Chapter II Model Categories

The closed model axioms have a list of basic abstract consequences, including an expanded notion of homotopy and a Whitehead theorem. The associated homotopy category is defined to be the result of of formally inverting the weak equivalences within the ambient closed model category, but can be constructed (in the CW-complex style) by taking homotopy classes of maps between objects which are fibrant and cofibrant. These topics are presented in the first section of this chapter.

The simplicial set category has rather more structure than just that of a closed model category: the set $\hom(X, Y)$ of maps between simplicial sets X and Y is the set of vertices of the function complex $\mathbf{Hom}(X, Y)$, and the collection of all such function complexes determines a simplicial category. We've already seen that the function complexes satisfy an exponential law and respect cofibrations and fibrations in a suitable sense. The existence of the function complexes and the interaction with the closed model structure can be abstracted to a definition of a simplicial model category, which is given in Section 2, along with various examples. Basic consequences of the additional simplicial structure are presented, with further examples, in Sections 3 and 4.

Sections 5, 6 and 7 are about detection principles for simplicial model structures. Generally speaking, a simplicial model category structure for the category $s\mathcal{C}$ of simplicial objects in a category $\mathcal{C}$ is induced from the simplicial model structure on the simplicial set category in the presence of an adjoint pair of functors

$$F : \mathbf{S} \rightleftarrows s\mathcal{C} : G,$$

(or a collection of adjoint pairs) if G has a little extra structure (such as preservation of filtered colimits) in addition to being a right adjoint — this is Theorem 5.1. In one major stream of examples, the category $\mathcal{C}$ is some algebraic species, such as groups or abelian groups, and G is a forgetful functor. There is, however, an extra technical condition on Theorem 5.1, namely that every cofibration of $s\mathcal{C}$ having the left lifting property with respect to all fibrations should be a weak equivalence. This condition can often be verified by brute force, as can be done in the presence of a small object argument for the factorization axioms (eg. simplicial abelian groups), but there is a deeper criterion, namely the existence of a natural fibrant model (Lemma 6.1). The other major source of examples has to do with G being a representable functor of the form $G = \hom(Z,\ )$, where Z is either small in the sense that $\hom_{\mathcal{C}}(Z,\ )$ respects filtered colimits, or is a disjoint union of small objects. In this setting, Kan's Ex^{∞}-construction (see Section III.4) is used to construct the natural fibrant models required by

Lemma 6.1. This line of argument is generalized significantly in Section 6, at the cost of the introduction of cofibrantly generated closed model categories and transfinite small object arguments.

Section 8 is an apparent return to basics. We develop a criterion for a pair of adjoint functors between closed model categories to induce adjoint functors on the homotopy category level, known as Quillen's total derived functor theorem. Quillen's result is, at the same time, a non-abelian version of the calculus of higher direct images, and a generalization of the standard result that cohomology is homotopy classes of maps taking values in Eilenberg-Mac Lane spaces.

The category of simplicial sets, finally, has even more structure: it is a *proper* simplicial model category, which means that, in addition to everything else, weak equivalences are preserved by pullback over fibrations and by pushout along cofibrations. This property is discussed in Section 9. Properness is the basis of the standard results about homotopy cartesian diagrams, as well as being of fundamental importance in stable homotopy theory. We discuss homotopy cartesian diagrams in the context of Gunnarsson's axiomatic approach to the glueing and coglueing lemmas [40].

1. Homotopical algebra.

Recall that a *closed model category* $\mathcal{C}$ is a category which is equipped with three classes of morphisms, called cofibrations, fibrations and weak equivalences which together satisfy the following axioms:

CM1: The category $\mathcal{C}$ is closed under all finite limits and colimits.

CM2: Suppose that the following diagram commutes in $\mathcal{C}$:

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "h"'] & & Y \arrow[dl, "f"] \\ & Z. & \end{tikzcd}$$

If any two of f, g and h are weak equivalences, then so is the third.

CM3: If f is a retract of g and g is a weak equivalence, fibration or cofibration, then so is f.

CM4: Suppose that we are given a commutative solid arrow diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ V \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where i is a cofibration and p is a fibration. Then the dotted arrow exists, making the diagram commute, if either i or p is also a weak equivalence.

CM5: Any map $f : X \to Y$ may be factored:

(a) $f = p \cdot i$ where p is a fibration and i is a trivial cofibration, and
(b) $f = q \cdot j$ where q is a trivial fibration and j is a cofibration.

This set of axioms has a list of standard consequences which amplifies the interplay between cofibrations, fibrations and weak equivalences, giving rise to collection of abstract techniques that has been known as homotopical algebra since Quillen introduced the term in [75]. This theory is is really an older friend in modern dress, namely obstruction theory made axiomatic. The basic results, along with their proofs, are sketched in this section.

We begin with the original meaning of the word "closed":

Lemma 1.1. *Suppose that $\mathcal{C}$ is a closed model category. Then we have the following:*

1. *A map $i : U \to V$ of $\mathcal{C}$ is a cofibration if and only if i has the left lifting property with respect to all trivial cofibrations.*
2. *The map i is a trivial cofibration if and only if it has the right lifting property with respect to all fibrations.*
3. *A map $p : X \to Y$ of $\mathcal{C}$ is a fibration if and only if it has the right lifting property with respect to all trivial cofibrations.*
4. *The map p is a trivial fibration if and only if it has the right lifting property with respect to all cofibrations.*

The point of Lemma 1.1 is that the various species of cofibrations and fibrations determine each other via lifting properties.

Proof: We shall only give a proof of the first statement; the other proofs are similar.

Suppose that i is a cofibration, p is a trivial fibration, and that there is a commutative diagram

(1.2)
$$\begin{tikzcd} U \arrow[r, "\alpha"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ V \arrow[r, "\beta"'] & Y \end{tikzcd}$$

Then there is a map $\theta : V \to X$ such that $p\theta = \beta$ and $\theta i = \alpha$, by **CM4**. Conversely suppose that $i : U \to V$ is a map which has the left lifting property with respect to all trivial fibrations. By **CM5**, i has a factorization

$$\begin{tikzcd} U \arrow[r, "j"] \arrow[dr, "i"'] & W \arrow[d, "q"] \\ & V \end{tikzcd}$$

where j is a cofibration and q is a trivial fibration. But then there is a commutative diagram

$$\begin{tikzcd} U \arrow[r, "j"] \arrow[d, "i"'] & W \arrow[d, "q"] \\ V \arrow[ur] \arrow[r, equal] & V \end{tikzcd}$$

and so i is a retract of j. **CM3** then implies that i is a cofibration. ■

The proof of the Lemma 1.1 contains one of the standard tricks that is used to prove that the axiom **CM4** holds in a variety of settings, subject to having an adequate proof of the factorization axiom **CM5**. Lemma 1.1 also immediately implies the following:

COROLLARY 1.3.

(1) *The classes of cofibrations and trivial cofibrations are closed under composition and pushout. Any isomorphism is a cofibration.*
(2) *The classes of fibrations and trivial fibrations are closed under composition and pullback. Any isomorphism is a fibration.*

The statements in Corollary 1.3 are axioms in Quillen's original definition of a model category [75].

Quillen defines a *cylinder object* for an object A in a closed model category $\mathcal{C}$ to be a commutative triangle

$$\begin{tikzcd} A \sqcup A \arrow[d, "i"'] \arrow[dr, "\nabla"] & \\ \tilde{A} \arrow[r, "\sigma"'] & A \end{tikzcd} \tag{1.4}$$

where $\nabla : A \sqcup A \to A$ is the canonical fold map which is defined to be the identity on A on each summand, i is a cofibration, and σ is a weak equivalence. Then a *left homotopy* of maps $f, g : A \to B$ is a commutative diagram

$$\begin{tikzcd} A \sqcup A \arrow[d, "i"'] \arrow[dr, "{(f,g)}"] & \\ \tilde{A} \arrow[r, "h"'] & B \end{tikzcd}$$

where (f, g) is the map on the disjoint union which is defined by f on one summand and g on the other, and the data consisting of

$$i = (i_0, i_1) : A \sqcup A \to \tilde{A}$$

comes from *some choice* of cylinder object for A.

There are many choices of cylinder object for a given object A of a closed model category $\mathcal{C}$: any factorization of $\nabla : A \sqcup A \to A$ into a cofibration followed by a trivial fibration that one might get out of **CM5** gives a cylinder object for A. In general, the object A needs to be cofibrant for its cylinder objects to be homotopically interesting:

Lemma 1.5.

(1) *Suppose that A is a cofibrant object of a closed model category $\mathcal{C}$, and that the diagram (1.4) is a cylinder object for A. Then the maps $i_0, i_1 : A \to \tilde{A}$ are trivial cofibrations.*

(2) *Left homotopy of maps $A \to B$ in a closed model category $\mathcal{C}$ is an equivalence relation if A is cofibrant.*

Proof: Denote the initial object of $\mathcal{C}$ by $\emptyset$.

For the first part, observe that the diagram

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & A \arrow[d, "in_R"] \\
A \arrow[r, "in_L"'] & A \sqcup A
\end{tikzcd}$$

is a pushout since cofibrations are closed under pushout by Lemma 1.1, and the unique map $\emptyset \to A$ is a cofibration by assumption. It follows that the inclusions in_L and in_R are cofibrations, so that the compositions $i_0 = (i_0, i_1) \circ in_L$ and $i_1 = (i_0, i_1) \circ in_R$ are cofibrations as well. Finally, the maps i_0 and i_1 are weak equivalences by **CM2**, since the map σ is a weak equivalence.

To prove the second statement, first observe that if $\tau : A \sqcup A \to A \sqcup A$ is the automorphism which flips summands, then the diagram

$$\begin{tikzcd}
A \sqcup A \arrow[d, "i \circ \tau"'] \arrow[dr, "\nabla"] & \\
\tilde{A} \arrow[r, "\sigma"'] & A,
\end{tikzcd}$$

which is constructed from (1.4) by twisting by τ, is a cylinder object for A. This implies that the left homotopy relation is symmetric.

Subject to the same definitions, the map $f\sigma : A \to B$ is clearly a left homotopy from $f : A \to B$ to itself, giving reflexivity.

Suppose given cylinder objects

$$\begin{tikzcd}
A \sqcup A \arrow[d, "(i_0^\epsilon, i_1^\epsilon)"'] \arrow[dr, "\nabla"] & \\
A_\epsilon \arrow[r, "\sigma^\epsilon"'] & A,
\end{tikzcd}$$

where $\epsilon = 0, 1$. Form the pushout

$$\begin{tikzcd}
A \arrow[r, "i_0^1"] \arrow[d, "i_1^0"'] & A_1 \arrow[d, "i_{1*}"] \\
A_0 \arrow[r, "i_{0*}"'] & \tilde{A}
\end{tikzcd}$$

Then the map

$$A \sqcup A \xrightarrow{(i_{0*}i_0^0, i_{1*}i_1^1)} \tilde{A}$$

is a composite

$$A \sqcup A \xrightarrow{i_0^0 \sqcup 1} A_0 \sqcup A \xrightarrow{(i_{0*}, i_{1*}i_1^1)} \tilde{A}.$$

The map $i_0^0 \sqcup 1$ is a cofibration by the first statement of the lemma, and there is a pushout diagram

$$\begin{tikzcd}
A \sqcup A \arrow[r, "i_1^0 \sqcup 1"] \arrow[d, "(i_0^1, i_1^1)"'] & A_0 \sqcup A \arrow[d, "(i_{0*}, i_{1*}i_1^1)"] \\
A_1 \arrow[r, "i_{1*}"'] & \tilde{A}.
\end{tikzcd}$$

In particular, there is a cylinder object for A

$$\begin{tikzcd}
A \sqcup A \arrow[d, "(i_{0*}i_0^0, i_{1*}i_1^1)"'] \arrow[dr, "\nabla"] & \\
A_\epsilon \arrow[r, "\sigma_*"'] & A.
\end{tikzcd}$$

It follows that if there are left homotopies $h_0 : A_0 \to B$ from f_0 to f_1 and $h_1 : A_1 \to B$ from f_1 to f_2, then there is an induced left homotopy $h_* : \tilde{A} \to B$ from f_0 to f_2. ■

A *path object* for an object B of a closed model category $\mathcal{C}$ is a commutative triangle

(1.6)
$$\begin{tikzcd} & \hat{B} \arrow[d, "p = (p_0, p_1)"] \\ B \arrow[ur, "s"] \arrow[r, "\Delta"'] & B \times B \end{tikzcd}$$

where Δ is the diagonal map, s is a weak equivalence, and p (which is given by p_0 on one factor and by p_1 on the other) is a fibration.

Once again, the factorization axiom **CM5** dictates that there is an ample supply of path objects for each object of an arbitrary closed model category. If a simplicial set X is a Kan complex, then the function complex $\mathbf{hom}(\Delta^1, X)$ is a path object for X, and the function space Y^I is a path object for each compactly generated Hausdorff space Y.

There is a notion of right homotopy which corresponds to path objects: two maps $f, g : A \to B$ are said to be *right homotopic* if there is a diagram

$$\begin{tikzcd} & \hat{B} \arrow[d, "(p_0, p_1)"] \\ A \arrow[ur, "h"] \arrow[r, "(f,g)"'] & B \times B \end{tikzcd}$$

where the map (p_0, p_1) arises from some path object (1.6), and (f, g) is the map which projects to f on the left hand factor and g on the right hand factor.

LEMMA 1.7.

(1) *Suppose that B is a fibrant object of a closed model category $\mathcal{C}$, and that $\hat{B}$ is a path object for B as in (1.6). Then the maps p_0 and p_1 are trivial fibrations.*

(2) *Right homotopy of maps $A \to B$ in $\mathcal{C}$ is an equivalence relation if B is fibrant.*

Lemma 1.7 is *dual* to Lemma 1.5 in a precise sense. If $\mathcal{C}$ is a closed model category, then its opposite $\mathcal{C}^{op}$ is a closed model category whose cofibrations (respectively fibrations) are the opposites of the fibrations (respectively cofibrations) in $\mathcal{C}$. A map in $\mathcal{C}^{op}$ is a weak equivalence for this structure if and only if its opposite is a weak equivalence in $\mathcal{C}$. Then Lemma 1.7 is an immediate consequence of the instance of Lemma 1.5 which occurs in $\mathcal{C}^{op}$. This sort of duality is ubiquitious in the theory: observe, for example, that the two statements of Corollary 1.3 are dual to each other.

Left and right homotopies are linked by the following result:

PROPOSITION 1.8. *Suppose that A is cofibrant. Suppose further that*

$$\begin{tikzcd} A \sqcup A \arrow[d, "{(i_0, i_1)}"'] \arrow[dr, "{(f,g)}"] & \\ \tilde{A} \arrow[r, "h"'] & B \end{tikzcd}$$

is a left homotopy between maps $f, g : A \to B$, and that

$$\begin{tikzcd} & \hat{B} \arrow[d, "{p = (p_0, p_1)}"] \\ B \arrow[ur, "s"] \arrow[r, "\Delta"'] & B \times B \end{tikzcd}$$

is a fixed choice of path object for B. Then there is a right homotopy

$$\begin{tikzcd} & \hat{B} \arrow[d, "{(p_0, p_1)}"] \\ A \arrow[ur, "H"] \arrow[r, "{(f,g)}"'] & B \times B. \end{tikzcd}$$

This result has a dual, which the reader should be able to formulate independently. Proposition 1.8 and its dual together imply

COROLLARY 1.9. *Suppose given maps $f, g : A \to B$, where A is cofibrant and B is fibrant. Then the following are equivalent:*

(1) *f and g are left homotopic.*
(2) *f and g are right homotopic with respect to a fixed choice of path object.*
(3) *f and g are right homotopic.*
(4) *f and g are left homotopic with respect to a fixed choice of cylinder object.*

In other words, all possible definitions of homotopy of maps $A \to B$ are the same if A is cofibrant and B is fibrant.

PROOF OF PROPOSITION 1.8: The map i_0 is a trivial cofibration since A is cofibrant, and (p_0, p_1) is a fibration, so that there is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "sf"] \arrow[d, "i_0"'] & \hat{B} \arrow[d, "{(p_0,p_1)}"] \\ \tilde{A} \arrow[ur, "K"] \arrow[r, "{(f\sigma, h)}"'] & B \times B \end{tikzcd}$$

for some choice of lifting K. Then the composite $K \circ i_1$ is the desired right homotopy. ■

We can now, unambiguously, speak of homotopy classes of maps between objects X and Y of a closed model category $\mathcal{C}$ which are both fibrant and cofibrant. We can also discuss homotopy equivalences between such objects. The classical Whitehead Theorem asserts that any weak equivalence $f : X \to Y$ of CW-complexes is a homotopy equivalence. CW-complexes are spaces which are both cofibrant and cofibrant. The analogue of this statement in an arbitrary closed model category is the following:

THEOREM 1.10 (WHITEHEAD). *Suppose that $f : X \to Y$ is a morphism of a closed model category $\mathcal{C}$ such that the objects X and Y are both fibrant and cofibrant. Suppose also that f is a weak equivalence. Then the map f is a homotopy equivalence.*

PROOF: Suppose, first of all, that f is a trivial fibration, and that

$$\begin{tikzcd} X \sqcup X \arrow[r, "\nabla"] \arrow[d, "{(i_0,i_1)}"'] & X \\ \tilde{X} \arrow[ur, "\sigma"] & \end{tikzcd}$$

is a cylinder object for X. Then one proves that f is a homotopy equivalence by finding, in succession, maps θ and h making the following diagrams commute:

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y \arrow[ur, "\theta"] \arrow[r, equal] & Y \end{tikzcd}$$

$$\begin{tikzcd}
X \sqcup X \arrow[r, "(\theta \circ f, 1)"] \arrow[d, "(i_0, i_1)"'] & X \arrow[d, "f"] \\
\tilde{X} \arrow[ur, "h"] \arrow[r, "f \circ \sigma"'] & Y.
\end{tikzcd}$$

Dually, if f is a trivial cofibration, then f is a homotopy equivalence.

Every weak equivalence $f : X \to Y$ between cofibrant and fibrant objects has a factorization

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\
& Z \arrow[ur, "p"'] &
\end{tikzcd}$$

in which i is a trivial cofibration and p is a trivial fibration. The object Z is both cofibrant and fibrant, so i and p are homotopy equivalences. ■

Suppose that X and Y are objects of a closed model category $\mathcal{C}$ which are both cofibrant and fibrant. Quillen denotes the set of homotopy classes of maps between such objects X and Y by $\pi(X, Y)$. There is a category $\pi\mathcal{C}_{cf}$ associated to any closed model $\mathcal{C}$: the objects are the cofibrant and fibrant objects of $\mathcal{C}$, and the morphisms from X to Y in $\pi\mathcal{C}_{cf}$ are the elements of the set $\pi(X, Y)$.

For each object X of $\mathcal{C}$, use **CM5** to choose, in succession, maps

$$* \xrightarrow{i_X} QX \xrightarrow{p_X} X$$

and

$$QX \xrightarrow{j_X} RQX \xrightarrow{q_X} *,$$

where i_X is a cofibration, p_X is a trivial fibration, j_X is a trivial cofibration, and q_X is a fibration. We can and will presume that π_X is the identity map if X is cofibrant, and that j_X is the identity map if QX is fibrant. Then RQX is an object which is both fibrant and cofibrant, and RQX is weakly equivalent to X, via the maps p_X and j_X.

