Chapter III Classical results and constructions

This chapter is a rather disparate collection of stories, most of them old and well known, but told from a modern point of view.

The first section contains several equivalent descriptions of the fundamental groupoid, one of which (ie. left adjoint to a classifying space functor) is powerful enough to show that the fundamental groupoid of a classifying space of a small category is the free groupoid on that category (Corollary 1.2), as well as prove the Van Kampen theorem (Theorem 1.4).

The second section, on simplicial abelian groups, contains a complete development of the Dold-Kan correspondence. This correspondence is an equivalence of categories between chain complexes and simplicial abelian groups; the result appears as Corollary 2.3. We also give an elementary description of the proper simplicial model structure for the category of simplicial abelian groups (Theorem 2.6, Proposition 2.11, and Remark 2.12), and then use this structure to derive the standard isomorphism

$$H^n(X, A) \cong [X, K(A, n)]$$

relating cohomology to homotopy classes of maps which take values in an Eilenberg-Mac Lane space (Corollary 2.17). We close Section 2 by showing that every simplicial abelian group is non-canonically a product of Eilenberg-Mac Lane spaces up to homotopy equivalence (Proposition 2.18).

We have included Section 3, on the Hurewicz theorem, as further evidence for the assertion that many results in the Algebraic Topology canon have very clean simplicial homotopy proofs. We use Postnikov towers and the Serre spectral sequence, both of which appear here for the first time in the book and are described more fully in later chapters. The Hurewicz homomorphism itself has a very satisfying functorial description in this context: it is the adjunction map $X \to \mathbb{Z}X$ from a simplicial set X to the corresponding free abelian simplicial group $\mathbb{Z}X$.

Section 4 contains a modernized treatment of Kan's Ex^∞ functor. This functor gives a combinatorial (even intuitionistic), natural way of mapping a simplicial set into a Kan complex, via a weak equivalence. The construction is therefore preserved by left exact functors which have right adjoints — these appear throughout topos theory [38]. We have seen similar applications already in Chapter II in connection with detecting simplicial model structures on categories of simplicial objects. The main theorem here is Theorem 4.7. The proof is an updated version of the original: the fundamental groupoid trick in Lemma 4.2 may be new, but the heart of the matter is Lemma 4.6.

There are two different suspension functors for a pointed simplicial set X, namely the smash product $S^1 \wedge X$ (where $S^1 = \Delta^1/\partial\Delta^1$ is the simplicial circle), and the Kan suspension ΣX. These are homotopy equivalent but not isomorphic constructions which have natually homeomorphic realizations. They further represent two of the standard subdivisions for the suspension of a pointed simplicial complex. Both have their uses; in particular, the Kan suspension is more easily related to the classifying space of a simplicial group which appears in Chapter V, and hence to Eilenberg-Mac Lane spectra [52]. We give a full treatment of the Kan suspension in Section 5, essentially to have it "in the bank" for later. Along the way, we say formally what it means for a simplicial set to have an extra degeneracy. This last idea has been in the folklore for a long time — it means, most succinctly, that the identity map on a simplicial set factors through a cone.

1. The fundamental groupoid, revisited.

Recall from Section I.8 that the classical fundamental groupoid $\pi|X|$ of the realization of a simplicial set X coincides with the groupoid $\pi_f S|X|$ associated to the singular complex $S|X|$. In that section, there is a remark to the effect that this groupoid is equivalent to the free groupoids GP_*X and $G(\mathbf{\Delta} \downarrow X)$ which are associated, respectively, to the path category P_*X and the simplex category $\mathbf{\Delta} \downarrow X$ for X. This claim has the following precise form:

Theorem 1.1. *The groupoids $G(\mathbf{\Delta} \downarrow X)$, GP_*X and $\pi|X|$ are naturally equivalent as categories.*

Proof: A functor $f : G \to H$ between groupoids is an equivalence if and only if

(1) the induced function $f : \hom_G(a, b) \to \hom_H(f(a), f(b))$ is a bijection for every pair of objects a, b of G, and
(2) for every object c of H there is a morphism $c \to f(a)$ in H.

The groupoids $GP_*S|X|$ and $\pi|X|$ are naturally isomorphic. The 1-simplices of $S|X|$ are paths of $|X|$, and $[d_1\sigma] = [d_0\sigma] \circ [d_2\sigma]$ for every 2-simplex $\sigma : |\Delta^2| \to |X|$ of $S|X|$. It follows that there is a functor $\gamma_X : GP_*S|X| \to \pi|X|$ which is defined by sending a path to its associated homotopy class. The inverse of γ_X is constructed by observing that homotopic paths in $|X|$ represent the same element of $GP_*S|X|$.

The next step is to show that the functor GP_* takes weak equivalences of simplicial sets to equivalences of groupoids. If f is a weak equivalence of $\mathbf{S}$, then f has a factorization $f = q \circ j$ where q is a trivial fibration and j is a trivial cofibration. The map q is left inverse to a trivial cofibration, and every trivial cofibration is a retract of a map which is a filtered colimit of pushouts of maps of the form $\Lambda^n_k \subset \Delta^n$. It suffices, therefore to show that GP_* takes pushouts of maps $\Lambda^n_k \subset \Delta^n$ to equivalences of categories.

The induced map $GP_*(\Lambda^n_k) \to GP_*(\Delta^n)$ is an isomorphism of groupoids if $n \geq 2$. The groupoid $GP_*(\Lambda^1_i)$ is a strong deformation retract of $GP_*(\Delta^1)$ in the category of groupoids for $i = 0, 1$. Isomorphisms and strong deformation retractions of groupoids are closed under pushout, and so GP_* takes weak equivalences of simplicial sets to equivalences of groupoids as claimed. It follows in particular that the groupoids GP_*X and $\pi|X|$ are naturally equivalent.

We can assign, to each 1-simplex $x : d_1x \to d_0x$, the morphism $(d^0)^{-1}(d^1)$ of $G(\mathbf{\Delta} \downarrow X)$ arising from the diagram

$$
\begin{tikzcd}
\Delta^0 \arrow[r, "d^1"] \arrow[dr, "d_1x"'] & \Delta^1 \arrow[d, "x"] & \Delta^0 \arrow[l, "d^0"'] \arrow[dl, "d_0x"] \\
& X. &
\end{tikzcd}
$$

This assignment defines a functor $\omega_X : GP_*X \to G(\mathbf{\Delta} \downarrow X)$, since the following diagram of simplicial maps commutes:

$$
\begin{tikzcd}
\Delta^0 \arrow[rr, "d^0"] \arrow[ddr, "d^1"'] & & \Delta^1 \arrow[d, "d^2"] & & \Delta^0 \arrow[ll, "d^1"'] \arrow[ddl, "d^1"] \\
& & \Delta^2 & & \\
& \Delta^1 \arrow[ur, "d^0"] & & \Delta^1 \arrow[ul, "d^1"'] & \\
& & \Delta^0 \arrow[ul, "d^0"] \arrow[ur, "d^0"'] & &
\end{tikzcd}
$$

Let $v_n : \Delta^0 \to \Delta^n$ denote the simplicial map which picks out the last vertex n of the ordinal number $\mathbf{n}$. Then the assignment

$$\Delta^n \xrightarrow{\sigma} X \mapsto \Delta^0 \xrightarrow{v_n} \Delta^n \xrightarrow{\sigma} X$$

is the object function of a functor $\mathbf{\Delta} \downarrow X \to P_*X$. Write $\nu_X : G(\mathbf{\Delta} \downarrow X) \to GP_*X$ for the induced functor on associated groupoids. Then the composite functor $\nu_X \circ \omega_X$ is the identity on GP_*X, and the composite $\omega_X \circ \nu_X$ is naturally isomorphic to the identity on $G(\mathbf{\Delta} \downarrow X)$. The natural isomorphism is determined by the maps

$$\begin{tikzcd} \Delta^0 \arrow[r, "\sigma \cdot v_n"] \arrow[d, "v_n"'] & X \\ \Delta^n \arrow[ur, "\sigma"'] & \end{tikzcd}$$

of the simplex category $\mathbf{\Delta} \downarrow X$. ∎

From now on, the fundamental groupoid of a simplicial set X, in any of its forms, will be denoted by πX.

A simplicial set map $f : X \to BC$ associates to each n-simplex x of X a functor $f(x) : \mathbf{n} \to C$ which is completely determined by the 1-skeleton of x and the composition relations arising from 2-simplices of the form

$$\Delta^2 \to \Delta^n \xrightarrow{x} X.$$

It follows that f can be identified with the graph morphism

$$\begin{tikzcd} X_1 \arrow[r, "f"] \arrow[d, shift left=2] \arrow[d, shift right=2] & \mathrm{Mor}(C) \arrow[d, shift left=2] \arrow[d, shift right=2] \\ X_0 \arrow[u] \arrow[r, "f"'] & \mathrm{Ob}(C) \arrow[u] \end{tikzcd}$$

subject to the relations $f(d_1\sigma) = f(d_0\sigma) \cdot f(d_2\sigma)$ arising from all 2-simplices σ of X. The path category P_*X is the category which is freely associated to the graph

$$X_1 \rightrightarrows X_0$$

and the 2-simplex relations, so that there is an adjunction isomorphism

$$\hom_{\mathbf{cat}}(P_*X, C) \cong \hom_{\mathbf{S}}(X, BC).$$

A small category C is completely determined by its set of arrows and composition relations. It follows that the path category P_*BC of the nerve BC is isomorphic to C. We therefore immediately obtain the following assertion:

COROLLARY 1.2. *Let C be a small category. Then the fundamental groupoid πBC of the nerve of C is equivalent, as a category, to the free groupoid GC on the category C.*

Corollary 1.2 may be used to give a direct proof of the fact that the fundamental group $\pi_1 BQ\mathcal{A}$ of the nerve of the Q-construction $Q\mathcal{A}$ on an exact category $\mathcal{A}$ is isomorphic to the 0^{th} K-group $K_0\mathcal{A}$ of $\mathcal{A}$ (see [48]).

REMARK 1.3. In the proof of Theorem 1.1, observe that the composite

$$S|X| \to BP_*S|X| \to BGP_*S|X| \xrightarrow{\gamma_X} \pi|X|$$

sends a path $\alpha : \Delta^1 \to S|X|$ of $|X|$ to its homotopy class $[\alpha]$ (rel $\partial\Delta^1$). It follows that the composite

$$S|X| \to BP_*S|X| \to BGP_*S|X|$$

induces isomomorphisms $\pi_1(S|X|, x) \cong \pi_1(BGP_*S|X|, x)$ for all choices of vertex $x \in S|X|$. We have also seen that the functor GP_* takes weak equivalences of simplicial sets to equivalences of groupoids, so that the commutativity of the diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\eta"'] & BP_*X \arrow[r] \arrow[d, "\eta_*"'] & BGP_*X \arrow[d, "\eta_*"] \\
S|X| \arrow[r] & BP_*S|X| \arrow[r] & BGP_*S|X|
\end{tikzcd}$$

implies that the composite

$$X \to BP_*X \to BGP_*X$$

induces isomorphisms

$$\pi_1(X, x) \cong \pi_1(BGP_*X, x) \cong \hom_{GP_*X}(x, x)$$

for any choice of vertex $x \in X$, if X is a Kan complex.

Note, in particular, that if X is connected, then there is a deformation retraction map of groupoids $r : GP_*X \to \hom_{GP_*X}(x, x)$ for any choice of vertex of X. This map r determines a composite map

$$X \to BP_*X \to BGP_*X \xrightarrow{r_*} B(\hom_{GP_*X}(x, x)),$$

which induces isomorphisms

$$\pi_1(X, y) \cong \pi_1(B \hom_{GP_* X}(x, x)) \cong \hom_{GP_* X}(x, x)$$

for all vertices y of X. It follows that there is a map

$$X \to B(\pi_1(X, x))$$

which induces isomorphisms on all fundamental groups if X is a connected Kan complex with base point x.

Again, the fundamental groupoid construction takes weak equivalences of simplicial sets to equivalences of groupoids. It follows, in particular, that there is a purely categorical definition of the fundamental group $\pi_1(X, x)$ of an arbitrary simplicial set X at a vertex x, given by

$$\pi_1(X, x) = \hom_{\pi X}(x, x).$$

This definition can be used, along with the observation that the fundamental groupoid construction $\pi = GP_*$ has a right adjoint and therefore preserves colimits, to give a rather short proof of the Van Kampen theorem:

Theorem 1.4 (Van Kampen). *Suppose that*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, hook, "j"'] & X \arrow[d] \\ B \arrow[r] & Y \end{tikzcd}$$

is a pushout diagram of simplicial sets, where the map j is a cofibration, and A, B and X are connected. Then, for any vertex x of A, the induced diagram

$$\begin{tikzcd} \pi_1(A, x) \arrow[r, "i_*"] \arrow[d, "j_*"'] & \pi_1(X, x) \arrow[d] \\ \pi_1(B, x) \arrow[r] & \pi_1(Y, x) \end{tikzcd}$$

is a pushout in the category of groups.

PROOF: The glueing lemma II.9.8 implies that we can presume that the map i is also a cofibration.

The induced maps $i_* : \pi A \to \pi X$ and $j_* : \pi A \to \pi B$ of fundamental groupoids are monomorphisms on objects, and so the strong deformation $r : \pi A \to \pi_1(A,x)$, by suitable choice of paths in X and$B$, can be extended to a strong deformation

$$\begin{tikzcd}
\pi B \arrow[d, "r"'] & \pi A \arrow[l, "i_*"'] \arrow[r, "j_*"] \arrow[d, "r"'] & \pi X \arrow[d, "r"] \\
\pi_1(B,x) & \pi_1(A,x) \arrow[l, "i_*"] \arrow[r, "j_*"'] & \pi_1(X,x)
\end{tikzcd}$$

in a suitable diagram category in groupoids, meaning that the homotopy $h : \pi A \to \pi A^{\mathbf{1}}$ giving the deformation extends to a homotopy

$$\begin{tikzcd}
\pi B \arrow[d, "h"'] & \pi A \arrow[l, "i_*"'] \arrow[r, "j_*"] \arrow[d, "h"'] & \pi X \arrow[d, "h"] \\
\pi B^{\mathbf{1}} & \pi A^{\mathbf{1}} \arrow[l, "i_*"] \arrow[r, "j_*"'] & \pi X^{\mathbf{1}}
\end{tikzcd}$$

But then, if

$$\begin{tikzcd}
\pi_1(A,x) \arrow[r, "i_*"] \arrow[d, "j_*"'] & \pi_1(X,x) \arrow[d] \\
\pi_1(B,x) \arrow[r] & G
\end{tikzcd}$$

is a pushout in the category of groups, it is also a pushout in the category of groupoids, and so the group G is a strong deformation retraction of the groupoid πY, since the diagram

$$\begin{tikzcd}
\pi A \arrow[r, "i_*"] \arrow[d, "j_*"'] & \pi X \arrow[d] \\
\pi B \arrow[r] & \pi Y
\end{tikzcd}$$

is a pushout in groupoids. The group G is therefore isomorphic to $\pi_1(Y,x)$. ∎

2. Simplicial abelian groups: the Dold-Kan correspondence.

Suppose that A is a simplicial abelian group, and define

$$NA_n = \bigcap_{i=0}^{n-1} \ker(d_i) \subset A_n.$$

The maps

$$NA_n \xrightarrow{(-1)^n d_n} NA_{n-1}$$

form a chain complex, on account of the simplicial identity

$$d_{n-1}d_n = d_{n-1}d_{n-1}.$$

Denote the corresponding chain complex by NA; this is the *normalized chain complex* associated to the simplicial abelian group A. The assignment $A \mapsto NA$ is plainly a functor from the category $s\mathbf{Ab}$ of simplicial abelian groups to the category $\mathbf{Ch}_+$ of chain complexes.

