresponding to $n + r$. The map induced on the r^{th} summand itself is the simplicial set map $BGl(R) \to BGl(R)$ which is induced by the group homomorphism $I_n \oplus ? : Gl(R) \to Gl(R)$ defined by $A \mapsto I_n \oplus A$, where I_n is the $n \times n$ identity matrix. As such, the group homomorphism $I_n \oplus ?$ is a filtered colimit of group homomorphisms $Gl_m(R) \to Gl_{m+n}(R)$. The key point is that

$I_n \oplus A$ is conjugate, via a suitable choice of permutation matrix, to $A \oplus I_n$ in $Gl_{m+n}(R)$. It follows that (vertical) components of the comparison map

$$\begin{tikzcd}
H_*(BGl_m(R),\mathbb{Z}) \arrow[r, "can_*"] \arrow[d, "(I_n\oplus ?)_*"'] & H_*(BGl_{m+1}(R),\mathbb{Z}) \arrow[r, "can_*"] \arrow[d, "(I_n\oplus ?)_*"'] & \dots \\
H_*(BGl_{m+n}(R),\mathbb{Z}) \arrow[r, "can_*"'] & H_*(BGl_{m+n+1}(R),\mathbb{Z}) \arrow[r, "can_*"'] & \dots
\end{tikzcd}$$

coincide with morphisms induced by canonical inclusions, and so the group homomorphism $I_n \oplus ?$ induces the identity map on $H_*(BGl(R),\mathbb{Z})$.

The space arising from the bisimplicial set $EM(R)\times_{M(R)} X(R)$ is contractible, since it is a filtered colimit of objects of the form $EM(R) \times_{M(R)} M(R)$. It follows that $X(R)$ has the homology of the loop space $\Omega d(BM(R))$, and that the component $\Omega d(BM(R))_0$ of $0 \in \mathbb{Z}$ is an H-space having the homology of $BGl(R)$. This component $\Omega d(BM(R))_0$ must therefore be a copy of Quillen's space $BGl(R)^+$.

Finally (without going into a lot of details), the monoidal structure on $M(R)$ is abelian up to coherent isomorphism, so effectively one is entitled to form a collection of connected objects

$$BM(R),\ BBM(R),\ B^3M(R),\dots$$

such that $B^{n+1}M(R)$ is a delooping of $B^nM(R)$ for all n, just like one could do if $M(R)$ happened to be a simplicial abelian group. The list of spaces corresponding to

$$\Omega BM(R),\ BM(R),\ BBM(R),\ B^3M(R),\dots$$

is the algebraic K-theory spectrum for the ring R; we have used the group completion theorem to identify its 0^{th} term.

There are several proofs of the group completion theorem in the literature: [48], [49], [73], [70]. The most elementary of these, and the one that will be given here, involves an analogue of the construction leading to the Serre spectral sequence, for maps of bisimplicial sets.

Suppose that $f : X \to Y$ is a map of bisimplicial sets, and consider all bisimplices $\sigma : \Delta^{r,s} \to Y$ of Y. Form the pullback diagram

$$\begin{tikzcd}
f^{-1}(\sigma) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\Delta^{r,s} \arrow[r] & Y
\end{tikzcd}$$

in the category of bisimplicial sets. The bisimplices $\Delta^{m,n} \to Y$ of Y are the category, of bisimplices
objects of the category $\mathbf{\Delta}^{\times 2} \downarrow Y$, called the *category of bisimplices* of Y. A morphism $\sigma \to \tau$ of this category is a commutative diagram of bisimplicial set maps

$$\begin{tikzcd}
\Delta^{r,s} \arrow[dd] \arrow[dr, "\sigma"] & \\
& Y. \\
\Delta^{m,n} \arrow[ur, "\tau"'] &
\end{tikzcd}$$

One sees immediately that the assignment $\sigma \mapsto f^{-1}(\sigma)$ defines a functor

$$f^{-1} : \mathbf{\Delta}^{\times 2} \downarrow Y \to \mathbf{S}^2.$$

Lemma 5.17. *Suppose that $f : X \to Y$ is a map of bisimplicial sets. Then the corresponding map*

$$\bigsqcup_{\substack{\sigma_0 \to \dots \to \sigma_r \\ \in B(\mathbf{\Delta}^{\times 2} \downarrow Y)}} f^{-1}(\sigma_0) \to X$$

of trisimplicial sets induces a weak equivalence of associated diagonal simplicial sets.