Any map $f : X \to Y$ lifts to a map $Qf : QX \to QY$, and then Qf extends to a map $RQf : RQX \to RQY$. The map Qf is not canonically defined: it is any morphism which makes the following diagram commute:

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & QY \arrow[d, "\pi_Y"] \\ QX \arrow[r, "f \circ \pi_X"'] \arrow[ur, "Qf"] & Y \end{tikzcd}$$

Note, however, that any two liftings $f_1, f_2 : QX \to QY$ of the morphism $f \circ \pi_X$ are left homotopic.

The argument for the existence of the morphism $RQf : RQX \to RQY$ is dual to the argument for the existence of Qf. If the maps $f_1, f_2 : QX \to QY$ are liftings of $f \circ \pi_X$ and $g_i : RQX \to RQY$ is an extension of the map $j_Y \circ f_i$ for $i = 1, 2$, then f_1 is left homotopic to f_2 by what we've already seen, and so the composites $j_Y \circ f_1$ and $j_Y \circ f_2$ are right homotopic, by Lemma 1.8. Observe finally that any right homotopy between the maps $j_Y f_1, j_Y f_2 : QX \to RQY$ can be extended to a right homotopy between the maps $g_1, g_2 : RQY \to RQY$. It follows that the assignment $f \mapsto RQf$ is well defined up to homotopy.

The *homotopy category* $\mathrm{Ho}(\mathcal{C})$ associated to a closed model category $\mathcal{C}$ can be defined to have the same objects as $\mathcal{C}$, and with morphism sets defined by

$$\hom_{\mathrm{Ho}(\mathcal{C})}(X, Y) = \pi(RQX, RQY).$$

There is a functor

$$\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$$

which is the identity on objects, and sends a morphism $f : X \to Y$ to the homotopy class $[RQf]$ which is represented by any choice of map $RQf : RQX \to RQY$ defined as above. If $f : X \to Y$ is a weak equivalence of $\mathcal{C}$, then $RQf : RQX \to RQY$ is a homotopy equivalence by the Whitehead Theorem, and so $\gamma(f)$ is an isomorphism of $\mathrm{Ho}(\mathcal{C})$.

This functor γ is universal with respect to all functors $F : \mathcal{C} \to \mathcal{D}$ which invert weak equivalences:

THEOREM 1.11. *Suppose that $F : \mathcal{C} \to \mathcal{D}$ is a functor such that $F(f)$ is an isomorphism of $\mathcal{D}$ for all weak equivalences $f : X \to Y$ of $\mathcal{C}$. Then there is a unique functor $F_* : \mathrm{Ho}(\mathcal{C}) \to \mathcal{D}$ such that $F_* \circ \gamma = F$.*

PROOF: The functor $F : \mathcal{C} \to \mathcal{D}$ takes (left or right) homotopic maps of $\mathcal{C}$ to the same map of $\mathcal{D}$, since it inverts weak equivalences. It follows that, if

$g : RQX \to RQY$ represents a morphism from X to Y in $\mathrm{Ho}(\mathcal{C})$, one can specify a well-defined morphism $F_*([g])$ of $\mathcal{D}$ by the assignment

$$F_*([g]) = F(\pi_Y)F(j_Y)^{-1}F(g)F(j_X)F(\pi_X)^{-1}. \tag{1.12}$$

This assignment plainly defines a functor $F_* : \mathrm{Ho}(\mathcal{C}) \to \mathcal{D}$ such that $F_*\gamma = F$.

Also, the morphisms $\gamma(\pi_X)$ and $\gamma(j_X)$ are both represented by the identity map on RQX, and so the composite

$$\gamma(\pi_Y)\gamma(j_Y)^{-1}\gamma(g)\gamma(j_X)\gamma(j_X)^{-1}$$

coincides with the morphism $[g] : X \to Y$ of $\mathrm{Ho}(\mathcal{C})$. The morphism $F_*([g])$ must therefore have the form indicated in (1.12) if the composite functor $F_*\gamma$ is to coincide with F. ∎

Remark 1.13. One can always formally invert a class Σ of morphisms of a category $\mathcal{C}$ to get a functor $\gamma : \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ which is initial among functors $F : \mathcal{C} \to \mathcal{D}$ which invert all members of the class of morphisms Σ (see Schubert's book [82]), provided that one is willing to construct $\mathcal{C}[\Sigma^{-1}]$ in some higher set theoretic universe. This means that the morphism "things" $\hom_{\mathcal{C}[\Sigma^{-1}]}(X, Y)$ of $\mathcal{C}[\Sigma^{-1}]$ may no longer be sets. The whole point of Theorem 1.11 is that we have found an explicit way to formally invert the class WE of weak equivalences of a closed model category $\mathcal{C}$ to obtain the category $\mathrm{Ho}(\mathcal{C})$ without invoking a higher universe. After the fact, all models of $\mathcal{C}[WE^{-1}]$ must be isomorphic as categories to $\mathrm{Ho}(\mathcal{C})$ on account of the universal property of the functor $\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$, so that all possible constructions have small hom sets.

Let $\pi\mathcal{C}_{cf}$ denote the category whose objects are the cofibrant fibrant objects of the closed model category $\mathcal{C}$, and whose sets of morphisms have the form

$$\hom_{\pi\mathcal{C}_{cf}}(X, Y) = \pi(X, Y).$$

It's obvious that the functor $\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ induces a fully faithful imbedding

$$\gamma_* : \pi\mathcal{C}_{cf} \to \mathrm{Ho}(\mathcal{C}),$$

and that every object of $\mathrm{Ho}(\mathcal{C})$ is isomorphic to an object which is in the image of the functor γ_*. In other words the category $\pi\mathcal{C}_{cf}$ of homotopy classes of maps between cofibrant fibrant objects of $\mathcal{C}$ is equivalent to the homotopy category $\mathrm{Ho}(\mathcal{C})$.

This observation specializes to several well-known phenomena. In particular, the category of homotopy classes of maps between CW-complexes is equivalent to the full homotopy category of topological spaces, and the homotopy category of simplicial sets is equivalent to the category of simplicial homotopy classes of maps between Kan complexes.

2. Simplicial categories.

A simplicial model category is, roughly speaking, a closed model category equipped with a notion of a mapping space between any two objects. This has to be done in such a way that it makes homotopy theoretic sense. Thus, besides the new structure, there is an additional axiom, which is called Axiom **SM7** (See 2.1 below).

The initial property one wants is the following: let $\mathbf{S}$ be the category of simplicial sets and let $\mathcal{C}$ be a model category, and suppose $A \in \mathcal{C}$ is cofibrant and $X \in \mathcal{C}$ is fibrant. Then, the space of maps in $\mathcal{C}$ should be a functor to simplicial sets

$$\mathbf{Hom}_{\mathcal{C}} : \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{S}$$

with the property that

$$\pi_0 \mathbf{Hom}_{\mathcal{C}}(A, X) \cong [A, X]_{\mathcal{C}} \ .$$

In addition, one would want to interpret $\pi_n \mathbf{Hom}_{\mathcal{C}}(A, X)$ in $\mathcal{C}$.

There are other desirable properties; for example, if A is cofibrant and $X \to Y$ a fibration in $\mathcal{C}$, one would want

$$\mathbf{Hom}_{\mathcal{C}}(A, X) \to \mathbf{Hom}_{\mathcal{C}}(A, Y)$$

to be a fibration of spaces – that is, of simplicial sets.

Before imposing the closed model category structure on $\mathcal{C}$, let us make the following definition:

Definition 2.1. *A category $\mathcal{C}$ is a simplicial category if there is a mapping space functor*

$$\mathbf{Hom}_{\mathcal{C}}(\cdot, \cdot) : \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{S}$$

with the properties that for A and B objects in $\mathcal{C}$

(1) $\mathbf{Hom}_{\mathcal{C}}(A, B)_0 = \hom_{\mathcal{C}}(A, B)$;
(2) *the functor $\mathbf{Hom}_{\mathcal{C}}(A, \cdot) : \mathcal{C} \to \mathbf{S}$ has a left adjoint*

$$A \otimes \cdot : \mathbf{S} \to \mathcal{C}$$

which is associative in the sense that there is a isomorphism

$$A \otimes (K \times L) \cong (A \otimes K) \otimes L$$

natural in $A \in \mathcal{C}$ and $K, L \in \mathbf{S}$;
(3) *The functor $\mathbf{Hom}_{\mathcal{C}}(\cdot, B) : \mathcal{C}^{op} \to \mathbf{S}$ has left adjoint*

$$\mathbf{hom}_{\mathcal{C}}(\cdot, B) : \mathbf{S} \to \mathcal{C}^{op} \ .$$

Of course, the adjoint relationship in (3) is phrased

$$\hom_{\mathbf{S}}(K, \mathbf{Hom}_{\mathcal{C}}(A,B)) \cong \hom_{\mathcal{C}}(A, \mathbf{hom}_{\mathcal{C}}(K,B)) \ .$$

Warning: The tensor product notation goes back to Quillen, and remains for lack of a better operator. But be aware that in this context we do not usually have a tensor product in the sense of algebra; that is, we don't have a pairing arising out of bilinear maps. Instead (see the next lemma), we have an adjoint to an internal hom functor, and this is the sole justification for the notation.

Some immediate implications of the definition are the following.

LEMMA 2.2.

(1) *For fixed* $K \in \mathbf{S}$, *there are adjoint functors*

$$\cdot \otimes K : \mathcal{C} \to \mathcal{C}$$

and

$$\mathbf{hom}_{\mathcal{C}}(K, \cdot) : \mathcal{C} \to \mathcal{C} \ .$$

(2) *For all* K *and* L *in* $\mathbf{S}$ *and* B *in* $\mathcal{C}$ *there is a natural isomorphism*

$$\mathbf{hom}_{\mathcal{C}}(K \times L, B) \cong \mathbf{hom}_{\mathcal{C}}(K, \mathbf{hom}_{\mathcal{C}}(L,B)).$$

(3) *For all* $n \geq 0$, $\mathbf{Hom}_{\mathcal{C}}(A,B)_n \cong \hom_{\mathcal{C}}(A \otimes \Delta^n, B)$.

PROOF: For fixed $K \in \mathbf{S}$ and $A \in \mathcal{C}$, the object $A \otimes K$ represents the functor

$$\hom_{\mathbf{S}}(K, \mathbf{Hom}_{\mathcal{C}}(A, \cdot)) : \mathcal{C} \to \text{Sets}.$$

A map $A \xrightarrow{f} A'$ yields a natural transformation on representing objects

$$f \otimes 1 : A \otimes K \to A' \otimes K$$

and we obtain the functor $\cdot \otimes K$. The functor $\mathbf{hom}_{\mathcal{C}}(K, \cdot)$ is obtained similarly. The adjointness follows from 2.1.2 and 2.1.3. This proves part 1. Part 2 then follows from the associativity built into 2.1.2. Part 3 follows from 2.1.2 and the fact that $\hom_{\mathbf{S}}(\Delta^n, X) \cong X_n$. ∎

REMARK: A consequence of Lemma 2.2.1 is that there is a composition pairing of simplicial sets

$$\mathbf{Hom}_{\mathcal{C}}(A,B) \times \mathbf{Hom}_{\mathcal{C}}(B,C) \to \mathbf{Hom}_{\mathcal{C}}(A,C)$$

defined as follows. If $f : A \otimes \Delta^n \to B$ is an n-simplex of $\mathbf{Hom}_{\mathcal{C}}(A, B)$ and $g : B \otimes \Delta^n \to C$ is an n-simplex of $\mathbf{Hom}_{\mathcal{C}}(B, C)$ then their pairing in $\mathbf{Hom}_{\mathcal{C}}(A, C)$ is the composition

$$A \otimes \Delta^n \xrightarrow{1 \otimes d} A \otimes (\Delta^n \times \Delta^n) \cong A \otimes \Delta^n \otimes \Delta^n \xrightarrow{f \otimes 1} B \otimes \Delta^n \xrightarrow{g} C.$$

Here $d : \Delta^n \to \Delta^n \times \Delta^n$ is the diagonal. This pairing is associative in the obvious sense and reduces to the composition pairing in $\mathcal{C}$ in simplicial degree zero. It is also unital in the sense that if $* \to \mathbf{Hom}_{\mathcal{C}}(A, A)$ is the vertex corresponding to the identity, then the following diagram commutes

$$\begin{tikzcd}
* \times \mathbf{Hom}_{\mathcal{C}}(A, B) \arrow[d] \arrow[dr, "="] & \\
\mathbf{Hom}_{\mathcal{C}}(A, A) \times \mathbf{Hom}_{\mathcal{C}}(A, B) \arrow[r] & \mathbf{Hom}_{\mathcal{C}}(A, B).
\end{tikzcd}$$

There is also a diagram using the identity of B. A shorthand way of encoding all this structure is to say that $\mathcal{C}$ is enriched over simplicial sets. ■

Another immediate consequence of the definition is the following result.

Lemma 2.3. *For a simplicial category $\mathcal{C}$ then the following extended adjointness isomorphisms hold:*

(1) $\mathbf{Hom}_{\mathbf{S}}(K, \mathbf{Hom}_{\mathcal{C}}(A, B)) \cong \mathbf{Hom}_{\mathcal{C}}(A \otimes K, B)$.
(2) $\mathbf{Hom}_{\mathbf{S}}(K, \mathbf{Hom}_{\mathcal{C}}(A, B)) \cong \mathbf{Hom}_{\mathcal{C}}(A, \mathbf{hom}(K, B))$.

Proof: This is an easy exercise using Lemma 2.2. ■

Note that, in fact, Definition 2.1 implies that there are functors

$$\cdot \otimes \cdot : \mathcal{C} \times \mathbf{S} \to \mathcal{C}$$

and

$$\mathbf{hom}_{\mathcal{C}}(\cdot, \cdot) : \mathbf{S}^{op} \times \mathcal{C} \to \mathcal{C}$$

satisfying 2.1.1, 2.1.2 and 2.2.1. In order to produce examples of simplicial categories, we note the following:

Lemma 2.4. *Let $\mathcal{C}$ be a category equipped with a functor*

$$\cdot \otimes \cdot : \mathcal{C} \times \mathbf{S} \to \mathcal{C}\ .$$

Suppose the following three conditions hold:

(1) *For fixed $K \in \mathbf{S}$, $\cdot \otimes K : \mathcal{C} \to \mathcal{C}$ has a right adjoint $\mathbf{hom}_{\mathcal{C}}(K, \cdot)$.*
(2) *For fixed A, the functor $A \otimes \cdot : \mathbf{S} \to \mathcal{C}$ commutes with arbitrary colimits and $A \otimes * \cong A$.*
(3) *There is an isomorphism $A \otimes (K \times L) \cong (A \otimes K) \otimes L$ natural is $A \in \mathcal{C}$ and $K, L \in \mathbf{S}$.*

Then $\mathcal{C}$ is a simplicial category with $\mathbf{Hom}_{\mathcal{C}}(A, B)$ defined by:

$$\mathbf{Hom}_{\mathcal{C}}(A, B)_n = \hom_{\mathcal{C}}(A \otimes \Delta^n, B)$$

Proof: We first prove 2.1.2 holds. If $K \in \mathbf{S}$, write K as the coequalizer in a diagram

$$\bigsqcup_q \Delta^{n_q} \rightrightarrows \bigsqcup_p \Delta^{n_p} \to K\ .$$

Then there is a coequalizer diagram

$$\bigsqcup_q A \otimes \Delta^{n_q} \rightrightarrows \bigsqcup_p A \otimes \Delta^{n_p} \to A \otimes K$$

Hence there is an equalizer diagram

$$\hom_{\mathcal{C}}(A \otimes K, B) \to \hom_{\mathcal{C}}(A \otimes (\bigsqcup_p \Delta^{n_p}), B) \rightrightarrows \hom_{\mathcal{C}}(A \otimes (\bigsqcup_q \Delta^{n_q}), B)\ .$$

This, in turn, is equivalent to the assertion that the equalizer of the maps

$$\hom_{\mathbf{S}}(\bigsqcup_p \Delta^{n_p}, \mathbf{hom}_{\mathcal{C}}(A, B)) \rightrightarrows \hom_{\mathbf{S}}(\bigsqcup_q \Delta^{n_q}, \mathbf{hom}_{\mathcal{C}}(A, B))$$

is the induced map

$$\hom_{\mathbf{S}}(K, \mathbf{hom}_{\mathcal{C}}(A, B)) \to \hom_{\mathbf{S}}(\bigsqcup_p \Delta^{n_p}, \mathbf{hom}_{\mathcal{C}}(A, B))$$

so 2.1.2 holds. If we let $\hom_{\mathcal{C}}(K, \cdot)$ be adjoint to $\cdot \otimes K$, as guaranteed by the hypotheses, 2.1.3 holds. Then finally, 2.1.1 is a consequence of the fact that $A \otimes * = A$. ■

We now give some examples. Needless to say $\mathbf{S}$ itself is a simplicial category with, for $A, B, K \in \mathbf{S}$

$$A \otimes K = A \times K$$

and (a tautology)

$$\mathbf{Hom}_{\mathbf{S}}(A, B) = \mathbf{Hom}_{\mathbf{S}}(A, B)$$

and

$$\mathbf{hom}_{\mathbf{S}}(K, B) = \mathbf{Hom}_{\mathbf{S}}(K, B) \ .$$

Only slightly less obvious is the following: let $\mathbf{S}_*$ denote the category of pointed (i.e., based) simplicial sets. Then $\mathbf{S}_*$ is a simplicial category with

$$A \otimes K = A \wedge K_+ = A \times K / * \times K$$

where $(\ )_+$ denote adding a disjoint basepoint

$$\mathbf{Hom}_{\mathbf{S}_*}(A, B)_n = \hom_{\mathbf{S}_*}(A \wedge \Delta^n_+, B)$$

and

$$\mathbf{hom}_{\mathbf{S}_*}(K, B) = \mathbf{Hom}_{\mathbf{S}}(K, B)$$

with basepoint given by the constant map

$$K \to * \to B.$$

Note that $\mathbf{Hom}_{\mathbf{S}_*}(A, B) \in \mathbf{S}$, but $\mathbf{hom}_{\mathbf{S}_*}(K, B) \in \mathbf{S}_*$.

This example can be radically generalized. Suppose $\mathcal{C}$ is a category that is co-complete; that is, $\mathcal{C}$ has all colimits. Let $s\mathcal{C}$ denote the simplicial objects in $\mathcal{C}$. Then if $K \in \mathbf{S}$, we may define, for $A \in s\mathcal{C}$, an object $A \otimes K \in s\mathcal{C}$ by

$$(A \otimes K)_n = \bigsqcup_{k \in K_n} A_n$$

where $\bigsqcup$ denotes the coproduct in $\mathcal{C}$, and if $\phi : \mathbf{n} \to \mathbf{m}$ is an ordinal number map $\phi^* : (A \otimes K)_m \to (A \otimes K)_n$ is given by

$$\bigsqcup_{k \in K_m} A_m \xrightarrow{\bigsqcup \phi^*} \bigsqcup_{k \in K_m} A_n \to \bigsqcup_{k \in K_n} A_n \ .$$

The first map is induced by $\phi^* : A_m \to A_n$, the second by $\phi^* : K_m \to K_n$.