The *Moore complex* of a simplicial abelian group A has the group A_n of n-simplices of A as n-chains, and has boundary $\partial : A_n \to A_{n-1}$ defined by

$$\partial = \sum_{i=0}^{n} (-1)^i d_i : A_n \to A_{n-1}.$$

Of course, one has to verify that $\partial^2 = 0$, but this is a consequence of the simplicial identities. The notation A will be used for the second purpose of denoting the Moore complex of the simplicial abelian group A – this could be confusing, but it almost never is.

Let DA_n denote the subgroup of A_n which is generated by the degenerate simplices. The boundary map ∂ of the Moore complex associated to A induces a homomorphism

$$\partial : A_n/DA_n \to A_{n-1}/DA_{n-1}.$$

The resulting chain complex will be denoted by $A/D(A)$, meaning "A modulo degeneracies". One sees directly from the definitions that there are chain maps

$$NA \xrightarrow{i} A \xrightarrow{p} A/D(A),$$

where i is the obvious inclusion and p is the canonical projection.

Theorem 2.1. *The composite*

$$NA \xrightarrow{p\cdot i} A/D(A)$$

is an isomorphism of chain complexes.

Proof: Write

$$N_jA_n = \bigcap_{i=0}^{j} \ker(d_i) \subset A_n,$$

and let $D_j(A_n)$ be the subgroup of A_n which is generated by the images of the degeneracies s_i for $i \leq j$. One shows that the composite

$$N_jA_n \hookrightarrow A_n \xrightarrow{p} A_n/D_j(A_n)$$

is an isomorphism for all n and $j < n$. Let ϕ denote this composite.

This is proved by induction on j. The case $j = 0$ is easy: any class $[x] \in A_n/s_0(A_{n-1})$ is represented by $x - s_0d_0x$, and $d_0(x - s_0d_0x) = 0$, so ϕ is onto; if $d_0x = 0$ and $x = s_0y$, then

$$0 = d_0x = d_0s_0y = y,$$

so $x = 0$.

Suppose that the map

$$\phi : N_kA_m \to A_m/D_k(A_m)$$

is known to be an isomorphism if $k < j$, and consider the map

$$\phi : N_jA_n \to A_n/D_j(A_n).$$

Form the commutative diagram

$$\begin{tikzcd}
N_{j-1}A_n \arrow[r, "\phi", "\cong"'] & A_n/D_{j-1}(A_n) \arrow[d, two heads] \\
N_jA_n \arrow[u, hook] \arrow[r, "\phi"'] & A_n/D_j(A_n).
\end{tikzcd}$$

On account ot the displayed isomorphism, any class $[x] \in A_n/D_j(A_n)$ can be represented by an element $x \in N_{j-1}A_n$. But then $x - s_j d_j x$ is in $N_j A_n$ and represents $[x]$, so the bottom map ϕ in the diagram is onto. The simplicial identities imply that the degeneracy $s_j : A_{n-1} \to A_n$ maps $N_{j-1}A_{n-1}$ into $N_{j-1}A_n$, and takes $D_{j-1}(A_{n-1})$ to $D_{j-1}(A_n)$, and so there is a commutative diagram

$$\begin{tikzcd} N_{j-1}A_{n-1} \arrow[r, "\phi", "\cong"'] \arrow[d, "s_j"'] & A_{n-1}/D_{j-1}(A_{n-1}) \arrow[d, "s_j"] \\ N_{j-1}A_n \arrow[r, "\cong", "\phi"'] & A_n/D_{j-1}(A_n). \end{tikzcd}$$

Furthermore, the sequence

$$0 \to A_{n-1}/D_{j-1}(A_{n-1}) \xrightarrow{s_j} A_n/D_{j-1}(A_n) \to A_n/D_j(A_n) \to 0$$

is exact. Thus, if $\phi(x) = 0$ for some $x \in N_j A_n$, then $x = s_j y$ for some $y \in N_{j-1}A_n$. But (again), $d_j x = 0$, so that

$$0 = d_j x = d_j s_j y = y,$$

so that $x = 0$, and our map is injective. ■

Every simplicial structure map $d^* : A_n \to A_m$ corresponding to a monomorphism $d : \mathbf{m} \hookrightarrow \mathbf{n}$ of ordinal numbers takes NA_n into NA_m. In fact, such maps are 0 unless d is of the form $d = d^n : \mathbf{n-1} \to \mathbf{n}$. Put a different way, suppose given a collection of abelian group homomorphisms

$$\partial : C_n \to C_{n-1}, \quad n \geq 0.$$

Associate to each ordinal number $\mathbf{n}$ the group C_n, and map each ordinal number monomorphism to an abelian group homomorphisms by the rule

$$d \mapsto \begin{cases} 0 & \text{if } d \text{ is not some } d^n, \text{ and} \\ C_n \xrightarrow{(-1)^n \partial} C_{n-1} & \text{if } d = d^n. \end{cases}$$

Then we get a contravariant functor on the category of ordinal number morphisms from such an assignment if and only if we started with a chain complex.

There is a simplicial abelian group whose n-simplices have the form

$$\bigoplus_{\mathbf{n} \twoheadrightarrow \mathbf{k}} NA_k.$$

Note that the direct sum is indexed over all ordinal number epimorphisms whose source is $\mathbf{n}$. The map

$$\theta^* : \bigoplus_{\mathbf{n} \twoheadrightarrow \mathbf{k}} NA_k \to \bigoplus_{\mathbf{m} \twoheadrightarrow \mathbf{r}} NA_r$$

associated to the ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$ is given on the summand corresponding to $\sigma : \mathbf{n} \to \mathbf{k}$ by the composite

$$NA_k \xrightarrow{d^*} NA_s \xrightarrow{in_t} \bigoplus_{\mathbf{m} \twoheadrightarrow \mathbf{r}} NA_r,$$

where

$$\mathbf{m} \overset{t}{\twoheadrightarrow} \mathbf{s} \overset{d}{\hookrightarrow} \mathbf{k}$$

is the epi-monic factorization of the composite

$$\mathbf{m} \xrightarrow{\theta} \mathbf{n} \overset{\sigma}{\twoheadrightarrow} \mathbf{k}.$$

Note as well that there is a morphism of simplicial abelian groups which is given in degree n by the map

$$\Psi : \bigoplus_{\mathbf{n} \twoheadrightarrow \mathbf{k}} NA_k \to A_n,$$

which is given at the summand corresponding to $\sigma : \mathbf{n} \to \mathbf{k}$ by the composite

$$NA_k \hookrightarrow A_k \xrightarrow{\sigma^*} A_n.$$

PROPOSITION 2.2. *The map Ψ is a natural isomorphism of simplicial abelian groups.*

PROOF: An induction on the degree n starts with the observation that $NA_0 = A_0$, and that there's only one map from the ordinal number $\mathbf{0}$ to itself. Suppose that Ψ is known to be an isomorphism in degrees less than n. Then any degeneracy $s_j x \in A_n$ is in the image of Ψ, because x is in the image of Ψ in degree $n-1$. On the other hand, Ψ induces an isomorphism of normalized complexes, so Ψ is epi in degree n by Theorem 2.1.

Suppose that (x_σ) maps to 0 under Ψ, where x_σ is the component of (x_σ) which corresponds to $\sigma : \mathbf{n} \twoheadrightarrow \mathbf{k}$. If $k < n$, then σ has a section $d : \mathbf{k} \to \mathbf{n}$, and the component of $d^*(x_\sigma)$ which corresponds to the identity on $\mathbf{k}$ is x_σ. But $\Psi d^*(x_\sigma) = 0$, so $d^*(x_\sigma) = 0$ by the inductive hypothesis, and so $x_\sigma = 0$ in NA_k. Thus, $x_\sigma = 0$ for $\sigma : \mathbf{n} \twoheadrightarrow \mathbf{k}$ with $k < n$. The remaining possibly non-trivial component is $x_{1_\mathbf{n}} \in NA_n$. But the restriction of Ψ to NA_n is the inclusion $NA_n \hookrightarrow A_n$, so that $x_{1_\mathbf{n}} = 0$ as well. ∎

Implicit in the above is a definition of a functor

$$\Gamma : \mathbf{Ch}_+ \to s\mathbf{Ab}$$

from chain complexes to simplicial abelian groups, with

$$\Gamma(C)_n = \bigoplus_{\mathbf{n} \twoheadrightarrow \mathbf{k}} C_k$$

for a chain complex C, and with simplicial structure maps given by the recipe above. The following result is now obvious from the work that we have done:

COROLLARY 2.3 (DOLD-KAN). *The functors*

$$N : s\mathbf{Ab} \to \mathbf{Ch}_+ \quad \textit{and} \quad \Gamma : \mathbf{Ch}_+ \to s\mathbf{Ab}$$

form an equivalence of categories.

PROOF: The natural isomorphism $\Gamma N(A) \cong A$ is Proposition 2.2, and the natural isomorphism $N\Gamma(C) \cong C$ can be derived from Theorem 2.1 by collapsing $\Gamma(C)$ by degeneracies. ∎

There is a subcomplex N_jA of the Moore complex A, which is defined by

$$N_jA_n = \begin{cases} \bigcap_{i=0}^{j} \ker(d_i) & \text{for } n \geq j+2, \\ NA_n & \text{for } n \leq j+1. \end{cases}$$

To see that these groups form a subcomplex, one has to verify that given $x \in N_jA_n$ with $n \geq j+2$, then

$$d_k\Big(\sum_{i=j+1}^{n} (-1)^i d_i(x)\Big) = 0$$

if $k \leq j$. This is a consequence of the simplicial identities $d_k d_i = d_{i-1} d_k$ that hold for $i > k$.

Observe that $N_0A = A$, that $N_{j+1}A \subset N_jA$, and that $NA = \cap_{j\geq 0} N_jA$. Let i_j denote the inclusion of $N_{j+1}A$ into N_jA.

Now define abelian group homomorphisms $f_j : N_jA_n \to N_{j+1}A_n$ by specifying that

$$f_j(x) = \begin{cases} x - s_{j+1}d_{j+1}(x) & \text{if } n \geq j+2, \text{ and} \\ x & \text{if } n \leq j+1. \end{cases}$$

One has to check that f_j takes values in $N_{j+1}A_n$, but this is an easy simplicial identity argument. The simplicial identities also imply that the collection of maps f_j defines a chain map $f_j : N_jA \to N_{j+1}A$. It is easily seen that the composite $f_j \cdot i_j$ is the identity on the chain complex $N_{j+1}A$.

Now define a abelian group homomorphisms $t_j : N_jA_n \to N_jA_{n+1}$ by

$$t_j(x) = \begin{cases} (-1)^j s_{j+1} & \text{if } n \geq j+1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Then a little more calculating in the night shows that

$$1 - i_jf_j = \partial t_j + t_j \partial : N_jA_n \to N_jA_n$$

in all degrees n, and that both sides of the equation are 0 in degrees $n \leq j+1$.

It follows that the composite abelian group homomorphisms

$$A_n = N_0A_n \xrightarrow{f_0} N_1A_n \xrightarrow{f_1} N_2A_n \xrightarrow{f_2} \cdots \xrightarrow{f_{n-2}} N_{n-1}A_n = NA_n$$

define a chain map $f : A \to NA$ such that $f \cdot i : NA \to NA$ is the identity. The collection of homomorphisms $T : A_n \to A_{n+1}$ defined by

$$T = i_0 \cdots i_{n-2}t_{n-1}f_{n-2} \cdots f_0 + i_0 \cdots i_{n-3}t_{n-2}f_{n-3} \cdots f_0 + \cdots + i_0t_1f_0 + t_0$$

specifies a chain homotopy $i \cdot f \simeq 1_A$.

The chain maps i, f and the chain homotopy T are natural with respect to morphisms of simplicial abelian groups A. We have proved

THEOREM 2.4. *The inclusion $i : NA \to A$ of the normalized chain complex in the Moore complex of a simplicial abelian group A is a chain homotopy equivalence. This equivalence is natural with respect to simplicial abelian groups A.*

The simplicial abelian group structure on A induces an abelian group structure on the set

$$\pi_n(A,0) = [(\Delta^n, \partial\Delta^n), (A,0)]$$

of homotopy classes of pairs of maps which satisfies an interchange law with respect to the canonical group structure for the homotopy group. It follows that the homotopy group structure and the induced abelian group structure coincide. In particular, there is a natural isomorphism

$$\pi_n(A,0) \cong H_n(NA)$$

for $n \geq 0$. Theorem 2.4 therefore immediately implies the following:

COROLLARY 2.5. *Suppose that A is a simplicial abelian group. Then there are isomorphisms*

$$\pi_n(A,0) \cong H_n(NA) \cong H_n(A),$$

where $H_n(A)$ is the n^{th} homology group of the Moore complex associated to A. These isomorphisms are natural in simplicial abelian groups A.

The group A_0 of vertices of A acts on the simplicial set underlying A, via the composite

$$A_0 \times A \xrightarrow{c\times 1} A \times A \xrightarrow{+} A,$$

where $c : A_0 \to A$ is the simplicial abelian group homomorphism given by inclusion of vertices, and $+$ is the abelian group structure on A. It follows that multiplication by a vertex a induces an isomorphism of homotopy groups

$$\pi_n(A,0) \xrightarrow[\cong]{+a_*} \pi_n(A,a)$$

for $n \geq 0$. It also follows that a homomorphism $f : A \to B$ of simplicial abelian groups is a weak equivalence of the underlying Kan complexes if and only if f induces a homology isomorphism (or quasi-isomorphism) $f_* : NA \to NB$ of the associated normalized chain complexes. Equivalently, f is a weak equivalence if and only if the induced map $f : A \to B$ of Moore complexes is a homology isomorphism.

Say that a simplicial abelian group homomorphism $f : A \to B$ is a *weak equivalence* if f is a weak equivalence of the underlying Kan complexes. We say that f is a *fibration* if the underlying simplicial set map is a Kan fibration. Finally, *cofibrations* in the category $s\mathbf{Ab}$ of simplicial abelian groups are morphisms which have the left lifting property with respect to all maps which are fibrations and weak equivalences.

THEOREM 2.6. *With these definitions, the category $s\mathbf{Ab}$ of simplicial abelian groups satisfies the axioms for a closed model category.*

This result is a consequence of Theorem II.5.1 — see also Remark II.6.2. One can also argue directly as follows:

PROOF: The limit axiom **CM1**, the weak equivalence axiom **CM2** and the retraction axiom **CM3** are trivial to verify.

A map $p : A \to B$ of simplicial abelian groups is a fibration if and only if it has the right lifting property with respect to all morphism $i : \mathbb{Z}\Lambda^n_k \to \mathbb{Z}\Delta^n$ induced by the inclusions $\Lambda^n_k \subset \Delta^n$. The simplicial abelian group $\mathbb{Z}\Lambda^n_k$ is a degreewise direct summand of $\mathbb{Z}\Delta^n$. Note as well that the each of these maps i is a weak equivalence, because the underlying inclusions are weak equivalences and therefore integral homology isomorphisms. A small object argument therefore implies that any map $f : C \to D$ of simplicial abelian groups has a factorization

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "j"'] & & D \\ & E \arrow[ur, "p"'] & \end{tikzcd}$$

such that p is a fibration and j is a morphism which has the left lifting property with respect to all fibrations, is a weak equivalence, and is a monomorphism in each degree. In particular, j is a cofibration and a weak equivalence, and so the corresponding factorization axiom is verified.

Similarly, a map $p : A \to B$ is a fibration and a weak equivalence if and only if it has the right lifting property with respect to all morphisms $\mathbb{Z}\partial\Delta^n \to \mathbb{Z}\Delta^n$ induced by the inclusions $\partial\Delta^n \subset \Delta^n$. It follows again by a small object argument that any map $f : C \to D$ has a factorization

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "i"'] & & D \\ & \overline{E} \arrow[ur, "q"'] & \end{tikzcd}$$

such that q is a fibration and a weak equivalence, and such that i is a cofibration and a levelwise monomorphism. We have therefore completely verified the factorization axiom **CM5**.