Proof: The bisimplicial set Y is a colimit of its bisimplices, and so X is a colimit of their pullbacks, giving rise to a coequalizer

$$\bigsqcup_{\sigma_0 \to \sigma_1} f^{-1}(\sigma_0) \rightrightarrows \bigsqcup_{\sigma} f^{-1}(\sigma) \to Y.$$

Also, the component of the simplicial set

$$\bigsqcup_{\sigma_0 \to \dots \to \sigma_r} f^{-1}(\sigma_0)(m,n)$$

corresponding to each bisimplex $x \in X(m,n)$ is contractible, since it's the nerve of a category having an initial object. It follows that diagonalizing the trisimplicial set map

$$\bigsqcup_{\substack{\sigma_0 \to \dots \to \sigma_r \\ \in B(\mathbf{\Delta}^{\times 2} \downarrow Y)}} f^{-1}(\sigma_0)(m,n) \to X(m,n)$$

first with respect to the variables r and m gives a bisimplicial set map

trisimplicial sets
n-fold simplicial sets *a{s}

$$\bigsqcup_{\substack{\sigma_0\to\cdots\to\sigma_r\\ \in B(\mathbf{\Delta}^{\times 2}\downarrow Y)}} f^{-1}(\sigma_0)(r,n) \to X(r,n)$$

which is a weak equivalence of simplicial sets in each vertical degree n, by Proposition 1.9. This same result then implies that the simplicial set map

$$\bigsqcup_{\substack{\sigma_0\to\cdots\to\sigma_r\\ \in B(\mathbf{\Delta}^{\times 2}\downarrow Y)}} f^{-1}(\sigma_0)(r,r) \to X(r,r)$$

is a weak equivalence. ■

Remark 5.18. There is a paradigm in the proof of Lemma 5.17 for the manipulation of trisimplicial sets. The diagonal of a trisimplicial set X is the simplicial set whose set of n-simplices is the set $X(n,n,n)$. This simplicial set defines the homotopy type arising from X, and it can be formed by iterating the diagonal construction for bisimplicial sets in three different ways. One picks the most convenient iteration for the problem at hand. Similar considerations apply, more generally, to n-fold simplicial sets

Consider the diagram of trisimplicial set maps

$$\begin{tikzcd}
f^{-1}(v) \arrow[r] \arrow[d] & \displaystyle\bigsqcup_{\sigma_0\to\cdots\to\sigma_r} f^{-1}(\sigma_0) \arrow[r, "\simeq"] \arrow[d] & X \arrow[d] \\
\Delta^{0,0} \arrow[r] \arrow[d, "\simeq"'] & \displaystyle\bigsqcup_{\sigma_0\to\cdots\to\sigma_r} \Delta^{m_0,n_0} \arrow[r, "\simeq"'] \arrow[d, "\simeq"] & Y \\
\ast \arrow[r, "v"'] & \displaystyle\bigsqcup_{\sigma_0\to\cdots\to\sigma_r} \ast, &
\end{tikzcd}$$

where $v : \Delta^{0,0} \to Y$ is a vertex of Y. Observe that $\ast = \Delta^{0,0}$ is the terminal object in the bisimplicial set category. The labelled horizontal maps induce weak equivalences of the corresponding diagonal simplicial sets by Lemma 5.17, while the corresponding vertical maps induce weak equivalences of diagonals by iterated application of Proposition 1.9. It follows from Lemma 5.11 that the pullback diagram

$$\begin{tikzcd} f^{-1}(v) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ * \arrow[r] & Y \end{tikzcd}$$

of bisimplicial sets induces a homology cartesian diagram of associated diagonals if each morphism of bisimplices

$$\begin{tikzcd} \Delta^{r_0,s_0} \arrow[dr, "\sigma_0"] \arrow[dd] & \\ & Y. \\ \Delta^{r_1,s_1} \arrow[ur, "\sigma_1"'] & \end{tikzcd}$$

induces an isomorphism

$$H_*(f^{-1}(\sigma_0)) \xrightarrow{\cong} H_*(f^{-1}(\sigma_1)).$$

By this, it is meant that there should be an induced isomorphism

$$H_*(d(f^{-1}(\sigma_0)), A) \xrightarrow{\cong} H_*(d(f^{-1}(\sigma_1)), A),$$

relative to some choice of coefficient group A. Let's agree to suppress mention of the diagonals and the coefficient groups in the rest of this section.

Proof of Theorem 5.15: We will show that each morphism

$$\begin{tikzcd} \Delta^{r,s} \arrow[dr, "\tau"] \arrow[dd, "(\zeta_1,\zeta_2)"'] & \\ & BM. \\ \Delta^{k,\ell} \arrow[ur, "\sigma"'] & \end{tikzcd} \tag{5.19}$$

induces an isomorphism

$$(\zeta_1, \zeta_2)_* : H_*(\pi^{-1}(\tau)) \xrightarrow{\cong} H_*(\pi^{-1}(\sigma)),$$

where $\pi : EM \times_M X \to BM$ is the canonical map.