Theorem 2.5. *Suppose that $\mathcal{C}$ is complete and complete. Then with this functor $\cdot \otimes \cdot : s\mathcal{C} \times \mathbf{S} \to s\mathcal{C}$, $s\mathcal{C}$ becomes a simplicial category with*

$$\mathbf{Hom}_{s\mathcal{C}}(A, B)_n = \hom_{s\mathcal{C}}(A \otimes \Delta^n, B) \ .$$

Proof: This is an application of Lemma 2.3. First note that it follows from the construction that there is a natural isomorphism

$$A \otimes (K \times L) \cong (A \otimes K) \otimes L.$$

And one has $A \otimes * \cong A$. Thus, we need only show that, for fixed $K \in \mathbf{S}$, the functor $\cdot \otimes K : \mathcal{C} \to \mathcal{C}$ has a right adjoint. To show this, one changes focus slightly. For $Y \in s\mathcal{C}$, define a functor

$$F_Y : \mathcal{C}^{op} \to \mathbf{S}$$

by

$$F_Y(A) = \hom_{\mathcal{C}}(A, Y) \ .$$

Then the functor $\mathcal{C}^{op} \to$ Sets given by

$$A \mapsto \mathbf{Hom}_{\mathbf{S}}(K, F_Y(A))_n = \hom_{\mathbf{S}}(K \times \Delta^n, \hom_{\mathcal{C}}(A, Y))$$

is representable. To see this, write $K \times \Delta^n$ as a coequalizer

$$\bigsqcup_q \Delta^{n_q} \rightrightarrows \bigsqcup_p \Delta^{n_p} \to K \times \Delta^n$$

then the representing object is defined by the equalizer diagram

$$\prod_q Y_{n_q} \leftleftarrows \prod_q Y_{m_q} \leftarrow \mathbf{hom}_{s\mathcal{C}}(K, Y)_n \ .$$

Letting the ordinal number vary yields an object $\mathbf{hom}_{s\mathcal{C}}(K, Y)$ and a natural isomorphism of simplicial sets

$$\hom_{\mathcal{C}}(A, \mathbf{hom}_{s\mathcal{C}}(K, Y)) \cong \mathbf{Hom}_{\mathbf{S}}(K, \hom_{\mathcal{C}}(A, Y)) \ , \tag{2.6}$$

or a natural equivalence of functors

$$F_{\mathbf{hom}_{s\mathcal{C}}(K,Y)}(\cdot) \cong \mathbf{Hom}_{\mathbf{S}}(K, F_Y(\cdot)) \ .$$

Now the morphisms $X \to Y$ in $s\mathcal{C}$ are in one-to-one correspondence with the natural transformations $F_X \to F_Y$, by the Yoneda lemma. In formulas, this reads

$$\hom_{s\mathcal{C}}(X, Y) \cong \mathrm{Nat}(F_X, F_Y) \ .$$

Now if $K \in \mathbf{S}$ and $X \in s\mathcal{C}$ we can define a new functor

$$F_X \otimes K : \mathcal{C}^{op} \to \mathbf{S}$$

by

$$(F_X \otimes K)(A) = F_X(A) \times K$$

we will argue below that

$$\mathrm{Nat}(F_{X \otimes K}, F_Y) \cong \mathrm{Nat}(F_X \otimes K, F_Y) \ .$$

Assuming this one has:

$$\begin{aligned} \hom_{s\mathcal{C}}(X, \mathbf{hom}_{s\mathcal{C}}(K, Y)) &\cong \mathrm{Nat}(F_X, F_{\mathbf{hom}_{s\mathcal{C}}(K,Y)}) \\ &\cong \mathrm{Nat}(F_X, \mathbf{Hom}_{\mathbf{S}}(K, F_Y)) \end{aligned}$$

by (2.6). Continuing, one has that this is isomorphic to

$$\mathrm{Nat}(F_X \otimes K, F_Y) \cong \mathrm{Nat}(F_{X \otimes K}, F_Y) \cong \hom_{s\mathcal{C}}(X \otimes K, Y)$$

so that

$$\hom_{s\mathcal{C}}(X, \mathbf{hom}_{s\mathcal{C}}(K, Y)) \cong \hom_{s\mathcal{C}}(X \otimes K, Y) \ .$$

as required. Thus we are left with

LEMMA 2.7. *There is an isomorphism*

$$\mathrm{Nat}(F_{X \otimes K}, F_Y) \cong \mathrm{Nat}(F_X \otimes K, F_Y) \ .$$

PROOF: It is easiest to show

$$\mathrm{Nat}(F_X \otimes K, F) \cong \hom_{s\mathcal{C}}(X \otimes K, Y) \ .$$

Given a natural transformation

$$\Phi : F_X \otimes K \to F_Y$$

note that

$$(F_X \otimes K)(X_n)_n = \prod_{k \in K_n} \hom_{\mathcal{C}}(X_n, X_n)\ .$$

Thus, for each $k \in K_n$, there is a map

$$\Phi(1)_k : X_n \to Y_n$$

corresponding to the identity in the factor corresponding to k. These assemble into a map

$$f_n : (X \otimes K)_n = \coprod_{k \in K_n} X_n \to Y_n\ .$$

We leave it to the reader to verify that yields a morphism

$$f : X \otimes K \to Y$$

of simplicial objects, and that the assignment $\Phi \to f$ yields the desired isomorphism. ■

EXAMPLES 2.8. One can now assemble a long list of simplicial categories: We note in particular

1) Let $\mathcal{C}$ be one of the following “algebraic” categories: groups, abelian groups, rings, commutative rings, modules over a ring R, algebras or commutative algebras over a commutative ring R, or Lie algebras. Then $s\mathcal{C}$ is a simplicial category.

2) Let $\mathcal{C}$ be the graded analog of one of the categories in the previous example. Then $s\mathcal{C}$ is a simplicial category. One might wish to examine the category of graded commutative algebras over a commutative ring R, for example.

3) Let $\mathcal{C} = \mathcal{CA}$ be the category of coalgebras over a field $\mathbb{F}$. Then $s\mathcal{CA}$ is a simplicial category.

4) Note that the hypotheses of $\mathcal{C}$ used Theorem 2.5 apply equally to $\mathcal{C}^{op}$. Thus $s(\mathcal{C}^{op})$ is also a simplicial category. But if $s(\mathcal{C}^{op})$ is a simplicial category, so is $(s(\mathcal{C})^{op})^{op}$. But this is the category $c\mathcal{C}$ of cosimplicial objects in $\mathcal{C}$. One must interpret the functors $\cdot \otimes \cdot$, $\hom_{c\mathcal{C}}(\cdot,\cdot)$, etc. in light of Theorem 2.5. Thus if $K \in \mathbf{S}$,

$$\mathbf{hom}_{c\mathcal{C}}(K, A)^n = \prod_{k \in K_n} A^n$$

and

$$\mathbf{Hom}_{c\mathcal{C}}(A, B)_n = \hom_{c\mathcal{C}}(A, \mathbf{hom}_{c\mathcal{C}}(\Delta^n, B))$$

and $A \otimes K$ is defined via Theorem 2.5.

To conclude this section, we turn to the following question: suppose given simplicial categories $\mathcal{C}$ and $\mathcal{D}$ and a functor $G : \mathcal{D} \to \mathcal{C}$ with left adjoint F. We want a criterion under which the simplicial structure is preserved.

LEMMA 2.9. *Suppose that for all $K \in \mathbf{S}$ and $A \in \mathcal{C}$ there is a natural isomorphism $F(A \otimes K) \cong F(A) \otimes K$. Then*

(1) *the adjunction extends to a natural isomorphism*

$$\mathbf{Hom}_{\mathcal{D}}(FA, B) \cong \mathbf{Hom}_{\mathcal{C}}(A, GB);$$

(2) *for all $K \in \mathbf{S}$ and $B \in \mathcal{D}$, there is a natural isomorphism*

$$G\mathbf{hom}_{\mathcal{D}}(K, B) \cong \mathbf{hom}_{\mathcal{C}}(K, GB) \ .$$

PROOF: Part (1) uses that $\mathbf{Hom}_{\mathcal{D}}(FA, B)_n \cong \hom_{\mathcal{D}}(FA \otimes \Delta^n, B)$. Part (2) is an exercise in adjunctions. ■

We give some examples.

EXAMPLES 2.10.

1) Let $G : \mathcal{D} \to \mathcal{C}$ have a left adjoint F. Extend this to a pair of adjoint functors by prolongation:

$$G : s\mathcal{D} \to s\mathcal{C}$$

with adjoint F. Thus $G(X)_n = G(X)_n$, and so on. Then, in the simplicial structure of Theorem 2.5, $F(X \otimes K) \cong F(X) \otimes K$, since F commutes with colimits.

2) Let $\mathcal{C}$ be an arbitrary simplicial category and $A \in \mathcal{C}$. Define

$$G : \mathcal{C} \to \mathbf{S}$$

by $G(B) = \mathbf{Hom}_{\mathcal{C}}(A, B)$. Then $F(X) = A \otimes X$ and the requirement on 2.9 is simply the formula

$$A \otimes (X \times K) \cong (A \otimes X) \otimes K \ .$$

3. Simplicial model categories.

If a category $\mathcal{C}$ is at once a simplicial category and a closed model category, we would like the mapping space functor to have homotopy theoretic content. This is accomplished by imposing the following axiom.

3.1 AXIOM **SM7**. *Let $\mathcal{C}$ be a closed model category and a simplicial category. Suppose $j : A \to B$ is a cofibration and $q : X \to Y$ is a fibration. Then*

$$\mathbf{Hom}_{\mathcal{C}}(B, X) \xrightarrow{(j^*, q_*)} \mathbf{Hom}_{\mathcal{C}}(A, X) \times_{\mathbf{Hom}_{\mathcal{C}}(A,Y)} \mathbf{Hom}_{\mathcal{C}}(B, Y)$$

is a fibration of simplicial sets, which is trivial if j or q is trivial.

A category satisfying this axiom will be called a simplicial model category. The next few sections will be devoted to producing a variety of examples, but in this section we will explore the consequences of this axiom.

PROPOSITION 3.2. *Let $\mathcal{C}$ be a simplicial model category and $q : X \to Y$ a fibration. Then if B is cofibrant*

$$q_* : \mathbf{Hom}_{\mathcal{C}}(B, X) \to \mathbf{Hom}_{\mathcal{C}}(B, Y)$$

is a fibration in $\mathbf{S}$. Similarly, if $j : A \to B$ is a cofibration and X is fibrant, then

$$j^* : \mathbf{Hom}_{\mathcal{C}}(B, X) \to \mathbf{Hom}_{\mathcal{C}}(A, X)$$

is a fibration.

PROOF: One sets A to be the initial object and Y to be the final object, respectively, in Axiom **SM7**. ∎

In other words, $\mathbf{Hom}_{\mathcal{C}}(\cdot, \cdot)$ has entirely familiar homotopical behavior. This is one way to regard this axiom. Another is that **SM7** is a considerable strengthening of the lifting axiom **CM4** of a closed model category.

PROPOSITION 3.3. *Axiom **SM7** implies axiom **CM4**; that is, given a lifting problem in simplicial category $\mathcal{C}$ satisfying **SM7***

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & X \arrow[d, "q"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

with j a cofibration and q a fibration, then the dotted arrow exists if either j or q is trivial.

Proof: Such a square is a zero-simplex in

$$\mathbf{Hom}_{\mathcal{C}}(A,X) \times_{\mathbf{Hom}_{\mathcal{C}}(A,Y)} \mathbf{Hom}_{\mathcal{C}}(B,Y)$$

and a lifting is a pre-image in the zero simplices of $\mathbf{Hom}_{\mathcal{C}}(B,X)$. Since trivial fibrations are surjective, the result follows. ■

But more is true: Axiom **SM7** implies that the lifting built in **CM4** is unique up to homotopy. To explain that, however, requires a few words about homotopy. First we record

Proposition 3.4. *Let $\mathcal{C}$ be a simplicial model category and $j : K \to L$ a cofibration of simplicial sets. If $A \in \mathcal{C}$ is cofibrant, then*

$$1 \otimes j : A \otimes K \to A \otimes L$$

is a cofibration in $\mathcal{C}$. If $X \in \mathcal{C}$ is fibrant

$$j^* : \mathbf{hom}_{\mathcal{C}}(L,X) \to \mathbf{hom}_{\mathcal{C}}(K,X)$$

is a fibration. If j is trivial, then so are $1 \otimes j$ and j^*.

Proof: For example, one needs to show $1 \otimes j$ has the left lifting property with respect to all trivial fibrations $q : X \to Y$ in $\mathcal{C}$. This is equivalent, by adjointness, to show j has the left lifting property with respect to

$$q_* : \mathbf{Hom}_{\mathcal{C}}(A,X) \to \mathbf{Hom}_{\mathcal{C}}(A,Y)$$

for all trivial fibrations q. But q_* is a trivial fibration of simplicial sets by **SM7**. The other three claims are proved similarly. ■

Recall the definitions of left and right homotopy from the section 1. The following implies that if A is cofibrant, then $A \otimes \Delta^1$ is a model for the the cylinder on A.

Lemma 3.5. *Let $\mathcal{C}$ be a simplicial model category and let $A \in \mathcal{C}$ be cofibrant. Then if $q : \Delta^1 \to *$ is the unique map*

$$1 \otimes q : A \otimes \Delta^1 \to A \otimes * \cong A$$

is a weak equivalence. Furthermore,

$$d_1 \sqcup d_0 : A \sqcup A \to A \otimes \Delta^1$$

is a cofibration and the composite

$$A \sqcup A \xrightarrow{d_0 \sqcup d_1} A \otimes \Delta^1 \xrightarrow{1 \otimes q} A$$

is the fold map.

Proof: The first claim follows from Proposition 3.4, since

$$d_1 : A \cong A \otimes \Delta^0 \to A \otimes \Delta^1$$

is a weak equivalence. The second claim follows from 3.4 also since $d_1 \bigsqcup d_0$ is equivalent to

$$1 \otimes j : A \otimes \partial\Delta^1 \to A \otimes \Delta^1$$

where $j : \partial\Delta \to \Delta^1$ is inclusion of the boundary. For the third claim one checks that $(1 \otimes q) \circ d_1 = (1 \otimes q) \circ d_0 = 1$. ■

Thus, if $\mathcal{C}$ is a simplicial model category and $A \in \mathcal{C}$ is cofibrant, then two morphisms $f, g : A \to X$ are homotopic if and only if there is a factoring

(3.6)
$$\begin{tikzcd} A \sqcup A \arrow[r, "d_1 \sqcup d_0"] \arrow[d, "f \sqcup g"'] & A \otimes \Delta^1 \arrow[dl, "H"] \\ X & \end{tikzcd}$$

This, too, is no surprise. As a further exercise, note that if one prefers right homotopy for a particular application, one could require a factoring

(3.7)
$$\begin{tikzcd} & \mathbf{hom}_{\mathcal{C}}(\Delta[1], X) \arrow[d, "j^*"] \\ A \arrow[ur, "G"] \arrow[r, "f \times g"'] & X \times X \cong \mathbf{hom}_{\mathcal{C}}(\partial\Delta[1], X) \end{tikzcd}$$

In using this formulation, one wants X to be fibrant so that j^* is a fibration.

To formulate the next notion, let A be cofibrant and $j : A \to B$ a cofibration. Given two maps $f, g : B \to X$ so that $j \circ f = j \circ g$, we say f and g are *homotopic under* A if there is a homotopy

$$H : B \otimes \Delta^1 \to X$$

so that $H \circ (j \otimes 1) : A \otimes \Delta^1 \to X$ is the constant homotopy on $j \circ f$. That is, $h \otimes (j \otimes 1)$ is the composite

$$A \otimes \Delta^1 \xrightarrow{1 \otimes q} A \otimes * \cong A \xrightarrow{j \circ f} X$$

where $q : \Delta^1 \to *$ is the unique map. There is an obvious dual motion of homotopic over Y.

The following result says that in a simplicial model category, the liftings required by axiom **CM4** are unique in a strong way.

PROPOSITION 3.8. *Let $\mathcal{C}$ be a simplicial model category and A a cofibrant object. Consider a commutative square*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & X \arrow[d, "q"] \\ B \arrow[r] & Y \end{tikzcd}$$

where j is a cofibration, q is a fibration and one of j or q is trivial. Then any two solutions $f, g : B \to X$ of the lifting problem are homotopic under A and over Y.

PROOF: The commutative square is a zero-simplex α in

$$\mathbf{Hom}_{\mathcal{C}}(A, X) \times_{\mathbf{Hom}_{\mathcal{C}}(A,Y)} \mathbf{Hom}_{\mathcal{C}}(B, Y)$$

Let $s_0\alpha$ be the corresponding degenerate 1-simplex. Then $s_0\alpha$ is the commutative square

$$\begin{tikzcd} A \otimes \Delta^1 \arrow[r] \arrow[d, "j \otimes 1"'] & X \arrow[d, "q"] \\ B \otimes \Delta^1 \arrow[r] & Y \end{tikzcd}$$

where the horizontal maps are the constant homotopies. Let

$$f, g \in \mathbf{Hom}_{\mathcal{C}}(B, X)_0 = \hom_{\mathcal{C}}(B, X)$$

be two solutions to the lifting problem. Thus $(i^*, q_*)f = (i^*, q_*)g = \alpha$. Then by **SM7**, there is a 1-simplex

$$\beta \in \mathbf{Hom}_{\mathcal{C}}(B, X)_1 \cong \hom_{\mathcal{C}}(B \otimes \Delta^1, X)$$

so that $d_1\beta = f$, $d_0\beta = g$, and $(i^*, q_*)\beta = s_0\alpha$. Then

$$\beta : B \otimes \Delta^1 \to X$$

is the required homotopy. ■

We now restate a concept from Section 1. For a simplicial model category $\mathcal{C}$, we define the homotopy category $Ho(\mathcal{C})$ as follows: the objects are the objects of $\mathcal{C}$ and the morphisms are defined by

$$[A, X]_{\mathcal{C}} = \hom_{\mathcal{C}}(B, Y)/\sim \tag{3.9}$$

where $q : B \to A$ is a trivial fibration with B cofibrant, $i : X \to Y$ is a trivial cofibration with Y fibrant and $f \sim g$ if and only if f is homotopic to g. It is an exercise that $\sim$ is in fact an equivalence relation, and it is a consequence of Proposition 3.8 that $[A, X]_{\mathcal{C}}$ is well-defined.

There is some ambiguity in the notation: $[A, X]_{\mathcal{C}}$ depends not only on $\mathcal{C}$, but on the particular closed model category structure.

In the sequel, $[\ ,\ ]$ means $[\ ,\ ]_{\mathbf{S}}$.

The following result gives homotopy theoretic content to the functors $\cdot \otimes K$ and $\mathbf{hom}_{\mathcal{C}}(K, \cdot)$.