By standard nonsense, any map α which is a cofibration and a weak equivalence is a retract of a map of the form j in the proof of the factorization axiom, so that α has the left lifting property with respect to all fibrations. This implies the lifting axiom **CM4**. ■

REMARK 2.7. It is a corollary of the proof of Theorem 2.6 that all cofibrations of $s\mathbf{Ab}$ are levelwise monomorphisms.

LEMMA 2.8. *Suppose that $f : A \to B$ is a homomorphism of simplicial abelian groups which is surjective in all degrees. Then f is a fibration of simplicial abelian groups.*

PROOF: Suppose given a commutative diagram of simplicial set maps

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\alpha"] \arrow[d, hook] & A \arrow[d, "f"] \\ \Delta^n \arrow[r, "\beta"'] & B. \end{tikzcd}$$

Then, by the assumptions, there is a simplex $\theta \in A_n$ such that $f(\theta) = \beta$. But then

$$\theta|_{\Lambda^n_k} - \alpha : \Lambda^n_k \to A$$

factors through the kernel K of f, and so $\theta|_{\Lambda^n_k} - \alpha$ extends to an n-simplex x of K in the sense that there is a commutative diagram of simplicial set maps

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\theta|_{\Lambda^n_k} - \alpha"] \arrow[d, hook] & K \\ \Delta^n \arrow[ur, "x"'] & \end{tikzcd}$$

Then $(\theta - x)|_{\Lambda^n_k} = \alpha$ and $f(\theta - x) = \beta$. ∎

LEMMA 2.9.

(1) *A homomorphism $f : A \to B$ of simplicial abelian groups is surjective (in all degrees) if and only if the associated chain complex map $f : NA \to NB$ is surjective in all degrees.*

(2) *The homomorphism $f : A \to B$ of simplicial abelian groups is a fibration if and only if the induced abelian group maps $f : NA_n \to NB_n$ are surjective for $n \geq 1$.*

Proof: For (1), note that the map $f : NA \to NB$ of normalized chain complexes is a retract of the map $f : A \to B$ of Moore complexes. Thus, if the simplicial abelian group homomorphism $f : A \to B$ is surjective in all degrees, then so is the associated map of normalized chain complexes.

Conversely, one shows that if $f : N_{j+1}A \to N_{j+1}B$ is surjective in all degrees, then the abelian group homomorphisms $f : N_jA_n \to N_jB_n$ are surjective for all $n \geq 0$. This is proved by induction on n. Take $x \in N_jB_n$. Then $x - s_{j+1}d_{j+1}x$ is in $N_{j+1}B_n$ and so is in the image of $f : N_{j+1}A_n \to N_{j+1}B_n$. Also, $d_{j+1} \in N_jB_{n-1}$ and is therefore in the image of $f : N_jA_{n-1} \to N_jB_{n-1}$ by the inductive assumption. It follows that x is in the image of $f : N_jA_n \to N_jB_n$.

For (2), suppose that $f : A \to B$ is a fibration of simplicial abelian groups. Then the existence of solutions to the lifting problems of the form

$$\begin{tikzcd}
\Lambda^n_n \arrow[r, "0"] \arrow[d, hook] & A \arrow[d, "f"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}$$

implies that $f : NA_n \to NB_n$ is surjective for $n \geq 1$.

Conversely, suppose that $f : NA_n \to NB_n$ is surjective in non-zero degrees. Form the diagram

$$\begin{tikzcd}
A \arrow[dr, "\theta"] \arrow[drr, "f"] \arrow[ddr] & & \\
& K(\pi_0 A, 0) \times_{K(\pi_0 B, 0)} B \arrow[r, "pr"'] \arrow[d] & B \arrow[d] \\
& K(\pi_0 A, 0) \arrow[r, "f_*"'] & K(\pi_0 B, 0),
\end{tikzcd}$$

where (for example) $K(\pi_0 A, 0)$ is the constant simplicial abelian group on the abelian group $\pi_0 A$. The hypotheses imply that applying the normalization functor to the map θ gives a surjective chain map

$$\theta : NA \to NK(\pi_0 A, 0) \times_{NK(\pi_0 B, 0)} NB \cong \pi_0 A[0] \times_{\pi_0 B[0]} NB,$$

and so (1) implies that the simplicial abelian group map

$$\theta : A \to K(\pi_0 A, 0) \times_{K(\pi_0 B, 0)} B$$

is surjective in all degrees and therefore a fibration by Lemma 2.8. The map $f_* : K(\pi_0 A, 0) \to K(\pi_0 B, 0)$ is a fibration, so that pr is a fibration. It follows that $f = pr \cdot \theta$ is a fibration. ■

COROLLARY 2.10. *The homomorphism $f : A \to B$ is a trivial fibration of simplicial abelian groups if and only if the induced morphism $f : NA \to NB$ of normalized chain complexes is surjective in all degrees, with acyclic kernel.*

It follows that the category $\mathbf{Ch}_+$ of chain complexes of abelian groups inherits a closed model structure from the simplicial abelian group category, in which the fibrations are the chain maps $f : C \to D$ such that f is surjective in degree n for $n \geq 1$, and where the weak equivalences are the *quasi-isomorphisms*, or rather the maps which induce isomorphisms in all homology groups. The cofibrations of $\mathbf{Ch}_+$ are those maps which have the left lifting property with respect to all trivial fibrations.

One can, alternatively, give a direct proof of the existence of this closed model structure on the chain complex category $\mathbf{Ch}_+$. The proof of the factorization axioms is a small object argument which is based on some rather elementary constructions. Explicitly, let $\mathbb{Z}[n]$ be the chain complex consisting of a copy of the integers $\mathbb{Z}$ in degree n and 0 elsewhere, and let $\mathbb{Z}\langle n+1\rangle$ be the chain complex

$$\cdots \to 0 \to \overset{n+1}{\mathbb{Z}} \xrightarrow{\partial = 1} \overset{n}{\mathbb{Z}} \to 0 \to \cdots .$$

Then maps $\mathbb{Z}[n] \to C$ classify n-cycles of C, and $\mathbb{Z}\langle n+1\rangle$ is the free chain complex on an element of degree $n+1$. Write x for the generator of $\mathbb{Z}[n]$ in degree n and write y for the generator of $\mathbb{Z}\langle n+1\rangle$ in degree $n+1$. There is a canonical map $j : \mathbb{Z}[n] \to \mathbb{Z}\langle n+1\rangle$ which is defined by $j(x) = \partial(y)$. Then $f : C \to D$ is a fibration if and only if f has the right lifting property with respect to all chain maps $0 \to \mathbb{Z}\langle n+1\rangle$ for $n \geq 0$. Further, one can show that f is a trivial fibration if and only if $f : C_0 \to D_0$ is surjective and f has the right lifting property with respect to all maps $j : \mathbb{Z}[n] \to \mathbb{Z}\langle n+1\rangle$, $n \geq 0$.

The chain complex category $\mathbf{Ch}_+$ has a natural cylinder construction. For any chain complex C, there is a chain complex C^I with

$$C^I_n = \begin{cases} C_n \oplus C_n \oplus C_{n+1} & \text{if } n \geq 1, \text{ and} \\ \{(x, y, z) \in C_0 \oplus C_0 \oplus C_1 | (x - y) + \partial z = 0\} & \text{if } n = 0. \end{cases}$$

and with

$$\partial(x, y, z) = (\partial x, \partial y, (-1)^n(x - y) + \partial z).$$

Then there is a commutative diagram of chain maps

$$\begin{tikzcd} & C^I \arrow[d, "p"] \\ C \arrow[ur, "s"] \arrow[r, "\Delta"'] & C \oplus C, \end{tikzcd}$$

where p is the fibration defined by $p(x, y, z) = (x, y)$ and s is a weak equivalence which is defined by $s(x) = (x, x, 0)$. It is an exercise to show that there is a homotopy $h : D \to C^I$ from f to g if and only if the maps $f, g : C \to D$ are chain homotopic.

PROPOSITION 2.11. *The category of simplicial abelian groups admits a simplicial model structure.*

PROOF: If K is a simplicial set and A is a simplicial abelian group, then there is a simplicial abelian group $A \otimes K$, which is defined by

$$A \otimes K = A \otimes \mathbb{Z}K,$$

where (in this case) $\mathbb{Z}K$ denotes the free simplicial abelian group associated to K. Equivalently, on the level of n-simplices, there is a canonical isomorphism

$$A \otimes K = \bigoplus_{\sigma \in K_n} A_n,$$

with simplicial structure maps induced in an obvious way from the corresponding maps for A and the simplicial set K. Dually, the simplicial abelian group structure on A induces a simplicial abelian group structure on the simplicial function space $\mathbf{Hom}(K, A)$. Finally, for simplicial abelian groups A and B, one defines the simplicial set $\mathbf{Hom}_{s\mathbf{Ab}}(A, B)$ to have n-simplices given by the set (actually abelian group) of simplicial abelian group homomorphisms $A \otimes \Delta^n \to B$. Then there are natural isomorphisms

$$\begin{aligned} \mathbf{Hom}_{s\mathbf{Ab}}(A \otimes K, B) &\cong \mathbf{Hom}(K, \mathbf{Hom}_{s\mathbf{Ab}}(A, B) \\ &\cong \mathbf{Hom}_{s\mathbf{Ab}}(A, \mathbf{Hom}(K, B)). \end{aligned}$$

The first of these isomorphisms follows from the exponential law

$$\hom_{s\mathbf{Ab}}(A \otimes K, B) \cong \hom_{\mathbf{S}}(K, \mathbf{Hom}_{s\mathbf{Ab}}(A, B)),$$

which itself is a specialization of the simplicial set exponential law. The second follows from the definition of $\mathbf{Hom}_{s\mathbf{Ab}}(A, B)$, together with the observation that the simplicial set K is a colimit of its simplices. It's obvious now that if $f : A \to B$ is a fibration of simplicial abelian groups and $i : K \hookrightarrow L$ is a cofibration of simplicial sets, then the induced map

$$\mathbf{Hom}(L, A) \xrightarrow{(i^*, f_*)} \mathbf{Hom}(K, A) \times_{\mathbf{Hom}(K,B)} \mathbf{Hom}(L, B)$$

is a fibration of simplicial abelian groups which is trivial if either i or f is trivial, just by remembering that the underlying simplicial set map has the same properties. This is Quillen's axiom **SM7(a)**, whence the simplicial model structure on $s\mathbf{Ab}$. ∎

Proposition 2.11 can also be proved by appealing to Theorem II.5.4.

REMARK 2.12. The category of simplicial abelian groups is also a *proper* simplicial model category (see Section II.8). In particular,

(1) weak equivalences are stable under pullback along fibrations, and
(2) the pushout of a weak equivalence along a cofibration is a weak equivalence.

The first requirement means that, in the pullback diagram

$$\begin{tikzcd} C \times_B A \arrow[r, "g_*"] \arrow[d] & A \arrow[d, "p"] \\ C \arrow[r, "g"'] & B, \end{tikzcd}$$

if p is a fibration and g is a weak equivalence, then g_* is a weak equivalence. This follows from the corresponding property for the category of simplicial sets.

The second requirement says that, given a pushout diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & C \arrow[d, "i_*"] \\ B \arrow[r, "f_*"'] & B \cup_A C, \end{tikzcd}$$

if i is a cofibration and f is a weak equivalence, then f_* is a weak equivalence. But of course i and i_* are monomorphisms which have the same cokernel, and so a comparison of long exact sequences in homology shows that f_* is a homology isomorphism.

The simplicial model structure on the simplicial abelian group category gives rise to a canonical choice of cylinder for cofibrant objects A. In effect, tensoring such an A with the commutative simplicial set diagram

$$\begin{tikzcd} \partial\Delta^1 \arrow[r] \arrow[d, "i"'] & \Delta^0 \\ \Delta^1 \arrow[ur] & \end{tikzcd}$$

gives a natural diagram of simplicial abelian group homomorphisms of the form

$$\begin{tikzcd} A \oplus A \arrow[r, "\nabla"] \arrow[d, "i_* = (d^0, d^1)"'] & A \\ A \otimes \Delta^1 \arrow[ur, "s"'] & \end{tikzcd}$$

The map i_* coincides up to isomorphism with the map

$$1 \otimes i : A \otimes \partial\Delta^1 \to A \otimes \Delta^1,$$

while the map d^0 coincides with

$$1 \otimes d^0 : A \otimes \Delta^0 \to A \otimes \Delta^1.$$

Since A is cofibrant, the simplicial model structure implies that $1 \otimes i$ is a cofibration and $1 \otimes d^0$ is a trivial cofibration. It follows that the map s is a weak equivalence.

The free simplicial abelian group $\mathbb{Z}X$ on a simplicial set X is a cofibrant simplicial abelian group, and there is a natural isomorphism $\mathbb{Z}X \otimes K \cong \mathbb{Z}(X \times K)$. It follows that an ordinary simplicial homotopy $X \times \Delta^1 \to B$ taking values in a simplicial abelian group B induces a homotopy $\mathbb{Z}X \otimes \Delta^1 \to B$ for the cylinder $\mathbb{Z}X \otimes \Delta^1$. Since $\mathbb{Z}X$ is cofibrant and B is fibrant, the existence of a simplicial homotopy $X \times \Delta^1 \to B$ between maps f and g is equivalent to the existence of a chain homotopy between the induced maps $f_*, g_* : \mathbb{Z}X \to B$ of Moore complexes: the induced maps $Nf_*, Ng_* : N\mathbb{Z}X \to NB$ must be chain homotopic by formal nonsense, and so the induced maps of Moore complexes must be chain homotopic by Theorem 2.4. There is a classical alternative method of seeing this point, based on the following

LEMMA 2.13. *Any "homotopy" $h : A \otimes \Delta^1 \to B$ from $f : A \to B$ to $g : A \to B$ gives rise to a chain homotopy between the associated maps of Moore complexes.*

PROOF: Let $\theta_j : \mathbf{n} \to \mathbf{1}$ be the ordinal number morphism such that $\theta_j(i) = 0$ if and only if $i \leq j$. Now, given the map h, define abelian group homomorphisms $h_j : A_n \to B_{n+1}$, $0 \leq j \leq n$, by specifying that

$$h_j(a) = h(s_j(a) \otimes \theta_j).$$

Then, as a consequence of the simplicial identities and the relations

$$d_i(\theta_j) = \begin{cases} \theta_{j-1} & \text{if } i \leq j, \\ \theta_j & \text{if } i > j, \end{cases}$$

and

$$s_i(\theta_j) = \begin{cases} \theta_{j+1} & \text{if } i \leq j, \\ \theta_j & \text{if } i > j \end{cases}$$

in the simplicial set Δ^1, one finds the relations

$$\begin{aligned} &d_0 h_0 = f, \\ &d_{n+1} h_n = g, \\ &d_i h_j = h_{j-1} d_i \qquad \text{if } i < j, \\ &d_{j+1} h_j = d_{j+1} h_{j+1}, \\ &d_i h_j = h_j d_{i-1} \qquad \text{if } i > j+1, \\ &s_i h_j = h_{j+1} s_i \qquad \text{if } i \leq j, \text{ and} \\ &s_i h_j = h_j s_{i-1} \qquad \text{if } i > j. \end{aligned}$$

It's now straightforward to verify that the collection of alternating sums

$$s = \sum_{i=0}^{n} (-1)^i h_i : A_n \to B_{n+1}$$

forms an explicit chain homotopy between the Moore complex maps f and g. ∎

It follows that every simplicial abelian group homomorphism $A \otimes \Delta^1 \to B$ gives rise to a chain homotopy of maps $NA \to NB$ between the associated normalized complexes. The converse is far from clear, unless A is cofibrant.