Recall that, in horizontal degree m, $\Delta^{k,\ell}_m$ may be identified with the simplicial set

$$\bigsqcup_{\mathbf{m}\to\mathbf{k}} \Delta^\ell.$$

There is a pullback diagram of simplicial sets

$$\begin{tikzcd}
\bigsqcup_{\mathbf{m}\to\mathbf{k}} (\Delta^\ell \times X) \arrow[r] \arrow[d] & M^{\times m} \times X \arrow[d, "pr"] \\
\bigsqcup_{\mathbf{m}\to\mathbf{k}} \Delta^\ell \arrow[r] & M^{\times m},
\end{tikzcd}$$

where the map on the left is a disjoint union of projections. Each ordinal number map $\theta : \mathbf{n} \to \mathbf{m}$ induces a simplicial map

$$\bigsqcup_{\mathbf{m}\to\mathbf{k}} (\Delta^\ell \times X) \xrightarrow{\theta^*} \bigsqcup_{\mathbf{n}\to\mathbf{k}} (\Delta^\ell \times X),$$

according to the horizontal structure of the bisimplicial set $\pi^{-1}(\sigma)$. There is a commutative diagram

$$\begin{tikzcd}
\Delta^\ell \times X \arrow[r, "\theta^*_\gamma"] \arrow[d, "in_\gamma"'] & \Delta^\ell \times X \arrow[d, "in_{\gamma\theta}"] \\
\bigsqcup_{\mathbf{m}\to\mathbf{k}} (\Delta^\ell \times X) \arrow[r, "\theta^*"'] & \bigsqcup_{\mathbf{n}\to\mathbf{k}} (\Delta^\ell \times X),
\end{tikzcd}$$

where θ^*_γ is defined on the simplex level by

$$(\zeta, x) \mapsto (\zeta, m_*(\zeta, x)).$$

Here, m_* is the composite simplicial set map

$$\Delta^\ell \times X \xrightarrow{\nu\times 1} M \times X \xrightarrow{m} X,$$

where m is the action of M on X and ν is some ℓ-simplex of M. The assumptions on the action m therefore imply that θ_γ^* is a homology isomorphism.

One of the spectral sequences for the homology of the diagonal of the bisimplicial set

$$\bigsqcup_{\mathbf{m}\to\mathbf{k}} (\Delta^\ell \times X)$$

has E_2-term

$$H_*(\bigoplus_{\omega:\mathbf{m}\to k} A(\omega)),$$

where A is a contravariant functor on the simplices of Δ^k: the group $A(\omega)$ is a copy of $H_q(\Delta^\ell \times X)$. We have just seen that every morphism

$$\begin{tikzcd}
\mathbf{m} \arrow[dr, "\gamma\theta"] \arrow[dd, "\theta"'] & \\
& \mathbf{k} \\
\mathbf{n} \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

induces an isomorphism $\theta^* : A(\gamma) \cong A(\gamma\theta)$. The morphism $1_\mathbf{k}$ is terminal in the simplex category for Δ^k, so there is a natural isomorphism which is defined by diagrams

$$\begin{tikzcd}
A(1_\mathbf{k}) \arrow[r, "\cong"] \arrow[d, "1"'] & A(\gamma) \arrow[d, "\theta^*"] \\
A(1_\mathbf{k}) \arrow[r, "\cong"'] & A(\gamma\theta).
\end{tikzcd}$$

Lemma 2.2 says that there are isomorphisms

$$H_i(\bigoplus_{\omega:\mathbf{m}\to k} A(\omega)) \cong \begin{cases} A(1_\mathbf{k}) & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

In particular, the inclusion

$$X \times \Delta^\ell \xrightarrow{in_v} \coprod_{\mathbf{0} \to \mathbf{k}} (X \times \Delta^\ell)$$

corresponding to a vertex v of Δ^k induces an isomorphism

$$H_q(X \times \Delta^\ell) \cong H_q(\pi^{-1}(\sigma))$$

Finally, observe that every bisimplex map of the form (5.19) induces a commutative diagram of simplicial set maps

$$\begin{tikzcd}
X \times \Delta^s \arrow[r, "1 \times \zeta_2"] \arrow[d, "in_w"'] & X \times \Delta^\ell \arrow[d, "in_{\zeta_1 w}"] \\
\coprod_{\mathbf{0} \to \mathbf{r}} (X \times \Delta^s) \arrow[r, "(\zeta_1, \zeta_2)_*"'] & \coprod_{\mathbf{0} \to \mathbf{k}} (X \times \Delta^\ell),
\end{tikzcd}$$

where w is a vertex of Δ^r. It follows that the map $(\zeta_1, \zeta_2)_*$ is a homology isomorphism. ■