Proposition 3.10. *Let $\mathcal{C}$ be a simplicial model category and A and B a cofibrant and a fibrant object of $\mathcal{C}$, respectively. Then*

$$[K, \mathbf{Hom}_{\mathcal{C}}(A, B)] \cong [A \otimes K, B]_{\mathcal{C}}$$

and

$$[K, \mathbf{Hom}_{\mathcal{C}}(A, B)] \cong [A, \mathbf{hom}_{\mathcal{C}}(K, B)]_{\mathcal{C}}\ .$$

Proof: Note that $\mathbf{Hom}_{\mathcal{C}}(A, B)$ is fibrant, by Proposition 3.2. Hence

$$[K, \mathbf{Hom}_{\mathcal{C}}(A, B)] = \hom_{\mathbf{S}}(K, \mathbf{Hom}_{\mathcal{C}}(A, B))/\sim$$

where $\sim$ means "homotopy" as above. But, since

$$A \otimes (K \times \Delta^1) = (A \otimes K) \otimes \Delta^1$$

we have that

$$\begin{aligned} \hom_{\mathbf{S}}(K, \mathbf{Hom}_{\mathcal{C}}(A, B))/\sim\ &\cong \hom_{\mathcal{C}}(A \otimes K, B)/\sim \\ &\cong [A \otimes K, B]_{\mathcal{C}} \end{aligned}$$

where we use Proposition 3.4 to assert that $A \otimes K$ is cofibrant. ∎

We now concern ourselves with developing a way of recognizing when **SM7** holds.

Proposition 3.11. *Let $\mathcal{C}$ be a closed model category and a simplicial category. Then the axiom* **SM7** *holds if and only if for all cofibrations $i : K \to L$ in* **S** *and cofibrations $j : A \to B$ in $\mathcal{C}$, the map*

$$(j \otimes 1) \cup (1 \otimes i) : (A \otimes L) \cup_{(A \otimes K)} (B \otimes K) \to B \otimes L\ .$$

is a cofibration which is trivial if either j or i is.

Proof: A diagram of the form

$$\begin{tikzcd}
K \arrow[r] \arrow[d] & \mathbf{Hom}_{\mathcal{C}}(B, X) \arrow[d] \\
L \arrow[r] & \mathbf{Hom}_{\mathcal{C}}(A, X) \times_{\mathbf{Hom}_{\mathcal{C}}(A,Y)} \mathbf{Hom}_{\mathcal{C}}(B, Y)
\end{tikzcd}$$

is equivalent, by adjointness, to a diagram

$$\begin{tikzcd}
(A \otimes L) \cup_{(A \otimes K)} (B \otimes K) \arrow[r] \arrow[d] & X \arrow[d] \\
B \otimes L \arrow[r] & Y\ .
\end{tikzcd}$$

The result follows by using the fact that fibrations and cofibrations are determined by various lifting properties. ■

From this we deduce

Corollary 3.12. (Axiom **SM7b**) *Let $\mathcal{C}$ be a closed model category and a simplicial category. The axiom* **SM7** *is equivalent to the requirement that for all cofibrations $j : A \to B$ in $\mathcal{C}$*

$$(A \otimes \Delta^n) \cup_{(A \otimes \partial\Delta^n)} (B \otimes \partial\Delta^n) \to B \otimes \Delta^n$$

is a cofibration (for $n \geq 0$) that is trivial if j is, and that

$$(A \otimes \Delta^1) \cup_{(A \otimes \{e\})} (B \otimes \{e\}) \to B \otimes \Delta^1$$

is the trivial cofibration for $e = 0$ or 1.

Proof: Let $i : K \to L$ be a cofibration of simplicial sets. Then, since i can be built by attaching cells to K, the first condition implies

$$(A \otimes L) \cup_{(A \otimes K)} (B \otimes K) \to (B \otimes L)$$

is a cofibration which is trivial if j is. The second condition and proposition I.4.2 (applied to B_2) yields that $(j \otimes 1) \cup (1 \otimes i)$ is trivial if i is. ■

In the usual duality that arises in these situations, we also have

Proposition 3.13. *Let $\mathcal{C}$ be a simplicial category and a model category and suppose $i : K \to L$ is a cofibration in S and $q : X \to Y$ a fibration in $\mathcal{C}$. Then the following are equivalent:*

(1) **SM7**,
(2) $\mathbf{hom}_{\mathcal{C}}(L, X) \to \mathbf{hom}_{\mathcal{C}}(K, X) \times_{\mathbf{hom}_{\mathcal{C}}(K,Y)} \mathbf{hom}_{\mathcal{C}}(L, Y)$ *is a fibration which is trivial if q or j is;*
(3) *(**SM7a**)* $\mathbf{hom}_{\mathcal{C}}(\Delta^n, X) \to \mathbf{hom}_{\mathcal{C}}(\partial\Delta^n, X) \times_{\mathbf{hom}_{\mathcal{C}}(\partial\Delta^n,Y)} \mathbf{hom}_{\mathcal{C}}(\Delta^n, Y)$ *is a fibration which is trivial if q is, and*

$$\mathbf{hom}_{\mathcal{C}}(\Delta^1, X) \to \mathbf{hom}_{\mathcal{C}}(e, X) \times_{\mathbf{hom}_{\mathcal{C}}(e,Y)} \mathbf{hom}_{\mathcal{C}}(\Delta^1, Y)$$

is a trivial fibration for $e = 0, 1$.

Example 3.14. *A simplicial model category structure on* **CGHaus** We can now show that the category **CGHaus** of compactly generated Hausdorff spaces is a simplicial model category. To supply the simplicial structure let $X \in$ **CGHaus** and $K \in \mathbf{S}$. Define

$$X \otimes K = X \times_{Ke} |K|$$

where $|\cdot|$ denotes the geometric realization and $\times_{Ke}$ the Kelly product, which is the product internal to the category **CGHaus**. Then if X and Y are in **CGHaus**, the simplicial set of maps between X and Y is given by

$$\mathbf{Hom}_{\mathbf{CGHaus}}(X, Y)_n = \hom_{\mathbf{CGHaus}}(X \times \Delta^n, Y)$$

regarded as a set. And the right adjoint to $\cdot \otimes K$ is given by

$$\mathbf{hom}_{\mathbf{CGHaus}}(K, X) = \mathbf{F}(|K|, X)$$

where $\mathbf{F}$ denotes the internal function space to **CGHaus**.

We have seen that **CGHaus** is a closed model category with the usual weak equivalences and Serre fibrations. In addition, Proposition 3.13 immediately implies that **CGHaus** is a simplicial model category.

It is worth pointing out that the realization functor $|\cdot|$ and its adjoint $S(\cdot)$ the singular set functor pass to the level of simplicial categories. Indeed, we've seen that if $X \in \mathbf{S}$ and $K \in \mathbf{S}$, then

$$|X \times K| = |X| \times_{Ke} |K|.$$

This immediately implies that if $Y \in \mathbf{CGHaus}$, then

$$S\mathbf{F}(|K|, Y) = \mathbf{Hom}_{\mathbf{S}}(K, SY)$$

and

$$\mathbf{Hom}_{\mathbf{CGHaus}}(|X|, Y) = \mathbf{Hom}_{\mathbf{S}}(X, SY).$$

4. Detecting weak equivalences.

LEMMA 4.1. *Suppose that $f : X \to Y$ is a morphism of a simplicial model category $\mathcal{C}$ which induces an isomorphism in the homotopy category* $\mathrm{Ho}(\mathcal{C})$. *Then f is a weak equivalence.*

PROOF: The map f induces an isomorphism in the homotopy category if and only if f induces a homotopy equivalence $RQX \to RQY$ of the associated cofibrant fibrant models. It suffices, therefore, to assume that X and Y are fibrant and cofibrant, and that $f : X \to Y$ is a homotopy equivalence.

Choose a factorization

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & Z \arrow[d, "p"] \\ & Y \end{tikzcd}$$

where j is a trivial cofibration and p is a fibration. The map j is a homotopy equivalence, so it suffices to show that the fibration $p : Z \to Y$ of cofibrant fibrant objects is a weak equivalence if it is a homotopy equivalence.

Choose a homotopy inverse $s : Y \to Z$ for the map p, and choose a homotopy $H : Z \otimes \Delta^1 \to Z$ from ps to the identity 1_Y. There is a diagram

$$\begin{tikzcd} Y \arrow[r, "s"] \arrow[d, "d^0"'] & Z \arrow[d, "p"] \\ Y \otimes \Delta^1 \arrow[r, "H"'] \arrow[ur, "H_*"] & Y. \end{tikzcd}$$

The composite map $H_* d^1$ is a section of p, and this composite is homotopic to s. We can therefore assume that the homotopy inverse s is a section of p.

Choose a homotopy $h : Z \otimes \Delta^1 \to Z$ from sp to 1_Z. The lifting γ exists in the diagram

$$\begin{tikzcd} \Lambda^2_0 \arrow[r, "{(\ , h, sph)}"] \arrow[d, hook] & \mathbf{Hom}(Z,Z) \arrow[d, "p_*"] \\ \Delta^2 \arrow[r, "s_1(ph)"'] \arrow[ur, "\gamma"] & \mathbf{Hom}(Z,Y) \end{tikzcd}$$

since p_* is a fibration. Write $k : Z \otimes \Delta^1 \to Z$ for the 1-simplex $d_0(\gamma)$ of $\mathbf{Hom}(X,X)$. Then k is a fibrewise homotopy from sp to 1_Z, meaning in particular that the following diagram commutes:

$$\begin{tikzcd} Z \otimes \Delta^1 \arrow[r, "k"] \arrow[d, "s^0"'] & Z \arrow[d, "p"] \\ Z \arrow[r, "p"'] & Y. \end{tikzcd}$$

Now suppose that the diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "j"'] & Z \arrow[d, "p"] \\ B \arrow[r, "\beta"'] & Y \end{tikzcd}$$

commutes, where j is a cofibration. One can form the diagram

$$\begin{tikzcd} (A \otimes \Delta^1) \cup (B \otimes \{0\}) \arrow[r, "{(k(\alpha \otimes 1), s\beta)}"] \arrow[d, "j_*"'] & Z \arrow[d, "p"] \\ B \otimes \Delta^1 \arrow[r, "s^0(\beta \otimes 1)"'] \arrow[ur, "K"] & Y \end{tikzcd}$$

since the map j_* is a trivial cofibration. Now check that the map θ defined by the composite

$$B \xrightarrow{d^0} B \otimes \Delta^1 \xrightarrow{K} Z$$

satisfies $p\theta = \beta$ and $\theta j = \alpha$. The map p therefore has the right lifting property with respect to all cofibrations, and is therefore a trivial fibration. ∎

It may seem unlikely, but Lemma 4.1 is probably the the most important technical consequence of having a simplicial model structure in hand, as opposed to just a closed model structure. It will be used many times in the sequel. Here is a sample application:

LEMMA 4.2. *Suppose that $f : A \to B$ is a map between cofibrant objects in a simplicial model category $\mathcal{C}$. Then f is a weak equivalence if and only if the induced map*

$$f^* : \mathbf{Hom}(B, Z) \to \mathbf{Hom}(A, Z)$$

is a weak equivalence of simplicial sets for each fibrant object Z of $\mathcal{C}$.

PROOF: We use the fact, which appears as Lemma II.9.4 below, that a map $f : A \to B$ between cofibrant objects in a closed model category has a factorization

$$\begin{tikzcd} & X \arrow[d, "q"] \\ A \arrow[ur, "j"] \arrow[r, "f"'] & B \end{tikzcd}$$

such that j is a cofibration and the map q is left inverse to a trivial cofibration $i : B \to X$.

If $f : A \to B$ is a weak equivalence, then the map $j : A \to X$ is a trivial cofibration, and hence induces a trivial fibration $j^* : \mathbf{Hom}(X, Z) \to \mathbf{Hom}(A, Z)$ for all fibrant objects Z. Similarly, the trivial cofibration i induces a trivial fibration i^*, so that the map $p^* : \mathbf{Hom}(B, Z) \to \mathbf{Hom}(X, Z)$ is a weak equivalence.

Suppose that the map $f^* : \mathbf{Hom}(B, Z) \to \mathbf{Hom}(A, Z)$ is a weak equivalence for all fibrant Z. To show that f is a weak equivalence, we can presume that the objects A and B are fibrant as well as cofibrant. In effect, there is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ \overline{A} \arrow[r, "\overline{f}"'] & \overline{B} \end{tikzcd}$$

in which the objects $\overline{A}$ and $\overline{B}$ are fibrant, and the vertical maps are trivial cofibrations, and then one applies the functor $\mathbf{Hom}(\ ,Z)$ for Z fibrant and invokes the previous paragraph.

Finally, suppose that A and B are fibrant as well as cofibrant, and presume that $f^* : \mathbf{Hom}(B,Z) \to \mathbf{Hom}(A,Z)$ is a weak equivalence for all fibrant Z. We can assume further that f is a cofibration, by taking a suitable factorization. The map $f^* : \mathbf{Hom}(B,A) \to \mathbf{Hom}(A,A)$ is therefore a trivial Kan fibration, and hence surjective in all degrees, so that there is a map $g : B \to A$ such that $g \cdot f = 1_A$. The maps $f \cdot g$ and 1_B are both preimages of the vertex f under the trivial fibration $f^* : \mathbf{Hom}(B,B) \to \mathbf{Hom}(A,B)$, so that there is a homotopy $f \cdot g \simeq 1_B$. In particular, f is a homotopy equivalence and therefore a weak equivalence, by Lemma 4.1. ∎

5. The existence of simplicial model category structures.

Here we concern ourselves with the following problem: Let $\mathcal{C}$ be a category and $s\mathcal{C}$ the category of simplicial objects over $\mathcal{C}$. Then, does $s\mathcal{C}$ have the structure of a simplicial model category? We will assume that there is a functor $G : s\mathcal{C} \to \mathbf{S}$ with a left adjoint

$$F : \mathbf{S} \to s\mathcal{C}\ .$$

Examples include algebraic categories such as the categories of groups, abelian groups, algebras over some ring R, commutative algebras, lie algebras, and so on. In these cases, G is a forgetful functor. See Example 2.10.

Define a morphism $f : A \to B$ is $s\mathcal{C}$ to be

a) a weak equivalence if Gf is a weak equivalence in $\mathbf{S}$;
b) a fibration if Gf is a fibration in $\mathbf{S}$;
c) a cofibration if it has the left lifting property with respect to all trivial fibrations in $s\mathcal{C}$.

A final definition is necessary before stating the result. Let $\{X_\alpha\}_{\alpha\in I}$ be a diagram in $\mathcal{C}$. Then, assuming the category $\mathcal{C}$ has enough colimits, there is a natural map

$$\varinjlim_I G(X_\alpha) \longrightarrow G(\varinjlim_I X_\alpha).$$

This is not, in general, an isomorphism. We say that G commutes with filtered colimits if this is an isomorphism whenever the index category I is filtered.

THEOREM 5.1. *Suppose $\mathcal{C}$ has all limits and colimits and that G commutes with filtered colimits. Then with the notions of weak equivalence, fibration, and cofibration defined above, $s\mathcal{C}$ is a closed model category provided the following assumption on cofibrations holds: every cofibration with the left lifting property with respect to fibrations is a weak equivalence.*

We will see that, in fact, $s\mathcal{C}$ is a simplicial model category with the simplicial structure of Theorem 2.5.

The proof of Theorem 5.1 turns on the following observation. As we have seen, a morphism $f : X \to Y$ is a fibration of simplicial sets if and only if it has the right lifting property with respect to the inclusions for all n, k

$$\Lambda^n_k \hookrightarrow \Delta^n$$

and f is a trivial fibration if and only if it has the right lifting property with respect to the inclusions $\partial\Delta^n \to \Delta^n$ of the boundary for all n. The objects $\Delta^n_k, \partial\Delta^n$, and Δ^n are *small* in the following sense: the natural map

$$\varinjlim_I \hom_{\mathbf{S}}(\Lambda^n_k, X_\alpha) \longrightarrow \hom_{\mathbf{S}}(\Lambda^n_k, \varinjlim_I X_\alpha)$$

is an isomorphism for all filtered colimits in $\mathbf{S}$. This is because Λ^n_k has only finitely many non-degenerate simplices. Similar remarks hold for $\partial\Delta^n$ and Δ^n.

LEMMA 5.2. *Any morphism $f : A \to B$ in $s\mathcal{C}$ can be factored*

$$A \xrightarrow{j} X \xrightarrow{q} B$$

where the morphism j is a cofibration and q is a trivial fibration.

PROOF: Coproducts of cofibrations are cofibrations, and given a pushout diagram

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d, "i"'] & A_2 \arrow[d, "j"] \\ A_1 \arrow[r] & B \end{tikzcd}$$

in $s\mathcal{C}$, then i a cofibration implies j is a cofibration, and that if $X \to Y$ is a cofibration in $\mathbf{S}$, then $FX \to FY$ is a cofibration in $s\mathcal{C}$. Inductively construct objects $X_n \in s\mathcal{C}$ with the following properties:

a) One has $A = X_0$ and there is a cofibration $j_n : X_n \to X_{n+1}$.

b) There are maps $q_n : X_n \to B$ so that $q_n = q_{n+1} \circ j_n$ and the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & X_n \arrow[d, "q_n"] \\ & B \end{tikzcd}$$

commutes, where $A \to X_n$ is the composite $j_{n-1} \circ \cdots \circ j_0$.

c) Any diagram

$$\begin{tikzcd} F\partial\Delta^m \arrow[r, "\varphi"] \arrow[d] & X_n \arrow[d, "q_n"] \\ F\Delta^m \arrow[r, "\psi"] & B \end{tikzcd}$$

can be completed to a diagram

$$\begin{tikzcd} F\partial\Delta^m \arrow[r, "\varphi"] \arrow[d] & X_n \arrow[d, "j_n"] \arrow[dr, "q_n"] & \\ F\Delta^m \arrow[r] & X_{n+1} \arrow[r, "q_{n+1}"'] & B \end{tikzcd}$$

where the bottom morphism is ψ.

Condition c) indicates how to construct X_{n+1} given X_n. Define $j_n : X_n \to X_{n+1}$ by the pushout diagrams

$$\begin{tikzcd} \coprod F\partial\Delta^m \arrow[r] \arrow[d] & X_n \arrow[d, "j_n"] \\ \coprod F\Delta^m \arrow[r] & X_{n+1} \end{tikzcd}$$

where the coproduct is over all diagrams of the type presented in c).

Then condition c) automatically holds. Further, $q_{n+1} : X_{n+1} \to B$ is defined and satisfies condition b) by the universal property of pushouts. Lastly, condition a) holds by the remarks at the beginning of the proof.

Now define $X = \varinjlim X_n$ and notice that we have a factoring

$$A \xrightarrow{j} X \xrightarrow{q} B$$

of the original morphism. The morphism j is a cofibration since directed colimits of cofibrations are cofibrations. We need only show $q : X \to B$ is a trivial fibration. This amounts to showing that any diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & GX \arrow[d, "Gq"] \\ \Delta^n \arrow[r] & GB \end{tikzcd}$$

can be completed. But $GX \cong \varinjlim_n GX_n$ by hypothesis on G, and the result follows by the small object argument. ■

The same argument, but using the trivial cofibrations in $\mathbf{S}$

$$\Lambda^n_k \hookrightarrow \Delta^n$$

for $n, k \geq 0$, proves the following lemma.

LEMMA 5.3. *Any morphism $f : A \to B$ in $s\mathcal{C}$ can be factored*

$$A \xrightarrow{j} X \xrightarrow{q} B$$

where q is a fibration and j is a cofibration which has the left lifting property with respect to all fibrations.