The following result establishes the relation between weak equivalence and homology isomorphism:

PROPOSITION 2.14. *The free abelian simplicial group functor $X \mapsto \mathbb{Z}X$ preserves weak equivalences.*

PROOF: The existence of a contracting simplicial homotopy of a simplicial set X onto a base point implies that the map $* \subset X$ induces a weak equivalence $\mathbb{Z}* \to \mathbb{Z}X$, since the simplicial contracting homotopy gives rise to a homotopy $\mathbb{Z}X \otimes \Delta^1 \to \mathbb{Z}X$ in the simplicial abelian group category. It follows that the canonical weak equivalences $\epsilon : S|\Delta^n| \to \Delta^n$ are homology isomorphisms in the sense that they induce weak equivalences $\epsilon_* : \mathbb{Z}S|\Delta^n| \to \mathbb{Z}\Delta^n$. An excision argument and an induction on n therefore implies that the maps $\epsilon : S|\operatorname{sk}_n X| \to \operatorname{sk}_n X$ are homology isomorphisms for any simplicial set X, so that the map $\epsilon : S|X| \to X$ is a homology isomorphism as well.

Any weak equivalence $f : X \to Y$ induces a homotopy equivalence of Kan complexes $S|f| : S|X| \to S|Y|$. This map $S|f|$ is a homology equivalence, so f is a homology equivalence. ∎

Write $\pi(\mathbb{Z}X, A)$ to denote homotopy classes of maps between the named objects in the simplicial abelian group category, computed with respect to the cylinder object $\mathbb{Z}X \otimes \Delta^1$.

The free abelian simplicial group functor $X \mapsto \mathbb{Z}X$ and the inclusion functor $i : s\mathbf{Ab} \subset \mathbf{S}$ both preserve weak equivalences. These functors are also adjoint. It follows that they induce corresponding functors $i : \operatorname{Ho}(s\mathbf{Ab}) \to \operatorname{Ho}(\mathbf{S})$ and $\mathbb{Z} : \operatorname{Ho}(\mathbf{S}) \to \operatorname{Ho}(s\mathbf{Ab})$, and that these functors are adjoint.

One way of seeing this (see also Section II.7) begins with the observation that there is a map $[X, A] \to [\mathbb{Z}X, A]_{s\mathbf{Ab}}$ (where $[\ ,\ ]_{s\mathbf{Ab}}$ denotes morphisms in the homotopy category $\operatorname{Ho}(s\mathbf{Ab})$) which is defined to be the composite

$$[X, A] \to [\mathbb{Z}X, \mathbb{Z}A]_{s\mathbf{Ab}} \xrightarrow{\epsilon_*} [\mathbb{Z}X, A]_{s\mathbf{Ab}},$$

where the first function is induced by the free simplicial abelian group functor, and $\epsilon : \mathbb{Z}A \to A$ is one of the adjunction maps. The corresponding composite

$$\pi(X, A) \to \pi(\mathbb{Z}X, \mathbb{Z}A) \xrightarrow{\epsilon_*} \pi(\mathbb{Z}X, A)$$

is an isomorphism, since $\mathbb{Z}(X \times \Delta^1) \cong \mathbb{Z}X \otimes \Delta^1$. All sources are cofibrant and all targets are fibrant, so there is a commutative diagram

$$\begin{tikzcd}
\pi(X,A) \arrow[r] \arrow[d, "\cong"'] & \pi(\mathbb{Z}X,\mathbb{Z}A) \arrow[r, "\epsilon_*"] \arrow[d, "\cong"'] & \pi(\mathbb{Z}X,A) \arrow[d, "\cong"] \\
{[X,A]} \arrow[r] & {[\mathbb{Z}X,\mathbb{Z}A]} \arrow[r, "\epsilon_*"'] & {[\mathbb{Z}X,A]}
\end{tikzcd}$$

in which all vertical maps are canonical bijections. It follows that the bottom horizontal composite is a bijection.

Let U denote the forgetful functor $s\mathbf{Ab} \to \mathbf{S}$. The adjointness of the induced functors on the homotopy category level is also a consequence of a formal principle (Brown's "adjoint functor lemma" [15, p.426], but see Section II.7): the adjunction maps $\epsilon : \mathbb{Z}U \to 1$ and $\eta : 1 \to U\mathbb{Z}$ induce natural transformations of the corresponding induced functors between the homotopy categories, and these transformations satisfy the triangle identities.

Now observe that there is a commutative diagram of isomorphisms

(2.15)
$$\begin{tikzcd}
\pi(\mathbb{Z}X,A) \arrow[d, "N", "\cong"'] \arrow[drr, dashed] & & \\
\pi_{Ch}(N\mathbb{Z}X,NA) & \pi_{Ch}(\mathbb{Z}X,NA) \arrow[l, "\cong", "i^*"'] \arrow[r, "\cong", "i_*"'] & \pi_{Ch}(\mathbb{Z}X,A)
\end{tikzcd}$$

in which the map labelled by N is induced by the normalization functor, and is well defined because $\mathbb{Z}X$ is cofibrant and A is fibrant, and the other labelled maps are induced by the chain homotopy equivalences $i : NA \subset A$ and $i : N\mathbb{Z}X \subset \mathbb{Z}X$. The dotted arrow takes the homotopy class of a simplicial abelian group map $f : \mathbb{Z}X \to A$ to the chain homotopy class which is represented by the map of Moore complexes induced by f. In particular, we have proved

PROPOSITION 2.16. *Suppose that X is a simplicial set and A is a simplicial abelian group. Then the group of simplicial homotopy classes $\pi(\mathbb{Z}X, A)$ can be canonically identified up to isomorphism with the group of chain homotopy classes $\pi_{Ch}(\mathbb{Z}X, A)$ between the associated Moore complexes.*

As a consequence of the proof of Proposition 2.16, we have the following:

COROLLARY 2.17. *Suppose that X is a simplicial set and B is an abelian group. Then there are canonical isomorphisms*

$$[X, K(B, n)] \cong H^n(X, B),$$

for $n \geq 0$.

PROOF: The simplicial abelian group $K(B, n)$ is $\Gamma B[n]$, where $B[n]$ is the chain complex which consists of the abelian group B concentrated in degree n. We know that the set $[X, K(B, n)]$ of morphisms in $\mathrm{Ho}(\mathbf{S})$ is canonically isomorphic to the set $\pi(X, K(B, n))$ of simplicial homotopy classes, which in turn is isomorphic to the set (really group) of homotopy classes

$$\pi(\mathbb{Z}X, K(B, n)) = \pi(\mathbb{Z}X, \Gamma B[n])$$

in the simplicial abelian group category. But from the diagram (2.15), there is are isomorphisms

$$\begin{aligned} \pi(\mathbb{Z}X, \Gamma B[n]) &\cong \pi_{Ch}(\mathbb{Z}X, N\Gamma B[n]) \\ &\cong \pi_{Ch}(\mathbb{Z}X, B[n]), \end{aligned}$$

and $\pi_{Ch}(\mathbb{Z}X, B[n])$ is the n^{th} cohomology group of the complex $\hom(\mathbb{Z}X, B)$. ∎

Suppose that C is a chain complex, let Z_n denote the subgroup of n-cycles and let $B_n = \partial(C_{n+1})$ be the subgroup of boundaries in C_n. Pick an epimorphism $p : F_n \twoheadrightarrow Z_n$, where F_n is a free abelian group. Then the kernel K_n of the composite

$$F_n \twoheadrightarrow Z_n \twoheadrightarrow H_n C$$

is free abelian, and the composite

$$K_n \subset F_n \xrightarrow{p} Z_n$$

factors through a map $p' : K_n \to B_n$. Since K_n is free abelian and the map $C_{n+1} \to B_n$ is surjective, the map p' lifts to a map $p : K_n \to C_{n+1}$. Write $F_n C$ for the chain complex

$$\cdots \to 0 \to \overset{n+1}{K_n} \hookrightarrow \overset{n}{F_n} \to 0 \to \cdots .$$

Then the epimorphism $F_n \to H_n C$ defines a quasi-isomorphism

$$q_n : F_n C \to H_n C[n],$$

while the maps labelled p define a chain map

$$p_n : F_nC \to C$$

which induces an isomorphism $H_n(F_nC) \cong H_nC$. It follows that there are quasi-isomorphisms

$$\bigoplus_{n\geq 0} H_nC[n] \xleftarrow{\oplus q_n} \bigoplus_{n\geq 0} F_nC \xrightarrow{\oplus p_n} C.$$

Note that the canonical map

$$\bigoplus_{n\geq 0} H_nC[n] \to \prod_{n\geq 0} H_nC[n]$$

is an isomorphism of chain complexes.

If the chain complex C happens to be the normalized complex NA of a simplicial abelian group A, then this construction translates through the functor Γ into weak equivalences of simplicial abelian groups

$$\prod_{n\geq 0} K(\pi_nA, n) \leftarrow \bigoplus_{n\geq 0} \Gamma F_nNA \to A.$$

These objects are fibrant in the category of simplicial sets, and so the weak equivalences induce homotopy equivalences of simplicial sets, proving

PROPOSITION 2.18. *Suppose that A is a simplicial abelian group. Then, as a simplicial set, A is non-canonically homotopy equivalent to the product of Eilenberg-Mac Lane spaces*

$$\prod_{n\geq 0} K(\pi_nA, n).$$

The key point in the argument for Proposition 2.18 is that a subgroup of a free abelian group is free, or at least projective. An analogous statement holds for modules over a principal ideal domain R so the simplicial abelian group A in the statement of the proposition can be replaced (at least) by a simplicial module over such a ring.

This is implicit above, but there is a natural short exact sequence

$$0 \to K(A, n) \to WK(A, n) \xrightarrow{p} K(A, n+1) \to 0$$

which are constructed by applying the functor Γ to the short exact sequence

$$0 \to A[n] \to A\langle n+1\rangle \to A[n+1] \to 0$$

of chain complexes. The simplicial abelian group $WK(A,n)$ is contractible, and the sequence

$$K(A,n) \to WK(A,n) \xrightarrow{p} K(A,n+1)$$

is one of the standard fibre sequences which is used to construct an equivalence $K(A,n) \simeq \Omega K(A,n+1)$ in the literature. There is one final observation about the Eilenberg-Mac Lane spaces $K(A,n) = \Gamma A[n]$ and the fibration p which is very commonly used:

LEMMA 2.19. *The map $p : WK(A,n) \to K(A,n+1)$ is a minimal fibration, and $K(A,n)$ is a minimal Kan complex, for all $n \geq 0$.*

PROOF: There is a relation $x \simeq_p y$ (in the sense of Section I.10) if and only if $(x-y) \simeq_p 0$, so it suffices to show that $z \simeq_p 0$ implies that $z = 0$ for any simplex z of $WK(A,n)$. But $z \simeq_p 0$ forces z to be in the fibre $K(A,n)$, and so to show that p is a minimal fibration we need only prove that $K(A,n)$ is a minimal Kan complex.

Suppose that z is an r-simplex of $K(A,n)$. If $z \simeq 0$ in $K(A,n)$ (rel $\partial\Delta^r$), then $d_i z = 0$ for all i, so that z is a normalized r-chain of $K(A,n)$. By the Dold-Kan correspondence, $NK(A,n)_r \cong A[n]_r$, which group is 0 if $r \neq n$, so it suffices to concentrate on the case $r = n$. There are identifications $NK(A,n)_n = K(A,n)_n = A$, and the resulting map $K(A,n)_n \to \pi_n(K(A,n),0)$ is an isomorphism. Thus $z \simeq 0$ (rel $\partial\Delta^n$) means that $z \mapsto 0$ in $\pi_n(K(A,n),0)$. But then $z = 0$. ∎

3. The Hurewicz map.

Suppose that X is a simplicial set. The *Hurewicz map* $h : X \to \mathbb{Z}X$ is alternate notation for the adjunction homomorphism which associates to an n-simplex x of X the corresponding generator of the free abelian group $\mathbb{Z}X_n$ on X_n. If X happens to be a connected Kan complex with base point $*$, then h induces a composite homomorphism of groups

$$\pi_n X \to \pi_n(\mathbb{Z}X, *) \to \pi_n(\mathbb{Z}X/\mathbb{Z}*, 0) \cong \tilde{H}_n(X, \mathbb{Z}),$$

where $\tilde{H}_*(X,\mathbb{Z})$ is reduced homology of X. The map

$$h_* : \pi_n X \to \tilde{H}_n(X, \mathbb{Z})$$

will denote this composite – h_* is the *Hurewicz homomorphism.* The simplicial set X will be a connected Kan complex with base point $*$ throughout this section, and all homology groups will be integral, so we shall write $\tilde{H}_* X$ to mean $\tilde{H}_*(X,\mathbb{Z})$. For ease of notation, write

$$\tilde{\mathbb{Z}}X = \mathbb{Z}X/\mathbb{Z}*\,.$$

The Hurewicz homomorphism $h_* : \pi_n X \to \tilde{H}_n X$ is the traditional map. To see this, let $S^n = \Delta^n/\partial\Delta^n$, with the obvious choice of base point. Then $\pi_n X$ is pointed simplicial homotopy classes of maps $S^n \to X$, and the homotopy group

$$\pi_n(\tilde{\mathbb{Z}}S^n) \cong \tilde{H}_n S^n$$

is a copy of the integers, canonically generated by the homotopy elemen κ_n which is represented by the composite map

$$S^n \xrightarrow{h} \mathbb{Z}S^n \to \tilde{\mathbb{Z}}S^n.$$

It is now easily seen, by drawing an appropriate commutative diagram, that if $\alpha : S^n \to X$ represents an element $[\alpha] \in \pi_n X$, then

$$h_*([\alpha]) = \alpha_*(\kappa_n) \in \tilde{H}_n X,$$

as in the standard definition.

Suppose that $p : E \to B$ is a Kan fibration, where B is a simply connected Kan complex with base point $*$, and let F be the fibre over $*$. There is a complete description of the Serre spectral sequence

$$E_2^{p,q} = H_p(B, H_q F) \Rightarrow H_{p+q} B$$

given below in IV.5.1, but it is non-standard. To recover the more usual form, filter the base B by skeleta $\mathrm{sk}_n\, B$, and form the pullback diagrams

$$\begin{tikzcd} F_n E \arrow[r, hook] \arrow[d, "p_*"'] & E \arrow[d, "p"] \\ \mathrm{sk}_n\, B \arrow[r, hook] & B \end{tikzcd}$$

to obtain a filtration $F_n E$ of the total space E and fibrations $p_* : F_n E \to \mathrm{sk}_n\, B$ for which $\pi_1\, \mathrm{sk}_n\, B$ acts trivially on the homology $H_* F$ of F.

Recall that $N_n B$ denotes the set of non-degenerate n-simplices of B. The pushout diagram

$$\begin{tikzcd} \bigsqcup_{x \in N_n B} \partial\Delta^n \arrow[r] \arrow[d, hook] & \mathrm{sk}_{n-1}\, B \arrow[d, hook] \\ \bigsqcup_{x \in N_n B} \Delta^n \arrow[r] & \mathrm{sk}_n\, B \end{tikzcd}$$

can be pulled back along p to obtain an identification of the filtration quotient $F_nE/F_{n-1}E$ with the wedge

$$\bigvee_{x \in N_n B} p^{-1}(\Delta^n)/p^{-1}(\partial\Delta^n).$$

The spectral sequence

$$H_r(\partial\Delta^n, H_sF) \Rightarrow H_{r+s}p^{-1}(\partial\Delta^n)$$

of IV.5 is used, along with the fact that H_*F splits off $H_*p^{-1}(\partial\Delta^n)$ to show that there is an isomorphism

$$H_*p^{-1}(\partial\Delta^n) \cong H_*F \oplus H_*F[n-1],$$

and that the map induced in homology by the inclusion $p^{-1}(\partial\Delta^n) \subset p^{-1}(\Delta^n)$ can be identified up to isomorphism with the projection

$$H_*F \oplus H_*F[n-1] \to H_*F$$

In particular, there is an isomorphism

$$\tilde{H}_*(p^{-1}(\Delta^n)/p^{-1}(\partial\Delta^n)) \cong H_*F[n],$$

and hence isomorphisms

$$\tilde{H}_*(F_nE/F_{n-1}E) \cong \bigoplus_{x \in N_n B} H_*F[n].$$

Thus,

$$\tilde{H}_{p+q}(F_pE/F_{p-1}E) \cong \bigoplus_{x \in N_p B} H_qF,$$

and it's a matter of chasing simplices through the boundary map of the normalized complex $N\mathbb{Z}B \otimes H_qF$ to see that the E_2-term of the spectral sequence for H_*E arising from the skeletal filtration for B has the form

$$E_2^{p,q} = H_p(B, H_qF).$$

All appeals to the Serre spectral sequence in the rest of this section will be specifically to this form.