PROOF OF THEOREM 5.1: The Axioms **CM1**–**CM3** are easily checked. The axiom **CM5b** is Lemma 5.2; the axiom **CM5a** follows from Lemma 5.3 and the assumption on cofibrations. For axiom **CM4**, one half is the definition of cofibration. For the other half, one proceeds as follows. Let

$$i : A \to B$$

be a trivial cofibration. Then by Lemma 5.3 we can factor the morphism i as

$$A \xrightarrow{j} X \xrightarrow{q} B$$

where j is a cofibration with the left lifting with respect to all fibrations, and q is a fibration. By the hypothesis on cofibrations, j is a weak equivalence. Since i is a weak equivalence, so is q. Hence, one can complete the diagram

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & X \arrow[d, "q"] \\ B \arrow[r, "="] & B \end{tikzcd}$$

and finds that i is a retract of j. Hence i has the left lifting property with respect to fibrations, because j does. This completes the proof. ∎

We next remark that, in fact, $s\mathcal{C}$ is a simplicial model category. For this, we impose the simplicial structure guaranteed by Theorem 2.5. Thus if $X \in s\mathcal{C}$ and $K \in \mathbf{S}$, we have that

$$(A \otimes K)_n = \bigsqcup_{k \in K_n} A_n.$$

From this, one sees that if $X \in \mathbf{S}$

$$F(X \times K) \cong F(X) \otimes K.$$

This is because F, as a left adjoint, preserves coproducts. Thus Lemma 2.9 applies and

$$G\mathbf{hom}_{s\mathcal{C}}(K, B) \cong \mathbf{hom}_{\mathbf{S}}(K, GB).$$

Theorem 5.4. *With this simplicial structure, $s\mathcal{C}$ becomes a simplicial model category.*

Proof: Apply Proposition 3.13.1. If $j : K \to L$ is a cofibration in $\mathbf{S}$ and $q : X \to Y$ is a fibration in $s\mathcal{C}$, the map

$$G\mathbf{hom}_{s\mathcal{C}}(L, X) \longrightarrow G(\mathbf{hom}_{s\mathcal{C}}(K, X) \times_{\mathbf{hom}_{s\mathcal{C}}(K,Y)} \mathbf{hom}_{s\mathcal{C}}(L, Y))$$

is isomorphic to

$$\mathbf{hom}_{\mathbf{S}}(L, GX) \longrightarrow \mathbf{hom}_{\mathbf{S}}(K, GX) \times_{\mathbf{hom}_{\mathbf{S}}(K,GY)} \mathbf{hom}_{\mathbf{S}}(L, GY)$$

by the remarks above and the fact that G, as a right adjoint, commutes with pullbacks. Since $\mathbf{S}$ is a simplicial model category, the result holds. ∎

5.5 A remark on the hypotheses. Theorem 5.1 and, by extension, Theorem 5.4 require the hypothesis that every cofibration with the left lifting property with respect to all fibrations is, in fact, a weak equivalence. This is so Lemma 5.3 produces the factoring of a morphism as a trivial cofibration followed by a fibration. In the next section we will give some general results about when this hypothesis holds; however, in a particular situation, one might be able to prove directly that the factoring produced in Lemma 5.3 actually yields a trivial cofibration. Then the hypothesis on cofibrations required by these theorems holds because any cofibration with the left lifting property with respect to all fibrations will be a retract of a trivial cofibration. Then one need say no more.

For example, in examining the proof of Lemma 5.3 (see Lemma 5.2), one sees that we would have a factorization of $f : A \to B$ as a trivial cofibration followed by a fibratiion provided one knew that 1.) $F(\Lambda^n_k) \to F(\Delta^n)$ was a weak equivalence or, more generally, that F preserved trivial cofibrations, and 2.) trivial cofibrations in $s\mathcal{C}$ were closed under coproducts, pushouts, and colimits over the natural numbers.

6. Examples of simplicial model categories.

As promised, we prove that a variety of simplicial categories satisfy the hypotheses necessary for Theorem 5.4 of the previous section to apply.

We begin with a crucial lemma.

LEMMA 6.1. *Assume that for every $A \in s\mathcal{C}$ there is a natural weak equivalence*

$$\varepsilon_A : A \to QA$$

where QA is fibrant. Then every cofibration with the left lifting property with respect to all fibrations is a weak equivalence.

PROOF: This is the argument given by Quillen, on page II.4.9 of [75]. Let $j : A \to B$ be the given cofibration. Then by hypothesis, we may factor

$$\begin{tikzcd} A \arrow[r, "\varepsilon_A"] \arrow[d, "j"'] & QA \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & * \end{tikzcd}$$

to get a map $u : B \to QA$ so that $uj = \varepsilon_A$. Then we contemplate the lifting problem

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] & \mathbf{hom}_{s\mathcal{C}}(\Delta^1, QB) \arrow[d, "q"] \\ B \arrow[r, "g"] \arrow[ur, dashed] & \mathbf{hom}_{s\mathcal{C}}(\partial\Delta^1, QB) \cong QB \times QB \end{tikzcd}$$

where q is induced by $\partial\Delta^1 \subseteq \Delta^1$, f is the composite

$$AA \xrightarrow{j} B \xrightarrow{\varepsilon_B} QB = \mathbf{hom}_{s\mathcal{C}}(*, QB) \to \mathbf{hom}_{s\mathcal{C}}(\Delta^1, QB)$$

and

$$g = (\varepsilon_B, Qj \circ u) .$$

Note that f is adjoint to the constant homotopy on

$$\varepsilon_B \circ j = Qj \circ \varepsilon_A : A \to QB .$$

Then q is a fibration since

$$G\mathbf{hom}_{s\mathcal{C}}(K, X) = \mathbf{hom}_{\mathbf{S}}(K, GX) \ ,$$

and $\mathbf{S}$ is a simplicial model category. Hence, since j is a cofibration, there exists

$$H : B \to \mathbf{hom}_{s\mathcal{C}}(\Delta^1, QB)$$

making both triangles commute. Then H is a right homotopy from

$$\varepsilon_B : B \to QB$$

to $Qj \circ u$, and this homotopy restricts to the constant homotopy on $\varepsilon_B \circ j = Qj \circ \varepsilon_A : A \to QB$. In other words, we have a diagram

$$\begin{tikzcd}
A \arrow[r, "\epsilon_A"] \arrow[d, "j"'] & QA \arrow[d, "Qj"] \\
B \arrow[ur, "u"] \arrow[r, "\epsilon_B"'] & QB
\end{tikzcd}$$

such that the upper triangle commutes and the lower triangle commutes up to homotopy. Apply the functor G to this diagram. Then G preserves right homotopies, and one checks directly on the level of homotopy groups that Gj is a weak equivalence, which, by definition, implies j is a weak equivalence. ∎

Example 6.2. Suppose every object of $s\mathcal{C}$ is fibrant. Then we may take $\varepsilon_A : A \to QA$ to be the identity. This happens, for example, if the functor $G : s\mathcal{C} \to \mathbf{S}$ factors through the sub-category of simplicial groups and simplicial group homomorphisms. Thus, Theorem 5.4 applies to

simplicial group s, simplicial abelian groups and simplicial R-modules, where G is the forgetful functor;

(1) more generally to simplicial modules over a simplicial ring R, where G is the forgetful functor,

(2) for a fixed commutative ring R; simplicial R-algebras, simplicial commutative R-algebras and simplicial Lie algebras over R. Again G is the forgetful functor.

Another powerful set of examples arises by making a careful choice of the form the functor G can take.

Recall that an object $A \in \mathcal{C}$ is small if $\hom_{\mathcal{C}}(A, \cdot)$ commutes with filtered colimits. Fix a small $Z \in \mathcal{C}$ and define

$$G : s\mathcal{C} \to \mathbf{S} \tag{6.3}$$

by

$$G(X) = \mathbf{hom}_{s\mathcal{C}}(Z, X) \ .$$

Then G has left adjoint

$$FK = Z \otimes K$$

and $G(\cdot)$ commutes with filtered colimits. Thus, to apply Theorem 5.4, we need to prove the existence of the natural transformation

$$\varepsilon : A \to QA$$

as in Lemma 6.1. Let

$$\mathrm{Ex} : \mathbf{S} \to \mathbf{S}$$

be Kan's Extension functor[1]. Then for all $K \in \mathbf{S}$ there is a natural map

$$\varepsilon_K : K \to \mathrm{Ex}\, K$$

which is a weak equivalence. Furthermore, most crucially for the application here, $\mathrm{Ex}(\cdot)$ commutes with all limits. This is because it's a right adjoint. Finally, if $\mathrm{Ex}^n K$ is this functor applied n times and

$$\mathrm{Ex}^n \varepsilon_K : \mathrm{Ex}^n K \longrightarrow \mathrm{Ex}^{n+1} K$$

the induced morphism, then $\mathrm{Ex}^\infty K = \varinjlim \mathrm{Ex}^n K$ is fibrant in $\mathbf{S}$ and the induced map

$$K \longrightarrow \mathrm{Ex}^\infty K$$

is a trivial cofibration.

LEMMA 6.4. *Suppose the category $\mathcal{C}$ is complete and cocomplete. Fix $n \geq 0$. Then there is a functor*

$$Q_0(\cdot)_n : s\mathcal{C} \longrightarrow \mathcal{C}$$

[1] This construction is discussed in Section III.4 below.

so that, for all $Z \in \mathcal{C}$, there is a natural isomorphism of sets

$$\hom_{\mathcal{C}}(Z,(Q_0A)_n) \cong \mathrm{Ex}(\mathbf{Hom}_{s\mathcal{C}}(Z,A))_n \ .$$

Proof: Recall that that the functor Ex on $\mathbf{S}$ is right adjoint to the the subdivision functor sd. Then one has a sequence of natural isomorphisms

$$\begin{aligned}
\mathrm{Ex}\,\mathbf{Hom}_{s\mathcal{C}}(Z,A)_n &\cong \hom_{\mathbf{S}}(\Delta^n, \mathrm{Ex}\,\mathbf{Hom}_{s\mathcal{C}}(Z,A)) \\
&\cong \hom_{\mathbf{S}}(\mathrm{sd}\,\Delta^n, \mathbf{Hom}_{s\mathcal{C}}(Z,A)) \\
&\cong \hom_{s\mathcal{C}}(Z \otimes \mathrm{sd}\,\Delta^n, A) \\
&\cong \hom_{s\mathcal{C}}(Z, \mathbf{hom}_{s\mathcal{C}}(\mathrm{sd}\,\Delta^n, A)) \\
&\cong \hom_{\mathcal{C}}(Z, \mathbf{hom}_{s\mathcal{C}}(\mathrm{sd}\,\Delta^n, A)_0).
\end{aligned}$$

The last isomorphism is due to the fact that Z is a constant simplicial object and maps out of a constant simplicial object are completely determined by what happens on zero simplices. Thus we can set

$$(Q_0A)_n = \mathbf{hom}_{s\mathcal{C}}(\mathrm{sd}\,\Delta^n, A)_0. \qquad \blacksquare$$

The simplicial object Q_0A defined by

$$\mathbf{n} \mapsto (Q_0A)_n = \mathbf{hom}_{s\mathcal{C}}(\mathrm{sd}\,\Delta^n, A)_0$$

is natural in A; that is, we obtain a functor $Q_0 : s\mathcal{C} \to s\mathcal{C}$. Since we regard $Z \in \mathcal{C}$ as a constant simplicial object in $s\mathcal{C}$

$$\begin{aligned}
\mathbf{Hom}_{s\mathcal{C}}(Z,Y)_n &\cong \hom_{s\mathcal{C}}(Z \otimes \Delta^n, Y) \\
&\cong \hom_{\mathcal{C}}(Z, Y_n)
\end{aligned}$$

one immediately has that

$$\mathbf{Hom}_{s\mathcal{C}}(Z, Q_0A) \cong \mathrm{Ex}\,\mathbf{Hom}_{S\mathcal{C}}(Z,A) \ .$$

Finally the natural transformation $\varepsilon_K : K \to \mathrm{Ex}^\infty K$ yields a natural map

$$\varepsilon_A : A \longrightarrow Q_0A$$

and, by iteration, maps

$$Q_0^n \varepsilon_A : Q_0^n A \longrightarrow Q_0^{n+1} A.$$

Define $QA = \varinjlim Q_0^n A$. The reader will have noticed that Q_0A and QA are independent of Z.

Now fix a small object $Z \in \mathcal{C}$ and regard Z as a constant simplicial object in $s\mathcal{C}$. Then we define a morphism $A \to B$ in $s\mathcal{C}$ to be a weak equivalence (or fibration) if and only if the induced map

$$\mathbf{Hom}_{s\mathcal{C}}(Z,A) \longrightarrow \mathbf{Hom}_{s\mathcal{C}}(Z,B)$$

is a weak equivalence (or fibration) of simplicial sets.

Proposition 6.5. *If Z is small, the morphism $\varepsilon_A : A \to QA$ is a weak equivalence and QA is fibrant.*

Proof: Since Z is small, we have that

$$\mathbf{Hom}_{s\mathcal{C}}(Z, QA) \cong \mathrm{Ex}^\infty \mathbf{Hom}_{s\mathcal{C}}(Z, A).$$

The morphism ε_A is a weak equivalence if and only if

$$\mathbf{Hom}_{s\mathcal{C}}(Z, A) \to \mathbf{Hom}_{s\mathcal{C}}(Z, QA) \cong \mathrm{Ex}^\infty \mathbf{Hom}_{s\mathcal{C}a}(Z, A)$$

is a weak equivalence and $QA \to *$ is a fibration if and only if

$$\mathbf{Hom}_{s\mathcal{C}}(Z, QA) \cong \mathrm{Ex}^\infty \mathbf{Hom}_{s\mathcal{C}}(Z, A) \to \mathbf{Hom}_{s\mathcal{C}}(Z, *) = *$$

is a fibration. Both of these facts follow from the properties of the functor $\mathrm{Ex}^\infty(\cdot)$. ∎

Corollary 6.6. *Let $\mathcal{C}$ be a complete and cocomplete category and $Z \in \mathcal{C}$ a small object. Then $s\mathcal{C}$ is a simplicial model category with $A \to B$ a weak equivalence (or fibration) if and only if*

$$\mathbf{Hom}_{s\mathcal{C}}(Z, A) \longrightarrow \mathbf{Hom}_{s\mathcal{C}}(Z, B)$$

is a weak equivalence (or fibration) of simplicial sets.

In practice one wants an intrinsic definition of weak equivalence and fibration, in the manner of the following example.

Example 6.7. All the examples of 6.2 can be recovered from Corollary 6.6. For example, $\mathcal{C}$ be the category of algebras over a commutative ring. Then $\mathcal{C}$ has a single projective generator; namely $A[x]$, the algebra on one generator. Then one sets $Z = A[x]$, which is evidently small, and one gets a closed model category structure from the previous result. However, if $B \in s\mathcal{C}$, then

$$\mathbf{Hom}_{s\mathcal{C}}(A[x], B) \cong B$$

in the category of simplicial sets, so one recovers the same closed model category structure as in Example 6.2.

If $\mathcal{C}$ is a category satisfying 5.1 with a single small projective generator, then $\mathcal{C}$ is known as a category of universal algebras. Setting Z to be the generator, one immediately gets a closed model category structure on $s\mathcal{C}$ from Corollary 6.6. This is the case for all the examples of 6.2.

To go further, we generalize the conditions of Theorem 5.1 a little, to require the existence of a collection of functors $G_i : s\mathcal{C} \to \mathbf{S}$, $i \in I$, each of which has a left adjoint $F_i : \mathbf{S} \to s\mathcal{C}$. We now say that a morphism $f : A \to B$ of $s\mathcal{C}$ is

a) a weak equivalence if $G_i f$ is a weak equivalence of simplicial sets for all $i \in I$;
b) a fibration if all induced maps $G_i f$ are fibrations of $\mathbf{S}$;
c) a cofibration if it has the left lifting property with respect to all trivial cofibrations of $s\mathcal{C}$.

Then Theorem 5.1 and Theorem 5.4 together have the following analogue:

Theorem 6.8. *Suppose that $\mathcal{C}$ has all small limits and colimits and that all of the functors $G_i : \mathcal{C} \to \mathbf{S}$ preserve filtered colimits. Then with the notions of weak equivalence, fibration and cofibration defined above, and if every cofibration with the left lifting property with respect to all fibrations is a weak equivalence, then $s\mathcal{C}$ is a simplicial model category.*

Proof: The proof is the same as that of Theorem 5.1, except that the small object arguments for the factorization axiom are constructed from all diagrams of the form

$$\begin{tikzcd} F_i\partial\Delta^m \arrow[r] \arrow[d] & A \arrow[d] \\ F_i\Delta^m \arrow[r] & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F_i\Lambda^n_k \arrow[r] \arrow[d] & A \arrow[d] \\ F_i\Delta^n \arrow[r] & B \end{tikzcd} \qquad \blacksquare$$

Theorem 6.8 will be generalized significantly in the next section — it is a special case of Theorem 7.8.

Now fix a set of small objects $Z_i \in \mathcal{C}$, $i \in I$, and regard each Z_i as a constant simplicial object in $s\mathcal{C}$. Then we define a morphism $A \to B$ in $s\mathcal{C}$ to be a weak equivalence (or fibration) if and only if the induced map

$$\mathbf{Hom}_{s\mathcal{C}}(Z_i, A) \longrightarrow \mathbf{Hom}_{s\mathcal{C}}(Z_i, B)$$

is a weak equivalence (or fibration) of simplicial sets. In the case where $\mathcal{C}$ is complete and cocomplete, we are still entitled to the construction of the natural map $\epsilon_A : A \to QA$ in $s\mathcal{C}$. Furthermore, each of the objects Z_i is small, so that Proposition 6.5 holds with Z replaced by Z_i, implying that the map ϵ_A is a weak equivalence and that QA is fibrant. Then an analogue of Lemma 6.1 holds for the setup of Theorem 6.8 (with G replaced by G_i in the proof), and we obtain the following result:

THEOREM 6.9. *Suppose that $\mathcal{C}$ is a small complete and cocomplete category, and let $Z_i \in \mathcal{C}$, $i \in I$, be a set of small objects. Then $s\mathcal{C}$ is a simplicial model category with $A \to B$ a weak equivalence (respectively fibration) if and only if the induced map*

$$\mathbf{Hom}_{s\mathcal{C}}(Z_i, A) \to \mathbf{Hom}_{s\mathcal{C}}(Z_i, B)$$

is a weak equivalence (respectively fibration) for all $i \in I$

EXAMPLE 6.10. Suppose that $\mathcal{C}$ is small complete and cocomplete, and has a set $\{P_\alpha\}$ of small projective generators. Theorem 6.9 implies that $\mathcal{C}$ has a simplicial model category structure, where $A \to B$ is a weak equivalence (or fibration) if

$$\mathbf{Hom}_{s\mathcal{C}}(P_\alpha, A) \to \mathbf{Hom}_{s\mathcal{C}}(P_\alpha, B)$$

is a weak equivalence (or fibration) for all α.

Note that the requirement that the objects P_α are projective generators is not necessary for the existence of the closed model structure. However, if we also assume that the category $\mathcal{C}$ has sufficiently many projectives in the sense that there is an effective epimorphism $P \to C$ with P projective for all objects $C \in \mathcal{C}$, then it can be shown that a morphism $f : A \to B$ of $s\mathcal{C}$ is a weak equivalence (respectively fibration) if every induced map

$$\mathbf{Hom}_{s\mathcal{C}}(P, A) \to \mathbf{Hom}_{s\mathcal{C}}(P, B)$$

arising from a projective object $P \in \mathcal{C}$ is a weak equivalence (respectively fibration) of simplicial sets. This is a result of Quillen [75, II.4], and its proof is the origin of the stream of ideas leading to Theorem 6.9. We shall go further in this direction in the next section.

To be more specific now, let $\mathcal{C}$ be the category of graded A-algebras for some commutative ring A and let, for $n \geq 0$,

$$P_n = A[x_n]$$

be the free graded algebra or an element of degree n. Then $\{P_n\}_{n\geq 0}$ form a set of projective generators for $\mathcal{C}$. Thus $s\mathcal{C}$ gets a closed model category structure and $B \to C$ in $s\mathcal{C}$ is a weak equivalence if and only if

$$(B)_n \to (C)_n$$

is a weak equivalence of simplicial sets for all n. Here $(\cdot)_n$ denotes the elements of degree n. This is equivalent to the following: if M is a simplicial graded A-module, define

$$\pi_* M = H_*(M, \partial)$$

where ∂ is the alternating of the face operators. Then $B \to C$ in $s\mathcal{C}$ is a weak equivalence if and only if

$$\pi_* B \to \pi_* C$$

is an isomorphism of bigraded A-modules.

This formalism works for graded groups, graded abelian groups, graded A-modules, graded commutative algebras, graded Lie algebras, and so on.

EXAMPLE 6.11. Let $\mathbb{F}$ be a field and let $\mathcal{C} = \mathcal{CA}$ be the category of coalgebras over $\mathbb{F}$. Then, by [87, p.46] every coalgebra $C \in \mathcal{CA}$ is the filtered colimit of its finite dimensional sub-coalgebras. Thus $\mathcal{CA}$ has a set of generators $\{C_\alpha\}$ where C_α runs over a set of representatives for the finite dimensional coalgebras. These are evidently small. Hence, $s\mathcal{CA}$ has a closed model category structure where $A \to B$ is a weak equivalence if and only if

$$\mathbf{Hom}_{s\mathcal{CA}}(C_\alpha, A) \to \mathbf{Hom}_{s\mathcal{CA}}(C_\alpha, B)$$

is a weak equivalence for all C_α. The significance of this example is that the C_α are not necessarily projective.

7. A generalization of Theorem 5.1.

The techniques of the previous sections are very general and accessible to vast generalization. We embark some ways on this journey here. First we expand on what it means for an object in a category to be small. Assume for simplicity that we are considering a category $\mathcal{C}$ which has all limits and colimits. We shall use the convention that a *cardinal number* is the smallest ordinal number in a given bijection class.

Fix an infinite cardinal number γ, and let $\mathbf{Seq}(\gamma)$ denote the well-ordered set of ordinals less than γ. Then $\mathbf{Seq}(\gamma)$ is a category with $\hom(s,t)$ one element if $s \leq t$ and empty otherwise. A γ-diagram in $\mathcal{C}$ is a functor $X : \mathbf{Seq}(\gamma) \to \mathcal{C}$. We will write $\varinjlim_\gamma X_s$ for the colimit. We shall say that X is a γ-diagram of cofibrations if each of the transition morphisms $X_s \to X_t$ is a cofibration of $\mathcal{C}$.

DEFINITION 7.1. *Suppose that β is an infinite cardinal. An object $A \in \mathcal{C}$ is β-small if for all γ-diagrams of cofibrations X in $\mathcal{C}$ with $\gamma \geq \beta$, the natural map*

$$\varinjlim_{\gamma} \hom_{\mathcal{C}}(A, X_s) \to \hom_{\mathcal{C}}(A, \varinjlim_{\gamma} X_s)$$

is an isomorphism. A morphism $A \to B$ of $\mathcal{C}$ is said to be β-small if the objects A and B are both β-small.

EXAMPLE 7.2. The small objects of the previous sections were ω-small, where ω is the first infinite cardinal. Compact topological spaces are also ω-small, but this assertion requires proof.

Supposethat $X : \mathbf{Seq}(\gamma) \to \mathbf{Top}$ is a γ-diagram of cofibrations. Then X is a retract of a γ-diagram of cofibrations $\overline{X}$, where each of the transition morphisms $\overline{X}_s \hookrightarrow \overline{X}_t$ is a relative CW-complex. In effect, set $\overline{X}_0 = X_0$, and set $\overline{X}_\alpha = \varinjlim_{s<\alpha} \overline{X}_s$ for limit ordinals $\alpha < \gamma$. Suppose given maps

$$X_s \xrightarrow{r_s} \overline{X}_s \xrightarrow{\pi_s} X_s$$

with $\pi_s r_s = 1$. Then $\overline{X}_{s+1}$ is defined by choosing a trivial fibration π_{s+1} and a relative CW-complex map $j_{s+1} : \overline{X}_s \hookrightarrow \overline{X}_{s+1}$ (ie. $\overline{X}_{s+1}$ is obtained from $\overline{X}_s$ by attaching cells, and j_{s+1} is the corresponding inclusion), such that the following diagram commutes:

$$\begin{tikzcd} \overline{X}_s \arrow[r, "j_{s+1}"] \arrow[d, "\pi_s"] & \overline{X}_{s+1} \arrow[d, "\pi_{s+1}"] \\ X_s \arrow[r, "i_{s+1}"'] & X_{s+1}, \end{tikzcd}$$

where the map $i_{s+1} : X_s \to X_{s+1}$ is the cofibration associated to the relation $s \leq s+1$ by the functor X. Then there is a lifting in the diagram

$$\begin{tikzcd} X_s \arrow[r, "j_{s+1} r_s"] \arrow[d, "i_{s+1}"'] & \overline{X}_{s+1} \arrow[d, "\pi_{s+1}"] \\ X_{s+1} \arrow[r, "1"'] \arrow[ur, "r_{s+1}"] & X_{s+1} \end{tikzcd}$$

so that the section r_s extends to a section r_{s+1} of the trivial fibration π_{s+1}. The inclusion

$$X_0 = \overline{X}_0 \hookrightarrow \varinjlim_{\gamma} \overline{X}_s$$

is a relative CW-complex map, and every compact subset of the colimit only meets finitely many cells outside of X_0. Every compact subset of $\varinjlim_\gamma \overline{X}_s$ is therefore contained in some subspace $\overline{X}_s$. It follows that every compact subset of $\varinjlim_\gamma X_s$ is contained in some X_s.

We next produce an appropriate generalization of saturation.

DEFINITION 7.3. *A class $\mathcal{M}$ of morphisms in $\mathcal{C}$ is β-saturated if it is closed under*

1) retracts*: Suppose there is a commutative diagram in $\mathcal{C}$*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "i"'] & X' \arrow[r] \arrow[d, "i'"] & X \arrow[d, "i"] \\ Y \arrow[r] & Y' \arrow[r] & Y \end{tikzcd}$$

with the horizontal composition the identity. Then if $i' \in \mathcal{M}$, then $i \in \mathcal{M}$.

2) coproducts*: if each $j_\alpha : X_\alpha \to Y_\alpha$ is in $\mathcal{M}$, then $\bigsqcup_\alpha j_\alpha : \bigsqcup_\alpha X_\alpha \to \bigsqcup_\alpha Y_\alpha$ is in $\mathcal{M}$;*

3) pushouts*: given a pushout diagram in $\mathcal{C}$*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "j"] \\ B \arrow[r] & Y, \end{tikzcd}$$

if i is in $\mathcal{M}$, then so is j.

4) colimits of β-sequences*: Suppose we are given a β-sequence*

$$X : \mathbf{Seq}(\beta) \to \mathcal{C}$$

with the following properties: a) for each successor ordinal $s+1 \in \mathbf{Seq}(\beta)$, the map $X_s \to X_{s+1}$ is in $\mathcal{M}$, and b) for each limit ordinal $s \in \mathbf{Seq}(\beta)$, the map $\varinjlim_{t<s} X_t \to X_s$ is in $\mathcal{M}$. Then

$$X_s \to \varinjlim_{\beta} X_s$$

is in $\mathcal{M}$ for all $s \in \mathbf{Seq}(\beta)$.

Up until now we have considered only saturated clesses of morphisms with $\beta = \omega$, the cardinality of a countable ordinal. In this case, one doesn't need the extra care required in making the definition of what it means to be closed under colimits.

LEMMA 7.4. *Let $\mathcal{C}$ be a closed model category. Then the class of cofibrations and the class of trivial cofibrations are both β-saturated for all β.*

PROOF: This is an exercise using the fact that cofibrations (or trivial cofibrations) are characterized by the fact that they have the left lifting property with respect to trivial fibrations (or fibrations). ■

The next step is to turn these concepts around.

DEFINITION 7.5. *Let $\mathcal{M}_0$ be a class of morphisms in $\mathcal{C}$. Then the β-saturation of $\mathcal{M}_0$ is the smallest β-saturated class of morphisms in $\mathcal{C}$ containing $\mathcal{M}_0$.*

We now come to the crucial axiom.

DEFINITION 7.6. *A closed model category is* cofibrantly generated *with respect to a cardinal β if the class of cofibrations and the class of trivial cofibrations are the β-saturations of* sets *of β-small morphisms $\mathcal{M}_0$ and $\mathcal{M}_1$ respectively.*

REMARKS 7.7.

1) Suppose that β and γ are cardinals such that $\beta \leq \gamma$. Then every γ-saturated class is β-saturated, because every sequence $X : \mathbf{Seq}(\beta) \to \mathcal{C}$ can be extended to a sequence $X_* : \mathbf{Seq}(\gamma) \to \mathcal{C}$ having the same colimit. It follows that the β-saturation of any set of morphisms is contained in its γ-saturation. Observe also that every β-small object is γ-small, directly from Definition 7.1. The size of the cardinal β in Definition 7.6 therefore doesn't matter, so long as it exists. One says that the closed model category $\mathcal{C}$ is *cofibrantly generated* in cases where the cardinal β can be ignored.

2) Until now, we've taken β to ω. Then the category of simplicial sets is cofibrantly generated, for example, by the usual small object argument. Similarly, modulo the care required for the assertion that finite CW-complexes are ω-small (Example 7.2), the category of topological spaces is cofibrantly generated with respect to ω. We will see larger cases later.

3) One could require one cardinal β_0 for cofibrations and β_1 for trivial cofibrations. However, $\beta = \max\{\beta_0, \beta_1\}$ would certainly work in either case, by 1).

To give the generalization of Theorem 5.1 we establish a situation. We fix a simplicial model category $\mathcal{C}$ and a simplicial category $\mathcal{D}$. Suppose we have a set of functors $G_i : \mathcal{D} \to \mathcal{C}$, indexed by the elements i in some set I, and suppose each G_i has a left adjoint F_i which preserves the simplicial structure in the sense that there is a natural isomorphism

$$F_i(X \otimes K) \cong F(X_i) \otimes K$$

for all $X \in \mathcal{C}$ and $K \in \mathbf{S}$. Define a morphism $f : A \to B$ in $\mathcal{D}$ to be a weak equivalence (or fibration) is

$$G_i f : G_i A \to G_i B$$

is a weak equivalence (or fibration).

Theorem 7.8. *Suppose the simplicial model category $\mathcal{C}$ is cofibrantly generated with respect to a cardinal β, and that*

(1) *all of the functors G_i commute with colimits over $\mathbf{Seq}(\beta)$, and*
(2) *the functors G_i take the β-saturation of the collection of all maps $F_j A \to F_j B$ arising from maps $A \to B$ in the generating family for the cofibrations of $\mathcal{C}$ and elements j of I to cofibrations of $\mathcal{C}$.*

Then if every cofibration in $\mathcal{D}$ with the left lifting property with respect to all fibrations is a weak equivalence, $\mathcal{D}$ is a simplicial model category.

Proof (Outline): There are no new ideas—only minor changes from the arguments of Section 5. The major difference is in how the factorizations are constructed. For example, to factor $X \to Y$ as $X \xrightarrow{j} Z \xrightarrow{q} Y$ where j is a cofibration which has the left lifting property with respect to all fibrations and q is a fibration, one forms a β-diagram $\{Z_s\}$ is $\mathcal{D}$ where

i) $Z_0 = X$;
ii) if $s \in \mathbf{Seq}(\beta)$ is a limit ordinal, $Z_s = \varinjlim_{t<s} Z_t$ and
iii) if $s+1$ is a successor ordinal, there is a pushout diagram

$$\begin{tikzcd}
\coprod_i \coprod_f F_i(A) \arrow[r] \arrow[d] & Z_s \arrow[d] \\
\coprod_i \coprod_f F_i(B) \arrow[r] & Z_{s+1}
\end{tikzcd}$$

where f runs over all diagrams

$$\begin{tikzcd} A \arrow[r] \arrow[d] & G_i(Z_s) \arrow[d] \\ B \arrow[r] & G_i(Y) \end{tikzcd}$$

where $A \to B$ is in the set $\mathcal{M}_1$ of β-small cofibrations in $\mathcal{C}$ whose β-saturation is all trivial cofibrations. ■

EXAMPLE 7.9. Suppose that $\mathcal{C}$ is a cofibrantly generated simplicial model category and I is a fixed small category. Write $\mathcal{C}^I$ for the category of functors $X : I \to \mathcal{C}$ and natural transformations between them. There are i-section functors $G_i : \mathcal{C}^I \to \mathcal{C}$ defined by $G_iX = X(i)$, $i \in I$, and each such G_i has a left adjoint $F_i : \mathcal{C} \to \mathcal{C}^I$ defined by

$$F_iD(j) = \bigsqcup_{i \to j \text{ in } I} D.$$

Say that a map $X \to Y$ of $\mathcal{C}^I$ is a *pointwise cofibration* if each of the maps $G_iX \to G_iY$ is a respectively cofibration of $\mathcal{C}$. If $A \to B$ is a generating cofibration for $\mathcal{C}$, the induced maps $F_iA(j) \to F_iB(j)$ are disjoint unions of cofibrations and hence are cofibrations of $\mathcal{C}$. The induced maps maps $F_iA \to F_iB$ are therefore pointwise cofibrations of $\mathcal{C}^I$. The functors G_j preserve all colimits, and so the collection of pointwise cofibrations of $\mathcal{C}^I$ is saturated (meaning β-saturated for some infinite cardinal β — similar abuses follow). The saturation of the collection of maps $F_iA \to F_iB$ therefore consists of pointwise cofibrations of $\mathcal{C}^I$.

A small object argument for $\mathcal{C}^I$ produces a factorization

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & Z \arrow[d, "q"] \\ & Y \end{tikzcd}$$

for an arbitrary map $f : X \to Y$ of $\mathcal{C}^I$, with q a fibration, and for which j is in the saturation of the collection of maps $F_iC \to F_iD$ arising from the generating set $C \to D$ for the class of trivial cofibrations of $\mathcal{C}$. But again, each induced map $F_iC(j) \to F_iD(j)$ is a trivial pointwise cofibration of $\mathcal{C}$, and the j-section functors preserve all colimits. The collection of maps of $\mathcal{C}^I$ which are trivial cofibrations in sections is therefore saturated, and hence contains the saturation of the maps $F_iC \to F_iD$. It follows that the map j is a weak equivalence as well as a cofibration. In particular, by a standard argument, every map of $\mathcal{C}^I$ which has the left lifting property with respect to all fibrations is a trivial cofibration.

It therefore follows from Theorem 7.8 that every diagram category $\mathcal{C}^I$ taking values in a cofibrantly generated simplicial model category has a simplicial model structure for which the fibrations and weak equivalences are defined pointwise. This result applies in particular to diagram categories $\mathbf{Top}^I$ taking values in topological spaces.

Here's the analog of Lemma 6.1:

Proposition 7.10. *Suppose there is a functor $Q : \mathcal{D} \to \mathcal{D}$ so that QX is fibrant for all X and there is a natural weak equivalence $\epsilon_X : X \to QX$. Then every cofibration with the left lifting property with respect to all fibrations is a weak equivalence.*

Proof: The argument is similar to that of Lemma 6.1; in particular, it begins the same way.

The map j has the advertised lifting property, so we may form the diagrams

$$\begin{tikzcd} A \arrow[r, "\epsilon_A"] \arrow[d, "j"'] & QA \arrow[d] \\ B \arrow[r] \arrow[ur, "u"] & * \end{tikzcd}$$

and

$$\begin{tikzcd} A \arrow[r, "s\epsilon_B j"] \arrow[d, "j"'] & \mathbf{hom}(\Delta^1, QB) \arrow[d] \\ B \arrow[r, "(\epsilon_B, Qj \circ u)"'] \arrow[ur] & QB \times QB \end{tikzcd}$$

where $s\epsilon_B j$ is the constant (right) homotopy on the composite

$$A \xrightarrow{j} B \xrightarrow{\epsilon_B} QB.$$

The functors G_i preserve right homotopies, so the diagram

$$\begin{tikzcd} A \arrow[r, "\epsilon_A"] \arrow[d, "j"'] & QA \arrow[d, "Qj"] \\ B \arrow[r, "\epsilon_B"'] \arrow[ur, "u"] & QB \end{tikzcd}$$

remains homotopy commutative after applying each of the functors G_i. It follows that the map $G_i(j)$ is a retract of the map $G_i(\epsilon_A)$ in the homotopy category $\mathrm{Ho}(\mathcal{C})$, and is therefore an isomorphism in $\mathrm{Ho}(\mathcal{C})$. But then a map in a simplicial model category which induces an isomorphism in the associated homotopy category must itself be a weak equivalence: this is Lemma 4.1. ■

EXAMPLE 7.11. The factorization axioms for a cofibrantly generated simplicial model category $\mathcal{C}$ can always be proved with a possibly transfinite small object argument (see, for example, the proofs of Proposition V.6.2 and Lemma IX.1.17). Such arguments necessarily produce factorizations which are natural in morphisms in $\mathcal{C}$, so that there is a natural fibrant model $X \hookrightarrow \tilde{X}$ for all objects X of $\mathcal{C}$. It follows that there are natural fibrant models for the objects of any diagram category $\mathcal{C}^I$ taking values in $\mathcal{C}$. We therefore obtain a variation of the proof of the existence of the closed model structure for $\mathcal{C}^I$ of Example 7.9 which uses Proposition 7.10. This means that the requirement in Theorem 7.8 that every cofibration which has the left lifting property with respect to all fibrations should be a weak equivalence is not particularly severe.

EXAMPLE 7.12. As an instance where the cofibrant generators are not ω-small, we point out that in Section IX.3 we will take the category of simplicial sets, with its usual simplicial structure and impose a new closed model category structure. Let E_* be any homology theory and we demand that a morphism $f : X \to Y$ in $\mathbf{S}$ be a

1) E_* equivalence if E_*f is an isomorphism
2) E_* cofibration if f is a cofibration as simplicial sets
3) E_* fibration if f has the right lifting property with respect to all E_* trivial cofibrations.