Now suppose that Y is a pointed Kan complex, and observe that the canonical path object PY is that pointed function complex

$$PY = \mathbf{Hom}_*(\Delta^1_*, Y),$$

where Δ^1_* is a copy of the standard 1-simplex Δ^1, pointed by the vertex 1. Furthermore, the loop space ΩY can be identified with the complex $\mathbf{Hom}_*(S^1, Y)$, and the path loop fibration for Y is the fibre sequence

$$\mathbf{Hom}_*(S^1, Y) \xrightarrow{\pi^*} \mathbf{Hom}_*(\Delta^1_*, Y) \xrightarrow{d_1} Y,$$

where $\pi : \Delta^1 \to S^1$ is the canonical map.

There is a canonical contracting pointed homotopy

$$h : \Delta^1_* \wedge \Delta^1_* \to \Delta^1_*,$$

which is defined by the picture

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] \arrow[dr] & 1 \arrow[d] \\
1 \arrow[r] & 1
\end{tikzcd}$$

in the poset $\mathbf{1}$. This homotopy h induces a contracting homotopy

$$\mathbf{Hom}_*(\Delta^1_*, Y) \wedge \Delta^1_* \xrightarrow{h_*} \mathbf{Hom}_*(\Delta^1_*, Y)$$

for the path space on Y, by adjointness.

Suppose that $f : X \to \mathbf{Hom}_*(S^1, Y)$ is a pointed map, and denote the composite

$$X \wedge \Delta^1_* \xrightarrow{\pi^* f \wedge 1} \mathbf{Hom}_*(\Delta^1_*, Y) \wedge \Delta^1_* \xrightarrow{h} \mathbf{Hom}_*(\Delta^1_*, Y)$$

by f_*. Then there is a commutative diagram of pointed simplicial set maps

$$\begin{tikzcd}
X \arrow[r, "d^1"] \arrow[d, "f"'] & X \wedge \Delta^1_* \arrow[r, "1 \wedge \pi"] \arrow[d, "f_*"'] & X \wedge S^1 \arrow[d, "\tilde{f}"] \\
\mathbf{Hom}_*(S^1, Y) \arrow[r, "\pi^*"'] & \mathbf{Hom}_*(\Delta^1_*, Y) \arrow[r, "d_1"'] & Y,
\end{tikzcd} \tag{3.1}$$

where the indicated map $\tilde{f}$ is the adjoint of f, or rather the composite

$$X \wedge S^1 \xrightarrow{f \wedge 1} \mathbf{Hom}_*(S^1, Y) \wedge S^1 \xrightarrow{ev} Y.$$

Note as well that $X \wedge \Delta^1_*$ is a model for the pointed cone on X (a different model for the cone is given in Section III.5 below).

Suppose that

$$F \to E \xrightarrow{p} B$$

is a fibre sequence of pointed Kan complexes, and that B is simply connected. The edge maps

$$\begin{tikzcd} F \arrow[r] \arrow[d] & E \arrow[r] \arrow[d, "p"'] & B \arrow[d, "1"] \\ * \arrow[r] & B \arrow[r] & B \end{tikzcd}$$

and the calculation of the E_2-terms for the Serre spectral sequences for the corresponding fibrations together imply that $E_n^{n,0}$ in the Serre spectral sequence for H_*E is the subgroup of H_nB consisting of elements x which can be "lifted along the staircase"

$$\begin{tikzcd} & & H_{n-1}F \arrow[d] \\ & H_n(F_nE/F_{n-1}E) \arrow[r, "\partial"] \arrow[d, "p_*"'] & H_{n-1}F_{n-1}E \\ H_n(\mathrm{sk}_n B) \arrow[r] \arrow[d] & H_n(\mathrm{sk}_n B/\mathrm{sk}_{n-1} B) & \\ H_nB & & \end{tikzcd}$$

to an element $z \in H_{n-1}F$, in the sense that there are elements $x_1 \in H_n(\mathrm{sk}_n B)$ and $x_2 \in H_n(F_nE/F_{n-1}E)$ such that $x_1 \mapsto x$, $x_1 \mapsto p_*(x_2)$, and $z \mapsto \partial(x_2)$. Furthermore, the image of such an x under the differential $d_n : E_2^{n,0} \to E_2^{0,n-1}$ is represented by the element z. But then comparing long exact sequences shows that the element x_2 is in the image of the map $H_n(F_nE/F) \to H_n(F_nE/F_{n-1}E)$.

As a consequence, the elements x of $E_n^{n,0}$ can be identified with elements of $H_n(B/*)$ which are in the image of the map $p_* : H_n(E/F) \to H_n(B/*)$, and for such x, if there is an element $x_1 \in H_n(E/F)$ such that $p_*(x_1) = x$, then $d_n(x)$ is represented by $\partial(x_1) \in H_{n-1}F$. This is a classical description of the *transgression*.

Suppose that Y is an n-connected pointed Kan complex, where $n \geq 1$, and consider the Serre spectral sequence for the path loop fibre sequence

$$\Omega Y \to PY \xrightarrow{d_1} Y.$$

Then $\tilde{H}_i Y = 0$ for $i \leq n$, while $\tilde{H}_j \Omega Y = 0$ for $j \leq n-1$. The point is that the assumption on the connectivity of Kan complex Y, for example, that there is a strong deformation retract Z of Y such that the n-skeleton $\mathrm{sk}_n Z$ is a point[1]. It follows that $E_2^{i,0} = E_i^{i,0}$ and $E_2^{0,i-1} = E_i^{0,i-1}$ for $i \leq 2n$, and the only possible non-trivial differential into or out of either group is the transgression, so that there is an exact sequence

$$0 \to E_\infty^{i,0} \to E_i^{i,0} \xrightarrow{d_i} E_i^{0,i-1} \to E_\infty^{0,i-1} \to 0.$$

The space PY is acyclic, so all E_∞-terms vanish in non-zero total degree, so we have shown that the transgression

$$H_i Y \xrightarrow{d_i} H_{i-1}\Omega Y$$

is an isomorphism for $i \leq 2n$ under the assumption that Y is n-connected.

The transgression $d_i : H_i Y \to H_{i-1}\Omega Y$ is, at the same time, related to the boundary map

$$\partial : \tilde{H}_i(X \wedge S^1) \xrightarrow{\cong} \tilde{H}_{i-1} X,$$

in the sense of the following result:

LEMMA 3.2. *Suppose that $f : X \to \Omega Y$ is a map of pointed simplicial sets, where Y is an n-connected Kan complex with $n \geq 1$. Then, for $i \leq 2n$, there is*

[1] This is an old idea. People sometimes say that complexes of the form Z are *n-reduced*. The construction of the deformation retraction (via an iterated homotopy extension property argument) is one of the early applications of the Kan complex concept, and should be done as an exercise.

a commutative diagram of the form

$$\begin{tikzcd} \tilde{H}_i(X \wedge S^1) \arrow[r, "\partial", "\cong"'] \arrow[d, "\tilde{f}_*"'] & \tilde{H}_{i-1}X \arrow[d, "f_*"] \\ \tilde{H}_i Y \arrow[r, "\cong", "d_i"'] & \tilde{H}_{i-1}\Omega Y, \end{tikzcd}$$

where $\tilde{f} : X \wedge S^1 \to Y$ is the adjoint of the map f.

Proof: Write $\Sigma X = X \wedge S^1$ and $CX = X \wedge \Delta^1_*$, as usual. Then the diagram (3.1) induces a commutative diagram

$$\begin{tikzcd} \tilde{H}_i\Sigma X \cong \tilde{H}_i(\Sigma X/*) \arrow[d, "\tilde{f}_*"'] & \tilde{H}_i(CX/X) \arrow[l, "\cong"'] \arrow[r, "\partial"] \arrow[d, "f_*"'] & \tilde{H}_{i-1}X \arrow[d, "f_*"] \\ \tilde{H}_i Y \cong \tilde{H}_i(Y/*) & \tilde{H}_i(PY/\Omega Y) \arrow[l, "d_{1*}"] \arrow[r, "\partial"'] & \tilde{H}_{i-1}\Omega Y \end{tikzcd}$$

The top composite in the diagram is the boundary map $\partial : \tilde{H}_i\Sigma X \to \tilde{H}_{i-1}X$. The bottom "composite" is the transgression, according to the discussion preceding the statement of the lemma. ■

Corollary 3.3. *Suppose that Y is an n-connected pointed Kan complex, with $n \geq 1$. Then, for $i \leq 2n$, the transgression $d_i : \tilde{H}_iY \to \tilde{H}_{i-1}\Omega Y$ can be identified with the composite*

$$\tilde{H}_i(Y) \xleftarrow[\cong]{\epsilon_*} \tilde{H}_i((\Omega Y) \wedge S^1) \xrightarrow[\cong]{\partial} \tilde{H}_{i-1}(\Omega Y).$$

Proof: The adjunction map $\epsilon : (\Omega Y) \wedge S^1 \to Y$ is the adjoint of the identity map on the loop space ΩY. ■

The link between the Hurewicz homomorphism and the transgression is the following:

Proposition 3.4. *Suppose that Y is an n-connected pointed Kan complex, with $n \geq 1$. Then, for $i \leq 2n$, there is a commutative diagram*

$$\begin{tikzcd} \pi_i Y \arrow[r, "\partial", "\cong"'] \arrow[d, "h_*"'] & \pi_{i-1}(\Omega Y) \arrow[d, "h_*"] \\ \tilde{H}_i Y \arrow[r, "\cong", "d_i"'] & \tilde{H}_{i-1}(\Omega Y) \end{tikzcd}$$

Proof: The instance of the diagram (3.1) corresponding to the identity map on ΩY and the naturality of the Hurewicz map together give rise to a commutative diagram

$$\begin{tikzcd} \tilde{\mathbb{Z}}\Omega Y \arrow[r, equal] \arrow[d] & \tilde{\mathbb{Z}}\Omega Y \arrow[d] & \Omega Y \arrow[l, "h"'] \arrow[d] \\ \tilde{\mathbb{Z}}C\Omega Y \arrow[r] \arrow[d] & \tilde{\mathbb{Z}}PY \arrow[d, "d_{1*}"'] & PY \arrow[l, "h"'] \arrow[d] \\ \tilde{\mathbb{Z}}((\Omega Y) \wedge S^1) \arrow[r, "\epsilon_*"'] & \tilde{\mathbb{Z}}Y & Y \arrow[l, "h"] \end{tikzcd}$$

The homomorphism $d_{1*} : \tilde{\mathbb{Z}}PY \to \tilde{\mathbb{Z}}Y$ has a factorization

$$\tilde{\mathbb{Z}}PY \xrightarrow{j} B \xrightarrow{q} \tilde{\mathbb{Z}}Y$$

in the category of simplicial abelian groups (or of simplicial sets — your choice), where j is a trivial cofibration and q is a fibration. Then, by comparing boundary maps for the resulting fibrations of simplicial sets, one finds a commutative diagram

$$\begin{tikzcd} \pi_i Y \arrow[rr, "\partial", "\cong"'] \arrow[d, "h_*"'] & & \pi_{i-1}(\Omega Y) \arrow[d, "h_*"] \\ \tilde{H}_i Y & \tilde{H}_i((\Omega Y) \wedge S^1) \arrow[l, "\cong", "\epsilon_*"'] \arrow[r, "\cong", "\partial"'] & \tilde{H}_{i-1}(\Omega Y). \end{tikzcd}$$

Now use Corollary 3.3. ■

Write $x \overset{r}{\sim} y$ for n-simplices x and y of a simplicial set Z if $x|_{\mathrm{sk}_r \Delta^n} = y|_{\mathrm{sk}_r \Delta^n}$, or equivalently if the composite simplicial set maps

$$\mathrm{sk}_r \Delta^n \subset \Delta^n \xrightarrow{\iota_x} Z$$

and

$$\mathrm{sk}_r \Delta^n \subset \Delta^n \xrightarrow{\iota_y} Z$$

coincide. The relation $\overset{r}{\sim}$ is clearly an equivalence relation on the simplices of Z, and we may form the quotient simplicial set $Z(r) = Z/\overset{r}{\sim}$ and write $p_r : Z \to Z(r)$. The simplicial set $Z(r)$ is called either the r^{th} *Moore-Postnikov section* of Z, or the r^{th} *Postnikov* section of Z. This construction is natural in Z: one can show quickly that if Z is a Kan complex, then so is $Z(r)$, and furthermore p_r induces isomorphisms $\pi_j(Z,x) \cong \pi_j(Z(r),x)$ for $j \leq r$ and all vertices x of Z, and $\pi_j(Z(r),x) = 0$ for $j > r$.

Postnikov sections will be discussed more thoroughly in Chapter VI.

If X is a pointed connected Kan complex, then the object $X(1)$ is a pointed connected Kan complex of type $K(\pi_1 X, 1)$, meaning that its homotopy groups consist of $\pi_1 X$ in dimension one and 0 elsewhere. There is, however, a more geometrically satisfying way to construct a space naturally having the homotopy type of $X(1)$ which uses the fundamental groupoid construction: we have seen in Section III.1 that there is a map

$$X \to B(\pi_1 X)$$

of Kan complexes which induces an isomorphism on fundamental groups. It follows, for example, that a connected Kan complex of type $K(\pi, 1)$ is weakly equivalent to the space $B\pi$.

The Hurewicz homomorphism $h_* : \pi_1(B\pi) \to \tilde{H}_1(B\pi)$ is isomorphic to the canonical group homomorphism $\pi \to \pi/[\pi,\pi]$ from π to its abelianization — this may be seen directly, or by invoking the following result:

LEMMA 3.5. *Suppose that Z is a Kan complex, such that the set Z_0 of vertices of Z consists of a single point. The the Hurewicz map*

$$h_* : \pi_1 Z \to \tilde{H}_1(Z)$$

can be identified up to isomorphism with the canonical homomorphism

$$\pi_1 Z \to \pi_1 Z/[\pi_1 Z, \pi_1 Z].$$

Proof: Since Z is reduced, the integral homology group $H_1(Z)$ is the quotient

$$H_1(Z) = \bigoplus_{\omega \in Z_1} \mathbb{Z}/\langle d_0\sigma - d_1\sigma + d_2\sigma | \sigma \in Z_2\rangle.$$

Up to this identification, the Hurewicz map

$$h : \pi_1 Z \to \bigoplus_{\omega \in Z_1} \mathbb{Z}/\langle d_0\sigma - d_1\sigma + d_2\sigma | \sigma \in Z_2\rangle$$

is defined by $[\omega] \mapsto [\omega]$, for 1-simplices ω of Z. One then shows, by chasing elements, that the map h is initial among all group homomorphisms $f : \pi_1 Z \to A$ which take values in abelian groups A. ∎

Corollary 3.6. *Suppose that X is a connected pointed Kan complex. Then the Hurewicz homomorphism $h_* : \pi_1 X \to \tilde{H}_1 X$ is isomorphic to the canonical homomorphism*

$$\pi_1 X \to \pi_1 X / [\pi_1 X, \pi_1 X]$$

from the fundamental group of X to its abelianization.