The E_* fibrant objects are the Bousfield local spaces. In this case the E_*-trivial cofibrations are the saturation of a set of E_* trivial cofibrations $f : A \to B$ where B is β-small with β some infinite cardinal greater than the cardinality of $E_*(pt)$.

One has functorial factorizations, so Example 7.11 can be repeated to show that $\mathbf{S}^I$ has a simplicial model category structure with $f : X \to Y$ a weak equivalence (or fibration) if and only if $X(i) \to Y(i)$ is an E_* equivalence (or E_* fibration) for all i.

8. Quillen's total derived functor theorem.

Given two closed model categories $\mathcal{C}$ and $\mathcal{D}$ and adjoint functors between them, we wish to know when these induce adjoint functors on the homotopy categories. This is Quillen's Total Derived Functor Theorem. We also give criteria under which the induced adjoint functors give an equivalence of the homotopy categories.

The ideas of this section are a generalization to non-abelian settings of an old idea of Grothendieck which can be explained by the following example. If R is a commutative ring and M, N are two R-modules, one might want to compute $\operatorname{Tor}_p^R(M, N)$, $p \geq 0$. However, there is a finer invariant, namely, the chain homotopy type of $M \otimes_R P_*$ where P_* is a projective resolution of N. One calls the chain homotopy equivalence class of $M \otimes_R P_*$ by the name $\operatorname{Tor}^R(M, N)$. This is the total derived functor. The individual Tor terms can be recovered by taking homology groups.

For simplicity we assume we are working with simplicial model categories, although many of the results are true without this assumption.

DEFINITION 8.1. *Let $\mathcal{C}$ be a simplicial model category and $\mathcal{A}$ any category. Suppose $F : \mathcal{C} \to \mathcal{A}$ is a functor that sends weak equivalences between cofibrant objects to isomorphisms. Define the total left derived functor*

$$\mathbf{L}F : \operatorname{Ho}(\mathcal{C}) \to \mathcal{A}.$$

by $\mathbf{L}F(X) = F(Y)$ where $Y \to X$ is a trivial fibration with Y cofibrant.

It is not immediately clear that $\mathbf{L}F$ is defined on morphisms or a functor. If $f : X \to X'$ is a morphism in $\mathcal{C}$ and $Y \to X$ and $Y' \to X'$ are trivial cofibrations with Y and Y' cofibrant, then there is a morphism g making the following diagram commute:

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "f"] & X' \end{tikzcd} \tag{8.2}$$

and we set $\mathbf{L}F(f) = \mathbf{L}F(g)$.

LEMMA 8.3. *The objects* $\mathbf{L}F(X)$ *and morphisms* $\mathbf{L}F(f)$ *are independent of the choices and* $\mathbf{L}F : \mathrm{Ho}(\mathcal{C}) \to \mathcal{A}$ *is a functor.*

PROOF: First note that $\mathbf{L}F(f)$ is independent of the choice of g in diagram (8.2). This is because any two lifts g and g' are homotopic and one has

$$\begin{tikzcd} F(Y) \sqcup F(Y) \arrow[r] \arrow[dr, "F(1)\sqcup F(1)"'] & F(Y \otimes \Delta^1) \arrow[r, "FH"] \arrow[d, "\cong"] & F(Y') \\ & F(Y) & \end{tikzcd}$$

where H is the homotopy. Next, if we let f be the identity in (8.2), the same argument implies $\mathbf{L}F(X)$ is independent of the choice of Y. Finally, letting $f = f_1 \circ f_2$ be a composite in diagram (8.2) the same argument shows $\mathbf{L}F(f_1 \circ f_2) = \mathbf{L}F(f_1) \circ \mathbf{L}F(f_2)$. ■

REMARK 8.4. For those readers attuned to category theory we note that $\mathbf{L}F$ is in fact a Kan extension in the following sense. Let $\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ be the localization functor and

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\gamma"] \arrow[d, "F"'] & \mathrm{Ho}(\mathcal{C}) \arrow[dl, dashed] \\ \mathcal{A} & \end{tikzcd}$$

the diagram of categories. There may or may not be a functor $\mathrm{Ho}(\mathcal{C}) \to \mathcal{A}$ completing the diagram; however, one can consider functors $T : \mathrm{Ho}(\mathcal{C}) \to \mathcal{A}$ equipped with a natural transformation

$$\varepsilon_T : T\gamma \to F.$$

The Kan extension is the final such functor T, if it exists. If R denotes this Kan extension, final means that given any such T, there is a natural transformation $\sigma : T \to R$ so that $\varepsilon_T = \varepsilon_R \sigma\gamma$. The Kan extension is unique if it exists. To see that it exists one applies Theorem 1, p. 233 of Mac Lane's book [66]. This result, in this context reads as follows: one forms a category $X \downarrow \gamma$ consisting of pairs (Z, f) where $Z \in \mathcal{C}$ and $f : X \to Z$ is a morphism in $\mathrm{Ho}(\mathcal{C})$. Then if

$$R(X) = \lim_{\substack{\longleftarrow \\ X\downarrow\gamma}} F(Z)$$

exists for all X, then R exists. However, the argument of Lemma 6.3 says that the diagram $F : (X \downarrow \gamma) \to \mathcal{A}$ has a terminal object. In fact, $X \downarrow \gamma$ has a terminal object, namely $X \to Y$ where $Y \to X$ is a trivial fibration (which has an inverse in $\mathrm{Ho}(\mathcal{C})$) with Y cofibrant. This shows that $R = \mathbf{L}F$. ■

Corollary 8.5. *Let $X \in \mathcal{C}$. If X is cofibrant, then $\mathbf{L}F(X) \cong F(X)$. If $Y \to X$ is any weak equivalence, with Y cofibrant, then $\mathbf{L}FX \cong FY$.*

Proof: The first statement is obvious and the second follows from

$$FY \cong \mathbf{L}F(Y) \xrightarrow{\cong} \mathbf{L}F(X)$$

since $Y \to X$ is an isomorphism in $\mathrm{Ho}(\mathcal{C})$. ■

Example 8.6. Let $\mathcal{C} = \mathcal{C}_*R$ be chain complexes of left modules over a ring R, and let $\mathcal{A} = n\mathcal{A}b$ be graded abelian groups. Define

$$F(C) = H_*(M \otimes_R C)$$

for some right module M. Then

$$\mathbf{L}F(C) = H_*(M \otimes_R D)$$

where $D \to C$ is a projective resolution of $\mathcal{C}$. There is a spectral sequence

$$\mathrm{Tor}_p^R(M, H_qC) \Rightarrow (\mathbf{L}F(C))_{p+q}.$$

In particular, if $H_*C = N$ concentrated in degree 0,

$$\mathbf{L}F(C) \cong \mathrm{Tor}_*^R(M, N),$$

bringing us back to what we normally mean by derived functors.

If $G : \mathcal{C} \to \mathcal{A}$ sends weak equivalences between fibrant objects to isomorphisms, one also gets a total right derived functor

$$\mathbf{R}G : \mathrm{Ho}(\mathcal{C}) \to \mathcal{A}.$$

It is also a Kan extension, suitably interpreted: it is initial among all functors $S : \mathrm{Ho}(\mathcal{C}) \to \mathcal{A}$ equipped with a natural transformation $\eta_S : F \to S\gamma$.

Now suppose we are given two simplicial model categories $\mathcal{C}$ and $\mathcal{D}$ and a functor $F : \mathcal{C} \to \mathcal{D}$ with a right adjoint G. The following is one version of the total derived functor theorem:

Theorem 8.7. *Suppose F preserves weak equivalences between cofibrant objects and G preserves weak equivalences between fibrant objects. Then $\mathbf{L}F : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$ and $\mathbf{R}G : \mathrm{Ho}(\mathcal{D}) \to \mathrm{Ho}(\mathcal{C})$ exists and $\mathbf{R}G$ is right adjoint to $\mathbf{L}F$.*

PROOF: That $\mathbf{L}F$ and $\mathbf{R}G$ exist is a consequence of Lemma 8.3 and its analog for total right derived functors. We need only prove adjointness.

If $X \in \mathcal{C}$, choose $Y \to X$ a trivial fibration with Y cofibrant. Hence $\mathbf{L}F(X) \cong F(Y)$. Now choose $F(Y) \to Z$, a trivial cofibration with Z fibrant. Then $\mathbf{R}G \circ \mathbf{L}F(X) \cong G(Z)$ and one gets a unit

$$\eta : X \to \mathbf{R}G \circ \mathbf{L}F(X)$$

by $X \leftarrow Y \to GF(Y) \to G(Z)$.

Similarly, let $A \in \mathcal{D}$. Choose $A \to B$ a trivial cofibration with B fibrant. Then $\mathbf{R}G(A) \cong G(B)$. Next choose $C \to G(B)$ a trivial fibration with C cofibrant. Then $\mathbf{L}F \circ \mathbf{R}G(A) \cong F(C)$ and one gets a counit $\varepsilon : \mathbf{L}F \circ \mathbf{R}G(A) \to A$ by

$$F(C) \to FGB \to B \leftarrow A.$$

We now wish to show

$$\mathbf{L}F(X) \xrightarrow{\mathbf{L}F\eta} \mathbf{L}F \circ \mathbf{R}G \circ \mathbf{L}F(X) \xrightarrow{\varepsilon_{\mathbf{R}G}} \mathbf{L}F(X)$$

is an isomorphism. In evaluating $\varepsilon_{\mathbf{R}G}$ we set $A = \mathbf{L}F(X) = F(Y)$, so that $B = Z$. Factor $Y \to GF(Y) \to G(Z)$ and $Y \xrightarrow{j} C \xrightarrow{q} G(Z)$ where j is a cofibration (so C is cofibrant) and q is a trivial fibration. Then $\varepsilon_{\mathbf{R}G}$ is given by

$$F(C) \xrightarrow{Fq} FGZ \to Z \leftarrow FY.$$

Furthermore there is a commutative square

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "j"'] & GF(Y) \arrow[d] \\ C \arrow[r] & G(Z) \end{tikzcd}.$$

and $Fj \cong P\mathbf{L}F\eta_X$. Expanding the diagram gives:

$$\begin{tikzcd} F(Y) \arrow[r] \arrow[d, "Fj"] & FGFY \arrow[r] \arrow[d] & FY \arrow[d, "\cong"] \\ F(C) \arrow[r, "Fq"] & FGZ \arrow[r] & Z \end{tikzcd}.$$

The line across the top is the identity and the indicated map is an isomorphism in the homotopy category. Hence we have proved the assertion.

The other assertion—that

$$\mathbf{R}G(A) \xrightarrow{\eta_\varepsilon} \mathbf{R}G \circ \mathbf{L}F \circ \mathbf{R}G(A) \xrightarrow{\mathbf{R}G_\varepsilon} \mathbf{R}G(A)$$

is an isomorphism—is proved similarly. The result now holds by standard arguments; e.g., [66] Thm. 2v), p. 81. ∎

An immediate corollary used several times in the sequel is:

Corollary 8.8. *Under the hypotheses of Theorem 8.7 assume further that for $X \in \mathcal{C}$ cofibrant and $A \in \mathcal{D}$ cofibrant*

$$X \to GA$$

is a weak equivalence if and only if its adjoint $FX \to A$ is a weak equivalence. Then $\mathbf{L}F$ and $\mathbf{R}G$ induce an adjoint equivalence of categories:

$$\mathrm{Ho}(\mathcal{C}) \cong \mathrm{Ho}(\mathcal{D}).$$

Proof: We need to check that $\eta : X \to \mathbf{R}G \circ \mathbf{L}F(X)$ is an isomorphism and $\varepsilon : \mathbf{L}F \circ \mathbf{R}G(A) \to A$ is an isomorphism. Using the notation established in the previous argument, we have a sequence of arrows that define η:

$$X \leftarrow Y \to GZ.$$

Now Y is cofibrant and Z is fibrant, and $FY \to Z$ is a weak equivalence; so $Y \to GZ$ is a weak equivalence and this shows η is an isomorphism. The other argument is identical. ∎

In practice one may not know *a priori* that F and G satisfy the hypotheses of Theorem 8.7. The following result is often useful. We shall assume that the model categories at hand are, in fact, simplicial model categories; however, it is possible to prove the result more generally.

Lemma 8.9. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between simplicial model categories, and suppose F has a right adjoint G. If G preserves fibrations and trivial fibrations, then F preserves cofibrations, trivial cofibrations and weak equivalences between cofibrant objects.*

PROOF: It follows from an adjointness argument that F preserves trivial cofibrations and cofibrations; for example, suppose $j : X \to Y$ is a cofibration in $\mathcal{C}$. To show Fj is a cofibration, one need only solve the lifting problem

$$\begin{tikzcd} FX \arrow[r] \arrow[d, "Fj"'] & A \arrow[d, "q"] \\ FY \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

for every trivial fibration q in $\mathcal{D}$. This problem is adjoint to

$$\begin{tikzcd} X \arrow[r] \arrow[d] & GA \arrow[d, "Gq"] \\ Y \arrow[r] \arrow[ur, dashed] & GB \end{tikzcd}$$

which has a solution by hypothesis.

Now suppose $f : X \to Y$ is a weak equivalence between cofibrant objects. Factor f as $X \xrightarrow{j} Z \xrightarrow{q} Y$ where j is a trivial cofibration and q a trivial fibration. We have just shown Fj is a weak equivalence. Also q is actually a homotopy equivalence: there is a map $s : Y \to Z$ so that $qs = 1_Y$ and $sq \simeq 1_Z$. Here X and Y are cofibrant. We claim that Fq is a homotopy equivalence, so that it is a weak equivalence by Lemma 4.1.

To see that Fq is a homotopy equivalence, note that $F(q)F(s) = 1_{FY}$. Next note that since Z is cofibrant, $Z \otimes \Delta^1$ is a cylinder object for Z and, since F preserves trivial cofibrations $F(Z \otimes \Delta^1)$ is a cylinder object for $F(Z)$. Hence $F(s)F(q) \simeq 1_{FZ}$. ■

REMARK 8.10. As usual, the previous result has an analog that reverses the roles of F and G; namely, if F preserves cofibrations and trivial cofibrations, then G preserves fibrations, trivial fibrations, and weak equivalences between fibrant objects. The proof is the same, *mutatis mutandis.*

EXAMPLE 8.11. Let I be a small category and S^I the category of I diagrams. Then S^I becomes a simplicial model category, where a morphism of diagrams $X \to Y$ is a weak equivalence or fibration if and only if $X(i) \to Y(i)$ is a weak equivalence of I-diagrams. The constant functor $S \to S^I$ preserves fibrations and weak equivalences, so (by Lemma 8.9), the left adjoint

$$F = \varinjlim_I : S^I \to S$$

preserves weak equivalences among cofibrant diagrams. Hence the total left derived functor

$$\mathbf{L}\varinjlim_{I} : \mathrm{Ho}(S^I) \to \mathrm{Ho}(S)$$

exists. This functor is the homotopy colimit and we write $\mathbf{L}\varinjlim_I = \underrightarrow{\mathrm{holim}}$. In a certain sense, made precise by the notion of Kan extensions in Remark 8.4, is the closest approximation to colimit that passes to the homotopy category. In any application it is useful to have an explicit formula for $\underrightarrow{\mathrm{holim}}_I X$ in terms of the original diagram X; this is given by the coend formula

$$\underrightarrow{\mathrm{holim}}_I X = \int^i B(i \downarrow I)^{\mathrm{op}} \otimes X(i).$$

These are studied in detail elsewhere in these notes.

This example, can be vastly generalized. If $\mathcal{C}$ is any cofibrantly generated simplicial model category, $\mathcal{C}^I$ becomes a simplicial model category and one gets

$$\underrightarrow{\mathrm{holim}}_I : \mathrm{Ho}(\mathcal{C}^I) \to \mathrm{Ho}(\mathcal{C})$$

in an analogous manner. ■

9. Homotopy cartesian diagrams.

We return, in this last section, to concepts which are particular to the category of simplicial sets and its close relatives. The theory of homotopy cartesian diagrams of simplicial sets is, at the same time, quite deep and essentially axiomatic. The axiomatic part of the theory is valid in arbitrary categories of fibrant objects such as the category of Kan complexes, while the depth is implicit in the passage from the statements about Kan complexes to the category of simplicial sets as a whole. This passage is non-trivial, even though it is completely standard, because it involves (interchangeably) either Quillen's theorem that the realization of a Kan fibration is a Serre fibration (Theorem I.10.10) or Kan's Ex^∞ construction (see III.4).

A *proper closed model category* $\mathcal{C}$ is a closed model category such that

P1 the class of weak equivalences is closed under base change by fibrations, and

P2 the class of weak equivalences is closed under cobase change by cofibrations.

In plain English, axiom **P1** says that, given a pullback diagram

$$\begin{tikzcd} X \arrow[r, "g_*"] \arrow[d] & Y \arrow[d, "p"] \\ Z \arrow[r, "g"'] & W \end{tikzcd}$$

of $\mathcal{C}$ with p a fibration, if g is a weak equivalence then so is g_*. Dually, axiom **P2** says that, given a pushout diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d] \\ C \arrow[r, "f_*"'] & D \end{tikzcd}$$

with i a cofibration, if f is a weak equivalence then so is f_*.

The category of simplicial sets is a canonical example of a proper closed model category (in fact, a proper simplicial model category). Furthermore, this is the generic example: most useful examples of proper closed model categories inherit their structure from simplicial sets. The assertion that the category of simplicial sets satisfies the two axioms above requires proof, but this proof is in part a formal consequence of the fact that every simplicial set is cofibrant and every topological space is fibrant. The formalism itself enjoys wide applicability, and will be summarized here, now.

A *category of cofibrant objects* is a category $\mathcal{D}$ with all finite coproducts (including an initial object φ), with two classes of maps, called weak equivalences and cofibrations, such that the following axioms are satisfied:

(A) Suppose given a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "h"'] & & Y \arrow[dl, "f"] \\ & Z & \end{tikzcd}$$

in $\mathcal{D}$. If any two of f, g and h are weak equivalences, then so is the third.

(B) The composite of two cofibrations is a cofibration. Any isomorphism is a cofibration.

(C) Pushout diagrams of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & B \arrow[d, "i_*"] \\ C \arrow[r] & D \end{tikzcd}$$

exist in the case where i is a cofibration. Furthermore, i_* is a cofibration which is trivial if i is trivial.

(D) For any object X there is at least one cylinder space $X \otimes I$.

(E) For any object X, the unique map $\emptyset \to X$ is a cofibration.

To explain, a *trivial cofibration* is a morphism of $\mathcal{D}$ which is both a cofibration and a weak equivalence. A *cylinder object* $X \otimes I$ for X is a commutative diagram

$$\begin{tikzcd} X \sqcup X \arrow[r, "\nabla"] \arrow[d, "i=(i_0, i_1)"'] & X \\ X \otimes I \arrow[ur, "\sigma"'] & \end{tikzcd}$$

in which i is a cofibration and σ is a weak equivalence, just like in the context of a closed model category (see Section 1 above). Each of the components i_ϵ of i must therefore be a trivial cofibration.