Proof: The space X has a strong deformation retract Z which is reduced. Now apply Lemma 3.5. ∎

Theorem 3.7 (Hurewicz). *Suppose that X is an n-connected Kan complex, where $n \geq 1$. Then the Hurewicz homomorphism $h_* : \pi_i X \to \tilde{H}_i X$ is an isomorphism if $i = n+1$ and an epimorphism if $i = n+2$.*

Proof: Suppose that F is the homotopy fibre of the map $p_{n+1} : X \to X(n+1)$. Then there are commutative diagrams

$$\begin{tikzcd}
\pi_{n+1} X \arrow[r, "\cong"] \arrow[d, "h_*"'] & \pi_{n+1} X(n+1) \arrow[d, "h_*"] \\
\tilde{H}_{n+1} X \arrow[r] & \tilde{H}_{n+1} X(n+1)
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi_{n+2} F \arrow[r, "\cong"] \arrow[d, "h_*"'] & \pi_{n+2} X \arrow[d, "h_*"] \\
\tilde{H}_{n+2} F \arrow[r] & \tilde{H}_{n+2} X
\end{tikzcd}$$

A Serre spectral sequence argument for the fibration p_{n+1} shows that the map $\tilde{H}_{n+1} X \to \tilde{H}_{n+1} X(n+1)$ is an isomorphism, since the space F is $(n+1)$-connected, and that it suffices to show

(1) that the Hurewicz homomorphism $h_* : \pi_{n+1} X(n+1) \to \tilde{H}_{n+1} X(n+1)$ is an isomorphism, and
(2) $\tilde{H}_{n+2} X(n+1) = 0$.

If these two statements are demonstrated (for all n), then the general statement that $h_* : \pi_{n+1}X \to \tilde{H}_{n+1}X$ is an isomorphism would be true, so that the map $h_* : \pi_{n+2}F \to \tilde{H}_{n+2}F$ would be an isomorphism as well. Furthermore, the assertion that $\tilde{H}_{n+2}X(n+1) = 0$ implies, via the Serre spectral sequence for p_{n+1}, that the map $\tilde{H}_{n+2}F \to \tilde{H}_{n+2}X$ is an epimorphism.

But the claim that $h_* : \pi_{n+1}X(n+1) \to \tilde{H}_{n+1}X(n+1)$ is an isomorphism reduces, by an inductive transgression argument involving Proposition 3.4, to Corollary 3.6, so statement (1) is proved. Similarly, statement (2) is reduced to showing that $\tilde{H}_3 Y = 0$ for any connected Kan complex Y of type $K(A, 2)$.

Let Y be such a Kan complex, and pick a strong deformation retraction map $r : Y \to W$ onto a 2-reduced subcomplex. Then there is a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "h"] \arrow[d, "r"'] & \tilde{\mathbb{Z}}Y \arrow[d, "r_*"] \\ W \arrow[r, "h"'] & \tilde{\mathbb{Z}}W \end{tikzcd}$$

The map r_* is a weak equivalence of simplicial abelian groups, hence of simplicial sets, and the complex $\tilde{\mathbb{Z}}W$ is 0 in degrees less than 2, as is its associated normalized chain complex $N\tilde{\mathbb{Z}}W$. There is a map of chain complexes $N\tilde{\mathbb{Z}}W \to \tilde{H}_2 Y[2]$ which induces an isomorphism in H_2. It follows that the induced composite

$$Y \xrightarrow{h} \tilde{\mathbb{Z}}Y \xrightarrow{r_*} \tilde{\mathbb{Z}}W \to \Gamma \tilde{H}_2 Y[2] = K(\tilde{H}_2 Y, 2)$$

is a weak equivalence.

We are therefore required only to show that $H_3 K(A, 2) = 0$ for any abelian group A. The functor $A \mapsto H_3 K(A, 2)$ preserves filtered colimits, so it suffices to presume that A is finitely generated. The functor $A \mapsto K(A, 2)$ takes finite direct sums to products of simplicial sets, and a Künneth exact sequence argument shows that $H_3 K(A \oplus B, 2) = 0$ if $H_3 K(A, 2) = H_3 K(B, 2) = 0$. Finally, a few more Serre spectral sequence arguments, combined with knowing that the circle has type $K(\mathbb{Z}, 1)$ imply (successively) that $H_3 K(\mathbb{Z}, 2) = 0$ and $H_3 K(\mathbb{Z}/n, 2) = 0$ for any n. ∎

There is a very easy way to form what one might call the n^{th} Postnikov section $C(n)$ of a chain complex C: define $C(n)$ to be the chain complex

$$\cdots \to 0 \to C_n / Im\ \partial \to C_{n-1} \to \cdots \to C_0.$$

Here, $Im\ \partial$ is the image of the boundary map $\partial : C_{n+1} \to C_n$. There is an obvious chain map $p_n : C \to C(n)$; this map induces an isomorphism in H_i for $i \leq n$, and $H_i C(n)$ is plainly trivial for $i > n$. There is also a natural chain map $i_n : H_n C[n] \to C(n)$, and this map is a homology isomorphism. We could have used this construction in place of the retraction onto the 2-reduced complex W in the proof of the Hurewicz Theorem. This construction is also used to prove the following:

COROLLARY 3.8. *Let Y be a connected Kan complex with $\pi_n Y \cong A$, and $\pi_j Y = 0$ for $j \neq n$, where $n \geq 2$. Then Y is naturally weakly equivalent to the space $K(A, n) = \Gamma A[n]$.*

The naturality in the statement of this result is with respect to maps $Y \to Z$ of connected Kan complexes having only one non-trivial homotopy group, in degree n.

PROOF: Let $p_{n*} : \tilde{\mathbb{Z}}Y \to \Gamma(N\tilde{\mathbb{Z}}Y(n))$ be the simplicial abelian group map induced by the n^{th} Postnikov section map $p_n : N\tilde{\mathbb{Z}}Y \to N\tilde{\mathbb{Z}}(n)$ of the associated normalized chain complex. Then the composite

$$Y \xrightarrow{h} \tilde{\mathbb{Z}}Y \xrightarrow{p_{n*}} \Gamma(N\tilde{\mathbb{Z}}Y(n))$$

is a natural weak equivalence, by the Hurewicz Theorem and the construction of the chain map p_n. But then the following maps are both natural weak equivalences

$$Y \xrightarrow{p_{n*}h} \Gamma(N\tilde{\mathbb{Z}}Y(n)) \xleftarrow{i_{n*}} \Gamma A[n] = K(A, n),$$

and the result is proved. ∎

REMARK 3.9. The construction in this last proof is the only known way of showing that a diagram of spaces having only one non-trivial presheaf of homotopy groups is weakly equivalent to a diagram of spaces $K(A, n) = \Gamma A[n]$, for some presheaf (aka. diagram) of abelian groups A.

THEOREM 3.10 (FREUDENTHAL). *Suppose that X is an n-connected pointed space, where $n \geq 0$. Then the canonical map $\eta : X \to \Omega(X \wedge S^1)$ induces a map $\pi_i X \to \pi_i \Omega(X \wedge S^1)$ which is an isomorphism if $i \leq 2n$ and an epimorphism if $i = 2n + 1$.*

PROOF: We shall suppose that $n \geq 1$, and leave the case $n = 0$ for the reader.

From the characterization of the transgression for the path-loop fibre sequence of Corollary 3.3, the map

$$\epsilon_* : \tilde{H}_i(\Omega X \wedge S^1) \to \tilde{H}_i X$$

is an isomorphism for $i \leq 2n$. One then uses the triangle identity

$$\begin{tikzcd} X \wedge S^1 \arrow[r, "\eta \wedge S^1"] \arrow[dr, "1"'] & \Omega(X \wedge S^1) \wedge S^1 \arrow[d, "\epsilon"] \\ & X \wedge S^1 \end{tikzcd}$$

to infer that $\eta_* : \tilde{H}_i X \to \tilde{H}_i \Omega(X \wedge S^1)$ is an isomorphism if $i \leq 2n+1$. The space $\Omega(X \wedge S^1)$ is simply connected by assumption, so a Serre spectral sequence argument says that the homotopy fibre F of the map $\eta : X \to \Omega(X \wedge S^1)$ has homology groups $\tilde{H}_i F$ which vanish for $i \leq 2n$. But F is a simply connected space, by the Hurewicz theorem together with the fact that η is a homology isomorphism in degree 2, so that F is $2n$-connected (by Hurewicz again), giving the result. ■

Theorem 3.10 is the classical Freudenthal suspension theorem, since the homomorphism $\eta_* : \pi_1 X \to \pi_i \Omega(X \wedge S^1)$ is isomorphic to the suspension homomorphism $\pi_i X \to \pi_{i+1}(X \wedge S^1)$. We shall finish this section with the relative Hurewicz theorem:

THEOREM 3.11. *Suppose that $f : X \to Y$ is a map with homotopy fibre F and homotopy cofibre Y/X. Suppose that F is n-connected for some $n \geq 0$ and that the total space X is simply connected. Then the homotopy fibre of the induced map $f_* : F \wedge S^1 \to Y/X$ is $(n+2)$-connected.*

PROOF: The diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & * \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

induces the map $f_* : F \wedge S^1 \to Y/X$ of associated homotopy cofibres.

The spaces X and Y are both simply connected, by the assumptions on the map f, and so the Serre exact sequence for the map f

$$H_{n+2} X \to H_{n+2} Y \xrightarrow{d_{n+2}} H_{n+1} F \to H_{n+1} X \to \dots$$

extends to a sequence of the form

$$H_{n+3}Y \xrightarrow{d_{n+3}} E_{n+3}^{0,n+2} \to H_{n+2}X \to H_{n+2}Y \to \dots .$$

The classical description of the transgression says that there is a comparison of exact sequences

$$\begin{tikzcd}
H_{n+3}X/F \arrow[r, "\partial"] \arrow[d] & H_{n+2}F \arrow[r] \arrow[d] & H_{n+2}X \arrow[r] \arrow[d, "="] & H_{n+2}X/F \arrow[r, "\partial"] \arrow[d] & \dots \\
H_{n+3}Y \arrow[r, "d_{n+3}"'] & E_{n+3}^{0,n+2} \arrow[r] & H_{n+2}X \arrow[r] & H_{n+2}Y \arrow[r, "d_{n+2}"'] & \dots
\end{tikzcd}$$

Chasing elements shows that the map $H_i(X/F) \to H_iY$ is an isomorphism for $i \leq n+2$; it is an epimorphism for $i = n+3$ because every element of $H_{n+3}Y$ is transgressive. Comparing Puppe sequences gives a diagram

$$\begin{tikzcd}
H_{n+3}X \arrow[r] \arrow[d, "="'] & H_{n+3}(X/F) \arrow[r] \arrow[d] & H_{n+3}(F \wedge S^1) \arrow[r] \arrow[d, "f_*"] & H_{n+3}(X \wedge S^1) \arrow[r] \arrow[d, "="] & \dots \\
H_{n+3}X \arrow[r] & H_{n+3}Y \arrow[r] & H_{n+3}(Y/X) \arrow[r] & H_{n+3}(X \wedge S^1) \arrow[r] & \dots
\end{tikzcd}$$

Then one chases some more elements to show that $f_* : H_i(F \wedge S^1) \to H_i(Y/X)$ is an epimorphism if $i = n+3$ and is an isomorphism in all lower degrees. The space Y/X is simply connected, so the homotopy fibre of $f_* : F \wedge S^1 \to Y/X$ must be $(n+2)$-connected. ■

One has a right to ask why Theorem 3.11 should be called a relative Hurewicz theorem. Here's the usual statement:

Corollary 3.12. *Suppose that A is a simply connected subcomplex of X, and that the pair (X, A) is n-connected for some $n \geq 1$. Then the Hurewicz map*

$$\pi_i(X, A) \xrightarrow{h} \tilde{H}_i(X, A)$$

is an isomorphism if $i = n+1$ and is surjective if $i = n+2$.

Proof: The relative homotopy group $\pi_i(X,A)$ is the homotopy group $\pi_{i-1}F$ of the homotopy fibre F of the inclusion $j : A \hookrightarrow X$, and the relative Hurewicz map h is the composite

$$\pi_i(X,A) = \pi_{i-1}F \xrightarrow{\Sigma} \pi_i(F \wedge S^1) \xrightarrow{j_*} \pi_i(X/A) \xrightarrow{h} \tilde{H}_i(X/A) = \tilde{H}_i(X,A)$$

It is standard to say that (X,A) is n-connected and mean that the homotopy fibre F is $(n-1)$-connected. The Freudenthal suspension theorem says that the suspension map $\Sigma : \pi_{i-1}F \to \pi_i(F \wedge S^1)$ is an isomorphism if $i = n+1$ and an epimorphism if $i = n+2$ (for all $n \geq 2$). Theorem 3.11 implies that $j_* : \pi_i(F \wedge S^1) \to \pi_i(X/A)$ is an isomorphism if $i = n+1$ and an epi if $i = n+2$. Finally, the space X/A is n-connected so that the ordinary Hurewicz map $h : \pi_i(X/A) \to \tilde{H}_i(X/A)$ is an isomorphism if $i = n+1$ and an epimorphism if $i = n+2$. ∎

Rather a lot of standard homotopy theory is amenable to proof by simplicial techniques. The reader may find it of particular interest to recast the Haussman-Husemoller treatment of acylic spaces and the Quillen plus construction [41] in this setting. In order to achieve this, it's helpful to know at the outset that the *universal cover* $\tilde{X}$ of a connected pointed Kan complex X is, generally, the homotopy fibre of the map $X \to B(\pi_1 X)$ — see as well the definition of *covering system* in Section VI.3. This means, in particular, that the homotopy type of $\tilde{X}$ can be recovered from the pullback diagram

$$\begin{tikzcd}
\tilde{X} \arrow[r] \arrow[d, "\pi"'] & E(\pi_1 X) \arrow[d] \\
X \arrow[r] & B(\pi_1 X),
\end{tikzcd}$$

so that the map $\pi : \tilde{X} \to X$ is a principal $\pi_1 X$-fibration.

4. The Ex^∞ functor.

Kan's Ex^∞ functor is a combinatorial construction which associates a Kan complex $\mathrm{Ex}^\infty X$ to an arbitrary simplicial set X, up to natural weak equivalence. It is constructed as an inductive limit of spaces $\mathrm{Ex}^n X$, in such a way that the m-simplices of $\mathrm{Ex}^{n+1} X$ are a finite inverse limit of sets of simplices of $\mathrm{Ex}^n X$. This means in particular that this construction has very useful analogues in categories other than simplicial sets. It remains interesting in its own right in the simplicial set context, since it involves subdivision in a fundamental way.

We give the details of this construction and establish its basic properties in this section. It is one of the few remaining areas of simplicial homotopy theory in which the original combinatorial flavour of the subject (see the proof of Lemma 4.7) has not been engulfed by the calculus of anodyne extensions.

Recall that the non-degenerate simplices of the standard n-simplex

$$\Delta^n = \hom_{\mathbf{\Delta}}(\ ,\mathbf{n})$$

are the monic ordinal number maps $\mathbf{m} \hookrightarrow \mathbf{n}$. There is exactly one such monomorphism for each subset of $\mathbf{n}$ of cardinality $m+1$. It follows that the non-degenerate simplices of Δ^n form a poset $P\Delta^n$, ordered by the face relation, and this poset is isomorphic to the non-empty subsets of the ordinal number $\mathbf{n}$, ordered by inclusion.

The poset has a nerve $BP\Delta^n$. We shall write

$$\operatorname{sd}\Delta^n = BP\Delta^n,$$

and call it the *subdivision* of Δ^n.

Lemma 4.1. *There is a homeomorphism*

$$h : |\operatorname{sd}\Delta^n| \xrightarrow{\cong} |\Delta^n|,$$

where h is the affine map which takes a vertex $\sigma = \{v_0, \dots, v_k\}$ of $\operatorname{sd}\Delta^n$ to the barycentre $\frac{1}{k+1}(v_0 + \dots + v_k)$ of the corresponding vertices.

In other words, $|\operatorname{sd}\Delta^n|$ is the barycentric subdivision of $|\Delta^n|$.

Proof: To see the co-ordinate transformation, take

$$\alpha_0 v_0 + \alpha_1 v_1 + \dots + \alpha_n v_n \in |\Delta^n|,$$

and rewrite it as

$$t_1 X_1 + t_2 X_2 + \dots t_r X_r,$$

where $0 < t_1 < t_2 < \dots < t_r$ and $X_i = v_{j_0} + v_{j_1} + \dots + v_{j_{n_j}}$.

Write

$$N_j = \sum_{k=j}^{r} (n_k + 1).$$

Then

$$\begin{aligned} & t_1X_1 + \cdots + t_rX_r \\ & = t_1(X_1 + \cdots + X_r) + (t_2 - t_1)(X_2 + \cdots + X_r) + \cdots + (t_r - t_{r-1})X_r \\ & = t_1N_1(\frac{1}{N_1})(X_1 + \cdots + X_r) + (t_2 - t_1)N_2(\frac{1}{N_2})(X_2 + \cdots + X_r) + \dots \\ & \qquad + (t_r - t_{r-1})N_r(\frac{1}{N_r})X_r. \end{aligned}$$

Note that

$$t_1N_1 + (t_2 - t_1)N_2 + \cdots + (t_r - t_{r-1})N_r = 1,$$

so that we've rewritten $\alpha_0 v_0 + \cdots + \alpha_n v_n$ as an affine sum of uniquely determined barycentres. ■

Any *function* $f : \mathbf{n} \to \mathbf{m}$ determines a map of posets $f_* : P\Delta^n \to P\Delta^m$ via $f_*(X) = f(X) =$ image of X under f. It follows that any poset morphism $\theta : \mathbf{n} \to \mathbf{m}$ determines a poset map $\theta_* : P\Delta^n \to P\Delta^m$, and hence induces a simplicial set map $\theta_* : \operatorname{sd}\Delta^n \to \operatorname{sd}\Delta^m$. This assignment is functorial, so we obtain a cosimplicial object $\mathbf{n} \mapsto \operatorname{sd}\Delta^n$ in the category of simplicial sets.

The *subdivision* $\operatorname{sd}X$ of a simplicial set X is defined by

$$\operatorname{sd}X = \varinjlim_{\sigma:\Delta^n \to X} \operatorname{sd}\Delta^n,$$

where the colimit is indexed over the simplex category $\mathbf{\Delta} \downarrow X$ for X.

The functor $X \mapsto \operatorname{sd}X$ is left adjoint to a functor $Y \mapsto \operatorname{Ex}Y$, where the simplicial set $\operatorname{Ex}Y$ is defined to have n-simplices given by the set of all simplicial set maps $\operatorname{sd}\Delta^n \to Y$.

There is a natural map $h : \operatorname{sd}\Delta^n \to \Delta^n$, called the *last vertex map*. It is specified as a map of posets $P\Delta^n \to \mathbf{n}$ by the assignment

$$[v_0, v_1, \dots, v_k] \mapsto v_k,$$

where $[v_0, \dots, v_k] : \mathbf{k} \to \mathbf{n}$ is a non-degenerate simplex of Δ^n specified by $i \mapsto v_i$. There is also a map of posets $g : \mathbf{n} \to P\Delta^n$ defined by $g(i) = [0, 1, \dots, i]$. Clearly $hg = 1$ and there is a relation

$$[v_0, v_1, \dots, v_k] \leq [0, 1, \dots, v_k]$$

in $P\Delta^n$. It follows that the last vertex map $h : \operatorname{sd}\Delta^n \to \Delta^n$ is a simplicial homotopy equivalence.

The natural maps $h : \operatorname{sd}\Delta^n \to \Delta^n$ induce a natural simplicial map

$$\eta = h^* : Y \to \operatorname{Ex}Y,$$

which is given on n-simplices by precomposition by the indicated map h.

Lemma 4.2. *The map $\eta : Y \to \operatorname{Ex} Y$ is a π_0 isomorphism, and induces a surjection on fundamental groupoids.*

Proof: The map η is an isomorphism on the vertex level. A 1-simplex of $\operatorname{Ex} Y$ is a diagram

$$x \xrightarrow{\alpha} y \xleftarrow{\beta} z$$

of 1-simplices of Y, and $\eta(\alpha)$ is the diagram

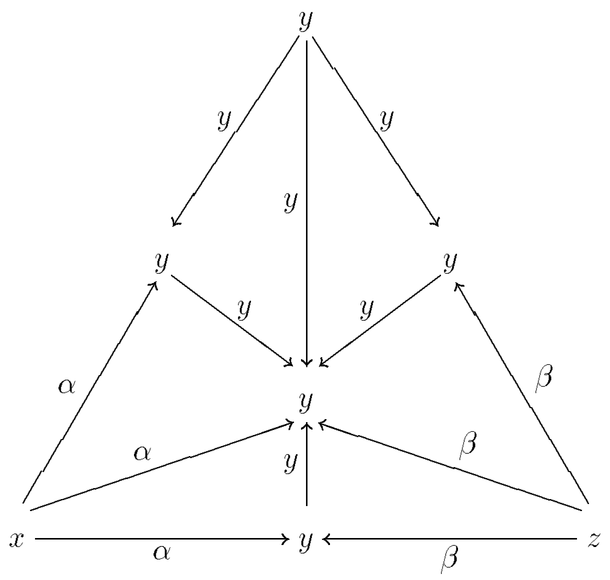
$$x \xrightarrow{\alpha} y \xleftarrow{s_0 y} y$$

for any 1-simplex $x \xrightarrow{\alpha} y$ of Y (incidentally, the notation means that $d_0(\alpha) = y$ and $d_1(\alpha) = x$). Thus, two vertices of $\operatorname{Ex} Y$ are related by a string of 1-simplices of $\operatorname{Ex} Y$ if and only if the corresponding vertices are related by a string on 1-simplices of Y, so that η is a π_0-isomorphism as claimed.

For each 1-simplex

$$x \xrightarrow{\alpha} y \xleftarrow{\beta} z$$

of $\operatorname{Ex} Y$, there is a 2-simplex σ of $\operatorname{Ex} Y$



Here, x is the 0^{th} vertex v_0 of σ, $v_1 = z$ and v_2 is the top copy of y. The two lower left 2-simplices of y are copies of $s_1\alpha$, the two lower right 2-simplices are copies of $s_1\beta$, and the two upper 2-simplices are constant simplices of Y associated to the vertex y. It follows that there is a relation

$$[\eta(\alpha)] = [x \xrightarrow{\alpha} y \xleftarrow{\beta} z][\eta(\beta)]$$

in the path category associated to $\mathrm{Ex}\,Y$. But then

$$[x \xrightarrow{\alpha} y \xleftarrow{\beta} z] = [\eta(\alpha)][\eta(\beta)]^{-1}$$

in the fundamental groupoid $\pi(\mathrm{Ex}\,Y)$. Every generator of $\pi(\mathrm{Ex}\,Y)$ is therefore in the image of the induced functor $\eta_* : \pi(Y) \to \pi(\mathrm{Ex}\,Y)$. ■

LEMMA 4.3. *The map $\eta : Y \to \mathrm{Ex}\,Y$ induces an isomorphism*

$$\eta_* : H_*(Y, \mathbb{Z}) \xrightarrow{\cong} H_*(\mathrm{Ex}\,Y, \mathbb{Z})$$

in integral homology.

PROOF: The natural maps $\eta : \Delta^k \to \mathrm{Ex}\,\Delta^k$ can be used to show that the Ex functor preserves homotopies. It follows that $\mathrm{Ex}\,\mathrm{sd}\,\Delta^n$ is contractible; in effect, the poset $P\Delta^n$ contracts onto the top non-degenerate simplex of Δ^n. It follows that $\mathrm{Ex}\,\mathrm{sd}\,\Delta^n$ has the homology of a point.

The natural map $\eta : Y \to \mathrm{Ex}\,Y$ induces an isomorphism $\pi_0 Y \cong \pi_0\,\mathrm{Ex}\,Y$, by Lemma 4.2. It follows that η induces a natural isomorphism

$$\eta_* : H_0(Y, \mathbb{Z}) \xrightarrow{\cong} H_0(\mathrm{Ex}\,Y, \mathbb{Z})$$

in the 0^{th} integral homology group. The simplices $\sigma : \Delta^n \to \mathrm{Ex}\,Y$ factor through maps $\mathrm{Ex}\,\sigma_* : \mathrm{Ex}\,\mathrm{sd}\,\Delta^n \to \mathrm{Ex}\,Y$. A standard acyclic models argument therefore implies that there is a natural chain map $\gamma : \mathbb{Z}\,\mathrm{Ex}\,Y \to \mathbb{Z}Y$ between the associated Moore complexes which induces the map $\eta_*^{-1} : H_0(\mathbb{Z}\,\mathrm{Ex}\,Y) \to H_0(\mathbb{Z}Y)$, and any two natural chain maps which induce η_*^{-1} are naturally chain homotopic. Similarly, the composite natural chain map

$$\mathbb{Z}\,\mathrm{Ex}\,Y \xrightarrow{\gamma} \mathbb{Z}Y \xrightarrow{\eta_*} \mathbb{Z}\,\mathrm{Ex}\,Y$$

is naturally chain homotopic to the identity map, and the models $\mathbb{Z}\Delta^n$ are used to show that the composite

$$\mathbb{Z}Y \xrightarrow{\eta_*} \mathbb{Z}\,\mathrm{Ex}\,Y \xrightarrow{\gamma} \mathbb{Z}Y$$

is naturally chain homotopic to the identity. The map $\eta_* : \mathbb{Z}Y \to \mathbb{Z}\,\mathrm{Ex}\,Y$ is therefore a natural chain homotopy equivalence, and so the map $\eta : Y \to \mathrm{Ex}\,Y$ is a homology isomorphism. ■

COROLLARY 4.4. *If the canonical map $Y \to *$ is a weak equivalence, then so is the map* $\operatorname{Ex} Y \to *$.

PROOF: The fundamental groupoid πY for Y is trivial, and η induces a surjection $\eta_* : \pi Y \to \pi \operatorname{Ex} Y$ by Lemma 4.2, so that $\operatorname{Ex} Y$ has a trivial fundamental groupoid as well. But $\operatorname{Ex} Y$ is acyclic by Lemma 4.3, so that $\operatorname{Ex} Y$ is weakly equivalent to a point, by the Hurewicz Theorem (Theorem 3.7). ■

LEMMA 4.5. *The functor* Ex *preserves Kan fibrations.*

PROOF: We show that the induced map $\operatorname{sd} \Lambda^n_k \hookrightarrow \operatorname{sd} \Delta^n$ is a weak equivalence.

The simplicial set $\operatorname{sd} \Lambda^n_k$ can be identified with the nerve of the poset of non-degenerate simplices of Λ^n_k, and the homeomorphism

$$h : |\operatorname{sd} \Delta^n| \xrightarrow{\cong} |\Delta^n|$$

restricts to a homeomorphism

$$|\operatorname{sd} \Lambda^n_k| \xrightarrow{\cong} |\Lambda^n_k|.$$

It follows that $\operatorname{sd} \Lambda^n_k \hookrightarrow \operatorname{sd} \Delta^n$ is a weak equivalence. ■

THEOREM 4.6. *The natural map $\eta : Y \to \operatorname{Ex} Y$ is a weak equivalence.*

PROOF: Let Y_f be a fibrant model for Y in the sense that there is a weak equivalence $\alpha : Y \to Y_f$. Pick a base point y for Y, and let PY_f be path space for Y_f corresponding to the base point $\alpha(y)$. Form the pullback diagram

$$\begin{tikzcd} \alpha^{-1} PY_f \arrow[r, "\alpha_*"] \arrow[d, "p_*"'] & PY_f \arrow[d, "p"] \\ Y \arrow[r, "\alpha"'] & Y_f, \end{tikzcd}$$

where $p : PY_f \to Y_f$ is the canonical fibration. Then α_* is a weak equivalence (Corollary II.9.6), so that $\alpha^{-1} PY_f$ is weakly equivalent to a point, and the fibre sequence

$$\Omega Y_f \to \alpha^{-1} PY_f \xrightarrow{p_*} Y.$$

gives rise to a comparison of fibre sequences

$$\begin{tikzcd}
\Omega Y_f \arrow[r] \arrow[d, "\eta"'] & \alpha^{-1} P Y_f \arrow[r, "p_*"] \arrow[d, "\eta"'] & Y \arrow[d, "\eta"] \\
\mathrm{Ex}\,\Omega Y_f \arrow[r] & \mathrm{Ex}\,\alpha^{-1} P Y_f \arrow[r, "\mathrm{Ex}\,p_*"'] & \mathrm{Ex}\,Y,
\end{tikzcd}$$

by Lemma 4.5. Computing in homotopy groups of realizations (and thereby implicitly using the fact that the realization of a Kan fibration is a Serre fibration — Theorem I.10.10), we find a commutative diagram

$$\begin{tikzcd}
\pi_1(Y,y) \arrow[r, "\cong"] \arrow[d, "\eta_*"'] & \pi_0(\Omega Y_f) \arrow[d, "\cong"', "\eta_*"] \\
\pi_1(\mathrm{Ex}\,Y,y) \arrow[r, "\cong"'] & \pi_0(\mathrm{Ex}\,\Omega Y_f).
\end{tikzcd}$$

One uses Corollary 4.4 to see that the indicated boundary maps are isomorphisms. It follows that η induces an isomorphism

$$\eta_* : \pi_1(Y,y) \xrightarrow{\cong} \pi_1(\mathrm{Ex}\,Y,y)$$

for all choices of base point y of Y. Inductively then, one shows that all maps

$$\eta_* : \pi_i(Y,y) \xrightarrow{\cong} \pi_i(\mathrm{Ex}\,Y,y)$$

are isomorphisms for all choices of base point y. ■

LEMMA 4.7. *For any map $\lambda : \Lambda^n_k \to \mathrm{Ex}\,Y$, the dotted arrow exists in the diagram*

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "\lambda"] \arrow[d] & \mathrm{Ex}\,Y \arrow[d, "\eta"] \\
\Delta^n \arrow[r, dashed] & \mathrm{Ex}^2\ Y.
\end{tikzcd}$$

Proof: The adjoint of the composite map

$$\Lambda^n_k \xrightarrow{\lambda} \operatorname{Ex} Y \xrightarrow{\eta} \operatorname{Ex}^2\ Y$$

is the composite

$$\operatorname{sd}\Lambda^n_k \xrightarrow{h} \Lambda^n_k \xrightarrow{\lambda} \operatorname{Ex} Y.$$

It therefore suffices to show that the dotted arrow h_* exists in the diagram

$$\begin{tikzcd} \operatorname{sd}\Lambda^n_k \arrow[r, "h"] \arrow[d, hook] & \Lambda^n_k \arrow[d, "\tilde{\eta}"] \\ \operatorname{sd}\Delta^n \arrow[r, dashed] & \operatorname{Ex}\operatorname{sd}\Lambda^n_k, \end{tikzcd}$$

making it commute, where $\tilde{\eta} : K \to \operatorname{Ex}\operatorname{sd} K$ is the counit of the adjunction $(\operatorname{sd}, \operatorname{Ex})$, and is defined by sending a simplex $\sigma : \Delta^n \to K$ to the simplex of $\operatorname{Ex}\operatorname{sd} K$ given by the induced map $\operatorname{sd}\sigma : \operatorname{sd}\Delta^n \to \operatorname{sd} K$.

Let $\sigma = (\sigma_0, \dots, \sigma_q)$ be a q-simplex of $\operatorname{sd}\Delta^n$, where the maps $\sigma_i : \mathbf{n}_i \hookrightarrow \mathbf{n}$ are simplices of Δ^n. Define a function $f_\sigma : \mathbf{q} \to \mathbf{n}$ by the assignment

$$f_\sigma(i) = \begin{cases} \sigma_i(n_i) & \text{if } \sigma_i \neq d_k\iota_n \text{ or } \iota_n, \text{ and} \\ k & \text{if } \sigma_i = d_k\iota_n \text{ or } \sigma_i = \iota_n. \end{cases}$$

The assignment $\sigma \mapsto f_\sigma$ is natural with respect to morphisms of ordinal numbers $\theta : \mathbf{q}' \to \mathbf{q}$ in the sense that $f_\sigma\theta = f_{\sigma\theta_*}$. There is a unique pair (X_σ, π_σ) consisting of a poset monic $X_\sigma : \mathbf{r} \hookrightarrow \mathbf{n}$ and a surjective function $\pi_\sigma : \mathbf{q} \twoheadrightarrow \mathbf{r}$ such that the diagram of functions

$$\begin{tikzcd} \mathbf{q} \arrow[rr, "f_\sigma"] \arrow[dr, "\pi_\sigma"'] & & \mathbf{n} \\ & \mathbf{r} \arrow[ur, hook, "X_\sigma"'] & \end{tikzcd}$$

commutes. Then a simplicial map $h_* : \operatorname{sd}\Delta^n \to \operatorname{Ex}\operatorname{sd}\Lambda^n_k$ is defined by the assignment $\sigma \mapsto \pi^*_\sigma(\tilde{\eta}(X_\sigma))$. Note that, whereas $\pi_\sigma : \mathbf{q} \to \mathbf{r}$ is only a function, it induces a poset map $\pi_{\sigma *} : P\Delta^q \to P\Delta^r$ and hence a simplicial map $\pi_{\sigma *}$: $\operatorname{sd}\Delta^q \to \operatorname{sd}\Delta^r$, so that the definition of the map h_* makes sense. ∎

For any simplicial set X, define $\operatorname{Ex}^\infty X$ to be the colimit in the simplicial set category of the string of maps

$$X \xrightarrow{\eta_X} \operatorname{Ex} X \xrightarrow{\eta_{\operatorname{Ex} X}} \operatorname{Ex}^2 X \xrightarrow{\eta_{\operatorname{Ex}^2 X}} \operatorname{Ex}^3 X \to \dots$$

Then the assignment $X \mapsto \operatorname{Ex}^\infty X$ defines a functor from the simplicial set category to itself, which is commonly called the Ex^∞ *functor* . Write $\nu : X \to \operatorname{Ex}^\infty X$ for the canonical natural map which arises from definition of $\operatorname{Ex}^\infty X$. The results of this section imply the following:

THEOREM 4.8.

(1) *The canonical map $\nu : X \to \operatorname{Ex}^\infty X$ is a weak equivalence, for any simplicial set X.*
(2) *For any X, the simplicial set $\operatorname{Ex}^\infty X$ is a Kan complex.*
(3) *The Ex^∞ functor preserves Kan fibrations.*

PROOF: The first statement is a consequence of Theorem 4.6. The second statement follows from Lemma 4.5. Statement (3) is implied by Lemma 4.7. ∎

5. The Kan suspension.

The ordinal number map $d^0 : \mathbf{n} \to \mathbf{n+1}$ induces an inclusion $d^0 : \Delta^n \to \Delta^{n+1}$. Let the vertex 0 be a base point for Δ^{n+1}, and observe that any simplicial map $\theta : \Delta^n \to \Delta^m$ uniquely extends to a simplicial map $\theta_* : \Delta^{n+1} \to \Delta^{m+1}$, which is pointed in the sense that $\theta_*(0) = 0$, and such that $\theta_* d^0 = d^0\theta$. Observe that $d^i_* = d^{i+1}$ and $s^j_* = s^{j+1}$ for all i and j.

The *cone* CY of a simplicial set Y is the pointed simplicial set

$$CY = \varinjlim_{\Delta^n \to Y} \Delta^{n+1},$$

where the colimit is indexed in the *simplex category* of all simplices $\Delta^n \to Y$, and is formed in the pointed simplicial set category. The maps $d^0 : \Delta^n \to \Delta^{n+1}$ induce a natural map $j : Y \to CY$.

The cone CY on a simplicial set Y is contractible: the contracting homotopies $h : \Delta^{n+1} \times \Delta^1 \to \Delta^{n+1}$ given by the transformations $0 \to i$ in the ordinal numbers $\mathbf{n+1}$ glue together along the simplices of Y to give a contracting homotopy

$h : CY \times \Delta^1 \to CY$ onto the base point of Y, since there are commutative diagrams

$$\begin{tikzcd} \Delta^{n+1} \times \Delta^1 \arrow[r, "h"] \arrow[d, "\theta_* \times 1"'] & \Delta^{n+1} \arrow[d, "\theta_*"] \\ \Delta^{m+1} \times \Delta^1 \arrow[r, "h"'] & \Delta^{m+1} \end{tikzcd}$$

for any ordinal number map $\theta : \mathbf{n} \to \mathbf{m}$.

It's now obvious that a simplicial set map $f : Y \to X$ can be extended to a map $g : CY \to X$ in the sense that there is a commutative diagram

$$\begin{tikzcd} CY \arrow[dr, "g"] & \\ Y \arrow[u, "j"] \arrow[r, "f"'] & X \end{tikzcd}$$

if and only if for each n-simplex x of Y there is an $(n+1)$-simplex $g(x)$ of X such that

(1) $d_0(g(x)) = f(x)$,

(2) $d_1 d_2 \cdots d_{n+1}(g(x))$ is some fixed vertex v of X for all simplices x of Y, and

(3) for all $i, j \geq 0$ and all simplices x of Y, we have

$$d_{i+1}(g(x)) = g(d_i x) \quad \text{and} \quad s_{j+1}(g(x)) = g(s_j x).$$

For the moment, given a simplicial set X, let $X_{-1} = \pi_0 X$ and write $d_0 : X_0 \to X_{-1}$ for the canonical map $X_0 \to \pi_0 X$.

A simplicial set X is said to have an *extra degeneracy* if there are functions $s_{-1} : X_n \to X_{n+1}$ for all $n \geq -1$, such that, in all degrees,

(1) $d_0 s_{-1}$ is the identity on X_n,

(2) for all $i, j \geq 0$, we have the identities

$$d_{i+1} s_{-1}(x) = s_{-1} d_i(x) \quad \text{and} \quad s_{j+1} s_{-1}(x) = s_{-1} s_j(x).$$

LEMMA 5.1. *Suppose that a simplicial set X has an extra degeneracy. Then the canonical map $X \to K(\pi_0 X, 0)$ is a homotopy equivalence.*

PROOF: It suffices to assume that X is connected. Then the association $x \mapsto s_{-1}(x)$ determines an extension

$$\begin{tikzcd} CX \arrow[dr, "s_{-1}"] & \\ X \arrow[u, "j"] \arrow[r, "1"'] & X \end{tikzcd}$$

of the identity map on X, according to the criteria given above. Also, CX contracts onto its base point, which point maps to $s_{-1}(*)$ in X, where $*$ denotes the unique element of $\pi_0 X$. ■

EXAMPLE 5.2. Suppose that G is a group. The *translation category* EG associated to the G-action $G \times G \to G$ has the elements of G for objects, and has morphisms of the form $h : g \to hg$. The nerve of this category is commonly also denoted by EG. Note in particular that an n-simplex of the resulting simplicial set EG has the form

$$g_0 \xrightarrow{g_1} g_1 g_0 \xrightarrow{g_2} \dots \xrightarrow{g_n} g_n \cdot \dots \cdot g_0,$$

and may therefore be identified with a string $(g_0, g_1, \dots, g_n)$ of elements of the group G. The simplicial set EG is plainly connected. It is also easily seen (by thinking in terms of strings of arrows) that the assignment

$$(g_0, g_1, \dots, g_n) \mapsto (e, g_0, g_1, \dots, g_n)$$

defines an extra degeneracy $s_{-1} : EG_n \to EG_{n+1}$ for EG, so that EG is contractible.

If K is a pointed simplicial set, then the *pointed cone* C_*K is defined by the pushout

$$\begin{tikzcd} C\Delta^0 \arrow[r, "C*"] \arrow[d] & CK \arrow[d] \\ * \arrow[r] & C_*K. \end{tikzcd}$$

Here, the map $C*$ is induced by the inclusion of the base point $* : \Delta^0 \to K$ in K.

The maps $d^0 : \Delta^n \to \Delta^{n+1}$ induce a natural pointed map $i : K \to C_*K$. The *Kan suspension* ΣK of K is defined to be the quotient

$$\Sigma K = C_*K/K.$$

The Kan suspension ΣK is a reduced simplicial set, and is a concrete model for the suspension of the associated pointed space $|K|$, in the sense that the realization $|\Sigma K|$ is naturally homeomorphic to the topological suspension of $|K|$. The existence of this homeomorphism is one of the reasons that the Kan suspension functor $\Sigma : \mathbf{S}_* \to \mathbf{S}_*$ preserves weak equivalences — one could also argue directly from the definitions by using the cofibre sequence

$$K \subset C_*K \to \Sigma K$$

and the contracting homotopy on the cone C_*K.

A pointed simplicial set map $\phi : \Sigma K \to Y$ consists of pointed functions

$$\phi_n : K_n \to Y_{n+1} \tag{5.3}$$

such that

(1) $d_1 \dots d_{n+1}\phi_n(x) = *$, and $d_0\phi_n(x) = *$ for each $x \in K_n$, and
(2) for each ordinal number map $\theta : \mathbf{n} \to \mathbf{m}$, the diagram of pointed functions

$$\begin{tikzcd} K_n \arrow[r, "\phi_n"] \arrow[d, "\theta^*"'] & Y_{n+1} \arrow[d, "(\tilde{\theta})^*"] \\ K_m \arrow[r, "\phi_m"'] & Y_{m+1} \end{tikzcd}$$

commutes.

Pointed simplicial set maps of the form $\psi : C_*K \to Y$ have a very similar characterization; one simply deletes the requirement that $d_0\psi_n(x) = *$. It follows that the pointed cone and Kan suspension functors preserve colimits of pointed simplicial sets.

An equivalent description of C_*K starts from the observation that the pointed simplicial set K is a member of a coequalizer diagram of the form

$$\bigvee_{\theta:\mathbf{n}\to\mathbf{m}} K_m \wedge \Delta^n_+ \rightrightarrows \bigvee_{n\geq 0} K_n \wedge \Delta^n_+ \to K$$

where, for example, $K_m \wedge \Delta^n_+$ is the wedge of the pointed set of m-simplices K_m, thought of as a discrete pointed simplicial set, with Δ^n_+, and Δ^n_+ is notation for the simplicial set $\Delta^n \sqcup \{*\}$, pointed by the disjoint vertex $*$. Then C_*K is defined by the coequalizer diagram

$$\bigvee_{\theta:\mathbf{n}\to\mathbf{m}} K_m \wedge \Delta^{n+1} \rightrightarrows \bigvee_{n\geq 0} K_n \wedge \Delta^{n+1} \to C_*K.$$

The set of m-simplices of Δ^{n+1} is the set of ordinal number maps of the form $\gamma : \mathbf{m} \to \mathbf{n+1}$. Each such γ fits into a pullback diagram

$$\begin{tikzcd} \mathbf{j} \arrow[r, "\gamma_*"] \arrow[d, "(d^0)^{n-j}"'] & \mathbf{n} \arrow[d, "d^0"] \\ \mathbf{m} \arrow[r, "\gamma"'] & \mathbf{n+1} \end{tikzcd}$$

in the ordinal number category, for some uniquely determined map $\gamma_* : \mathbf{j} \to \mathbf{n}$ if $\gamma^{-1}(\mathbf{n}) \neq \emptyset$. It follows that, as a pointed set, Δ^{n+1}_m has the form

$$\begin{aligned} \Delta^{n+1}_m &= \Delta^n_m \sqcup \Delta^n_{m-1} \sqcup \cdots \sqcup \Delta^n_0 \sqcup \{*\} \\ &= (\Delta^n_+)_m \vee (\Delta^n_+)_{m-1} \vee \cdots \vee (\Delta^n_+)_0, \end{aligned}$$

where the base point corresponds to the case $\gamma(\mathbf{m}) = 0$. Now take another map $\zeta : \mathbf{k} \to \mathbf{m}$; there is a pullback diagram of ordinal number maps

$$\begin{tikzcd} \zeta^{-1}(\mathbf{j}) \arrow[r, "\hat{\zeta}"] \arrow[d] & \mathbf{j} \arrow[d, "(d^0)^{m-j}"] \\ \mathbf{k} \arrow[r, "\zeta"'] & \mathbf{m} \end{tikzcd} \tag{5.4}$$

in the case where $\mathbf{r} = \zeta^{-1}(\mathbf{j}) \neq \emptyset$. It follows that the restriction of $\zeta^* : \Delta^{n+1}_m \to \Delta^{n+1}_k$ to the summand $(\Delta^n_+)_j$ is the map

$$\hat{\zeta}^* : (\Delta^n_+)_j \to (\Delta^n_+)_r$$

if $\zeta^{-1}(\mathbf{j}) \neq \emptyset$, and is the map to the base point otherwise.

Suppose that K is a pointed simplicial set. There is a a pointed simplicial set $\tilde{C}_*(K)$ whose set of n-simplices is given by

$$\tilde{C}_*(K)_n = K_n \vee K_{n-1} \vee \cdots \vee K_0.$$

The map $\zeta^* : \tilde{C}_*(K)_m \to \tilde{C}_*(K)_k$ associated to $\zeta : \mathbf{k} \to \mathbf{m}$ is given on the summand K_j by the composite

$$K_j \xrightarrow{\hat{\zeta}^*} K_r \hookrightarrow \tilde{C}_*(K_k)$$

in the case where $\zeta^{-1}(\mathbf{j}) \neq \emptyset$, $\mathbf{r} = \zeta^{-1}(\mathbf{j})$, and the map $\hat{\zeta}$ is defined by the diagram (5.4). If $\zeta^{-1}(\mathbf{j}) = \emptyset$, then the restriction of ζ^* to K_j maps to the base point.

One checks that $\tilde{C}_*(K)$ is indeed a pointed simplicial set, and that the construction is functorial in K. Furthermore, the functor preserves colimits, so that the diagram

$$\bigvee_{\theta:\mathbf{n}\to\mathbf{m}} \tilde{C}_*(K_m \wedge \Delta^n_+) \rightrightarrows \bigvee_{n\geq 0} \tilde{C}_*(K_n \wedge \Delta^n_+) \to \tilde{C}_* K.$$

is a coequalizer. On the other hand, the definitions imply that there are isomorphisms

$$\begin{aligned} \tilde{C}_*(Y \wedge \Delta^n_+) &\cong Y \wedge \tilde{C}_*(\Delta^n_+) \\ &\cong Y \wedge \Delta^{n+1}, \end{aligned}$$

which are natural in the pointed sets Y and simplices Δ^n. This is enough to prove

LEMMA 5.5. *There is a pointed simplicial set isomorphism*

$$\tilde{C}_*(K) \cong C_*(K),$$

which is natural in K.

The composite

$$K \xrightarrow{i} C_*(K) \cong \tilde{C}_*(K),$$

in degree n is the inclusion of the wedge summand K_n. The canonical map $i : K \to C_*(K)$ is therefore an inclusion. Collapsing by K in each degree also gives a nice description of the Kan suspension ΣK: the set of $(n+1)$-simplices of ΣK is given by the wedge sum

$$\Sigma K_{n+1} = K_n \vee K_{n-1} \vee \cdots \vee K_0.$$

It's also a worthy exercise to show that the maps $\eta_n : K_n \to \Sigma K_{n+1}$ corresponding to the identity map $\Sigma K \to \Sigma K$ under the association (5.3) are inclusions $K_n \to K_n \vee K_{n-1} \vee \cdots \vee K_0$ of wedge summands.