The definition of category of cofibrant objects is dual to the definition of category of fibrant objects given in Section I.9. All results about categories of fibrant objects therefore imply dual results for categories of cofibrant objects, and conversely. In particular, we immediately have the dual of one of the assertions of Proposition I.9.5:

Proposition 9.1. *The full subcategory of cofibrant objects $\mathcal{C}_c$ in a closed model category $\mathcal{C}$, together with the weak equivalences and cofibrations between them, satisfies the axioms* **(A)** - **(E)** *for a category of cofibrant objects.*

REMARK 9.2. One likes to think that a category of cofibrant objects structure (respectively a category of fibrant objects structure) is half of a closed model structure. This intuition fails, however, because it neglects the power of the axiom **CM4**.

COROLLARY 9.3.

(1) *The category of simplicial sets is a category of cofibrant objects.*
(2) *The category of compactly generated Hausdorff spaces is a category of fibrant objects.*

LEMMA 9.4. *Suppose that $f : A \to B$ is an arbitrary map in a category of cofibrant objects $\mathcal{D}$. Then f has a factorization $f = q \cdot j$, where j is a cofibration and q is left inverse to a trivial cofibration. In particular, q is a weak equivalence.*

PROOF: The proof of this result is the mapping cylinder construction. It's also dual of the classical procedure for replacing a map by a fibration.

Choose a cylinder object

$$\begin{tikzcd} A \sqcup A \arrow[r, "\nabla"] \arrow[d, "{(i_0, i_1)}"'] & A \\ A \otimes I \arrow[ur, "\sigma"'] & \end{tikzcd}$$

for A, and form the pushout diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i_0"'] & B \arrow[d, "i_{0*}"] \\ A \otimes I \arrow[r, "f_*"'] & B_*. \end{tikzcd}$$

Then $(f\sigma) \cdot i_0 = f$, and so there is a unique map $q : B_* \to B$ such that $q \cdot f_* = f\sigma$ and $q \cdot i_{0*} = 1_B$. Then $f = q \cdot (f_* i_1)$.

The composite map $f_* i_1$ is a cofibration, since the diagram

$$\begin{tikzcd} A \sqcup A \arrow[r, "f \sqcup 1_A"] \arrow[d, "{(i_0, i_1)}"'] & B \sqcup A \arrow[d, "{(i_{0*}, f_* i_1)}"] \\ A \otimes I \arrow[r, "f_*"'] & B_* \end{tikzcd}$$

is a pushout. ■

Lemma 9.5. *Suppose that*

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "i"'] & B \arrow[d] \\ C \arrow[r, "u_*"'] & D \end{tikzcd}$$

is a pushout in a category of cofibrant objects $\mathcal{D}$, such that i is a cofibration and u is a weak equivalence. Then the map u_* is a weak equivalence.

Proof: Trivial cofibrations are stable under pushout, so Lemma 9.4 implies that it suffices to assume that there is a trivial cofibration $j : B \to A$ such that $u \cdot j = 1_B$.

Form the pushout diagram

$$\begin{tikzcd} B \arrow[r, "j"] \arrow[d, "j"'] & A \arrow[dd, "j_*"] \\ A \arrow[d, "i"'] & \\ C \arrow[r, "\tilde{j}"'] & \tilde{D} \end{tikzcd}$$

Then $\tilde{j}$ is a trivial cofibration.

Let $f : \tilde{D} \to C$ be the unique map which is determined by the commutative diagram

$$\begin{tikzcd} B \arrow[r, "j"] \arrow[d, "j"'] & A \arrow[dd, "i"] \\ A \arrow[d, "i"'] & \\ C \arrow[r, "1_C"'] & C. \end{tikzcd}$$

Form the prism

$$
\begin{tikzcd}
A \arrow[rr, "u"] \arrow[d, "j_*"'] \arrow[drr, "i"] & & B \arrow[d] \arrow[ddr] & \\
\tilde{D} \arrow[rr, "\tilde{u}"] \arrow[dr, "f"'] & & B_* \arrow[dr, "f_*"'] & \\
& C \arrow[rr, "u_*"'] & & D
\end{tikzcd}
$$

such that the front and back faces are pushouts (ie. push out the triangle on the left along u). Then $\tilde{u}$ is a weak equivalence, since $\tilde{j}$ is a weak equivalence and $u \cdot j = 1_B$. It therefore suffices to show that the map f_* is a weak equivalence.

The bottom face

$$
\begin{tikzcd}
\tilde{D} \arrow[r, "\tilde{u}"] \arrow[d, "f"'] & B_* \arrow[d, "f_*"] \\
C \arrow[r, "u_*"'] & D
\end{tikzcd}
$$

is a pushout, and the map f is a weak equivalence. The morphism

$$
\begin{tikzcd}
& A \arrow[dl, "j_*"'] \arrow[dr, "i"] & \\
\tilde{D} \arrow[rr, "f"'] & & C
\end{tikzcd}
$$

is therefore a weak equivalence in the category $A \downarrow \mathcal{D}$, and the argument of Lemma 9.4 says that this map has a factorization in $A \downarrow \mathcal{D}$ of the form $f = q \cdot j$, where j is a cofibration and q is left inverse to a cofibration. It follows that pushing out along u preserves weak equivalences of $A \downarrow \mathcal{D}$, so that f_* is a weak equivalence of $\mathcal{D}$. ■

COROLLARY 9.6. *The category* $\mathbf{S}$ *of simplicial sets is a proper simplicial model category.*

Proof: Axiom **P2** is a consequence of Lemma 9.5 and Corollary 9.3. The category **CGHaus** of compactly generated Hausdorff spaces is a category of fibrant objects, so the dual of Lemma 9.5 implies Axiom **P1** for that category. One infers **P1** for the simplicial set category from the exactness of the realization functor (Proposition I.2.4), the fact that the realization functor preserves fibrations (Theorem I.10.9), and the fact that the canonical map $\eta : X \to S|X|$ is a weak equivalence for all simplicial sets X (see the proof of Theorem I.11.4). ■

Remark 9.7. Axiom **P1** for the category of simplicial sets can alternatively be seen by observing that Kan's Ex^∞ preserves fibrations and pullbacks (Lemma III.4.5), and preserves weak equivalences as well (Theorem III.4.6). Thus, given a pullback diagram

$$\begin{tikzcd} X \arrow[r, "g_*"] \arrow[d] & Y \arrow[d, "p"] \\ Z \arrow[r, "g"'] & W \end{tikzcd}$$

with p a fibration and g a weak equivalence, if we want to show that g_* is a weak equivalence, it suffices to show that the induced map $\mathrm{Ex}^\infty g_*$ in the pullback diagram

$$\begin{tikzcd} \mathrm{Ex}^\infty X \arrow[r, "\mathrm{Ex}^\infty g_*"] \arrow[d] & \mathrm{Ex}^\infty Y \arrow[d, "\mathrm{Ex}^\infty p"] \\ \mathrm{Ex}^\infty Z \arrow[r, "\mathrm{Ex}^\infty g"'] & \mathrm{Ex}^{infty} W \end{tikzcd}$$

is a weak equivalence. But all of the objects in this last diagram are fibrant and the map $\mathrm{Ex}^\infty g$ is a weak equivalence, so the desired result follows from the dual of Lemma 9.5.

The following result is commonly called the *glueing lemma*. The axiomatic argument for it that is given here is due to Thomas Gunnarsson [40].

LEMMA 9.8. *Suppose given a commutative cube*

(9.9)
$$\begin{tikzcd}
A_1 \arrow[rr, "j_1"] \arrow[dr, "i_1"] \arrow[dd, "f_A"'] & & B_1 \arrow[dr] \arrow[dd, "f_B" near start] & \\
& C_1 \arrow[rr] \arrow[dd, "f_C" near start] & & D_1 \arrow[dd, "f_D"] \\
A_2 \arrow[rr, "j_2" near end] \arrow[dr, "i_2"'] & & B_2 \arrow[dr] & \\
& C_2 \arrow[rr] & & D_2
\end{tikzcd}$$

in a category of cofibrant objects $\mathcal{D}$. Suppose further that the top and bottom faces are pushouts, that i_1 and i_2 are cofibrations, and that the maps f_A, f_B and f_C are weak equivalences. Then f_D is a weak equivalence.

PROOF: It suffices to assume that the maps j_1 and j_2 are cofibrations. To see this, use Lemma 9.4 to factorize j_1 and j_2 as cofibrations followed by weak equivalences, and then use Lemma 9.5 to analyze the resulting map of cubes.

Form the diagram

$$\begin{tikzcd}
A_1 \arrow[rr, "j_1"] \arrow[dr, "i_1"] \arrow[dddd, "f_A"'] & & B_1 \arrow[dr] \arrow[d, "f_{A*}"] & \\
& C_1 \arrow[rr, "j_{1*}"] \arrow[dddd, "f_C" near start] & B' \arrow[dr, "\theta"] \arrow[dd, "n_B"'] & D_1 \arrow[dd, "f_{C*}"] \\
& & & \\
& & & D' \arrow[dd, "n_D"] \\
A_2 \arrow[uuurr] \arrow[rr, "j_2"] \arrow[dr, "i_2"'] & & B_2 \arrow[dr, "i_{2*}"] & \\
& C_2 \arrow[uurr] \arrow[rr] & & D_2.
\end{tikzcd}$$

by pushing out the top face along the left face of the cube (9.9). The square

$$\begin{tikzcd} A_2 \arrow[r] \arrow[d, "i_2"'] & B' \arrow[d, "\theta"] \\ C_2 \arrow[r] & D' \end{tikzcd}$$

is a pushout, so the square

$$\begin{tikzcd} B' \arrow[r, "n_B"] \arrow[d, "\theta"'] & B_2 \arrow[d, "i_{2*}"] \\ D' \arrow[r, "n_D"'] & D_2 \end{tikzcd}$$

is a pushout, and θ is a cofibration. The map f_{A*} is a weak equivalence, since j_1 is a cofibration and f_A is a weak equivalence. Similarly, f_{C*} is a weak equivalence, since j_{1*} is a cofibration and f_C is a weak equivalence. The map $f_B = n_B \cdot f_{A*}$ is assumed to be a weak equivalence, so it follows that n_B is a weak equivalence. Then n_D is a weak equivalence, so $f_D = n_D \cdot f_{C*}$ is a weak equivalence. ∎

The dual of Lemma 9.8 is the *co-glueing lemma* for categories of fibrant objects:

Lemma 9.10. *Suppose given a commutative cube*

(9.11)
$$\begin{tikzcd}[row sep=small, column sep=small] A_1 \arrow[rr] \arrow[dr] \arrow[dd, "f_A"'] & & B_1 \arrow[dr, "p_1"] \arrow[dd, "f_B" near start] & \\ & C_1 \arrow[rr] \arrow[dd, "f_C" near start] & & D_1 \arrow[dd, "f_D"] \\ A_2 \arrow[rr] \arrow[dr] & & B_2 \arrow[dr, "p_2"] & \\ & C_2 \arrow[rr] & & D_2 \end{tikzcd}$$

in a category of fibrant objects $\mathcal{E}$. Suppose further that the top and bottom squares are pullbacks, that the maps p_1 and p_2 are fibrations, and that the maps f_B, f_C and f_D are weak equivalences. Then the map f_A is a weak equivalence.

Either of the two methods of proof of Corollary 9.6 given above (see also Remark 9.7), together with Lemma 9.10, now yield the following expanded consequence for the full category of simplicial sets:

COROLLARY 9.12. *Suppose given a commutative cube*

$$\begin{tikzcd}
X_1 \arrow[rr] \arrow[dr] \arrow[dd, "f_X"'] & & Y_1 \arrow[dr, "p_1"] \arrow[dd, "f_Y" near start] & \\
& W_1 \arrow[rr] \arrow[dd, "f_W" near start] & & Z_1 \arrow[dd, "f_Z"] \\
X_2 \arrow[rr] \arrow[dr] & & Y_2 \arrow[dr, "p_2"] & \\
& W_2 \arrow[rr] & & Z_2
\end{tikzcd}$$

of morphisms in the category $\mathbf{S}$ *of simplicial sets. Suppose further that the top and bottom squares are pullbacks, that the maps* p_1 *and* p_2 *are fibrations, and that the maps* f_Y, f_W *and* f_Z *are weak equivalences. Then the map* f_X *is a weak equivalence.*

Corollary 9.12 is the basis for the theory of homotopy cartesian diagrams in the category of simplicial sets. We say that a commutative square of simplicial set maps

(9.13)
$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\
W \arrow[r] & Z
\end{tikzcd}$$

is *homotopy cartesian* if for any factorization

(9.14)
$$\begin{tikzcd}
Y \arrow[rr, "f"] \arrow[dr, "i"'] & & Z \\
& \tilde{Y} \arrow[ur, "p"'] &
\end{tikzcd}$$

of f into a trivial cofibration i followed by a fibration p the induced map

$$X \xrightarrow{i_*} W \times_Z \tilde{Y}$$

is a weak equivalence.

In fact (and this is the central point), for the diagram (9.13) to be homotopy cartesian, it suffices to find only one such factorization $f = p \circ i$ such that the map i_* is a weak equivalence. This is a consequence of the following:

Lemma 9.15. *Suppose given a commutative diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ W \arrow[r] & Z \end{tikzcd}$$

of morphisms in the category of simplicial sets, and factorizations

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "i_j"'] & & Z \\ & Y_j \arrow[ur, "p_j"'] & \end{tikzcd}$$

of f as trivial cofibration i_j followed by a fibration p_j for $j = 1, 2$. Then the induced map $i_{1*} : X \to W \times_Z Y_1$ is a weak equivalence if and only if the map $i_{2*} : X \to W \times_Z Y_2$ is a weak equivalence.

Proof: There is a lifting θ in the diagram

$$\begin{tikzcd} Y \arrow[r, "i_2"] \arrow[d, "i_1"'] & Y_2 \arrow[d, "p_2"] \\ Y_1 \arrow[r, "p_1"'] \arrow[ur, "\theta"] & Z, \end{tikzcd}$$

by the closed model axioms. Form the commutative cube

$$\begin{tikzcd}
W \times_Z Y_1 \arrow[rr] \arrow[dr] \arrow[dd, "\theta_*"'] & & Y_1 \arrow[dr, "p_1"] \arrow[dd, "\theta" near start] & \\
& W \arrow[rr, crossing over] & & Z \arrow[dd, "1_Z"] \\
W \times_Z Y_2 \arrow[rr] \arrow[dr] & & Y_2 \arrow[dr, "p_2"] & \\
& W \arrow[rr] \arrow[from=uu, crossing over, "1_W" near start] & & Z.
\end{tikzcd}$$

Then the map θ_* is a weak equivalence by Corollary 9.12. There is a commutative diagram

$$\begin{tikzcd}
& W \times_Z Y_1 \arrow[dd, "\theta_*"] \\
X \arrow[ur, "i_{1*}"] \arrow[dr, "i_{2*}"'] & \\
& W \times_Z Y_2,
\end{tikzcd}$$

and the desired result follows. ■

REMARK 9.16. The argument of Lemma 9.15 implies that the definition of homotopy cartesian diagrams can be relaxed further: the diagram (9.13) is homotopy cartesian if and only if there is a factorization (9.14) such that p is a fibration and i is a weak equivalence, and such that the induced map $i_* : X \to W \times_Z \tilde{Y}$ is a weak equivalence.

The way that the definition of homotopy cartesian diagrams has been phrased so far says that the diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\
W \arrow[r, "g"'] & Z
\end{tikzcd}$$

is homotopy cartesian if a map induced by a factorization of the map f into a fibration following a trivial cofibration is a weak equivalence. In fact, it doesn't matter if we factor f or g:

Lemma 9.17. *Suppose given a commutative diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ W \arrow[r, "g"'] & Z \end{tikzcd}$$

in the category of simplicial sets. Suppose also that we are given factorizations

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "i"'] & & Z & W \arrow[rr, "g"] \arrow[dr, "j"'] & & Z \\ & \tilde{Y} \arrow[ur, "p"'] & & & \tilde{W} \arrow[ur, "q"'] & \end{tikzcd}$$

of f and g respectively such that i and j are trivial cofibrations and p and q are fibrations. Then the induced map $i_* : X \to W \times_Z \tilde{Y}$ is a weak equivalence if and only if the map $j_* : X \to \tilde{W} \times_Z Y$ is a weak equivalence.

Proof: There is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "i_*"] \arrow[d, "j_*"'] & W \times_Z \tilde{Y} \arrow[d, "j \times 1"] \\ \tilde{W} \times_Z Y \arrow[r, "1 \times i"'] & \tilde{W} \times_Z \tilde{Y}. \end{tikzcd}$$

The map p is a fibration, so the map $j \times 1$ is a weak equivalence, and q is a fibration, so $1 \times i$ is a weak equivalence, all by Corollary 9.12. ■

The coglueing lemma also has the following general consequence for homotopy cartesian diagrams:

COROLLARY 9.18. *Suppose given a commutative cube*

$$\begin{tikzcd}
X_1 \arrow[rr] \arrow[dr] \arrow[dd, "f_X"'] & & Y_1 \arrow[dr] \arrow[dd, "f_Y" near start] & \\
& W_1 \arrow[rr] \arrow[dd, "f_W" near start] & & Z_1 \arrow[dd, "f_Z"] \\
X_2 \arrow[rr] \arrow[dr] & & Y_2 \arrow[dr] & \\
& W_2 \arrow[rr] & & Z_2
\end{tikzcd}$$

of morphisms in the category $\mathbf{S}$ *of simplicial sets. Suppose further that the top and bottom squares are homotopy cartesian diagrams, and that the maps* f_Y, f_W *and* f_Z *are weak equivalences. Then the map* f_X *is a weak equivalence.*

A *homotopy fibre sequence* of simplicial sets is a homotopy cartesian diagram in $\mathbf{S}$

$$\begin{tikzcd}
X \arrow[r, "j"] \arrow[d] & Y \arrow[d, "f"] \\
* \arrow[r, "x"'] & Z.
\end{tikzcd}$$

In effect, one requires that the composite $f \circ j$ factor through the base point x of Z, and that if $f = p \circ i$ is a factorization of f into a trivial cofibration followed by a fibration, then the canonical map $X \to F$ is a weak equivalence, where F is the fibre of p over x. More colloquially (see also Remark 9.16), this means that X has the homotopy type of the fibre F of any replacement of the map f by a fibration up to weak equivalence. It is common practice to abuse notation and say that

$$X \xrightarrow{j} Y \xrightarrow{f} Z$$

is a homotopy fibre sequence, and mean that these maps are a piece of a homotopy cartesian diagram as above. Every fibration sequence

$$F \to E \to B$$

is plainly a homotopy fibre sequence.

Example 9.19. Suppose that

$$X \xrightarrow{j} Y \xrightarrow{f} Z$$

is a homotopy fibre sequence, relative to a base point x of Z, and that there is a vertex $y \in Y$ such that $f(y) = x$. Suppose that the canonical map $Y \to *$ is a weak equivalence. Then X is weakly equivalent to the loop space $\Omega\tilde{Z}$ for some (and hence any) fibrant model $\tilde{Z}$ for Z. To see this, choose a trivial cofibration $j : Z \to \tilde{Z}$, where $\tilde{Z}$ is a Kan complex, and use the factorization axioms to form the commutative square

$$\begin{tikzcd} Y \arrow[r, "j"] \arrow[d, "f"'] & \tilde{Y} \arrow[d, "p"] \\ Z \arrow[r, "j"'] & \tilde{Z}, \end{tikzcd}$$

where both maps labelled j are trivial cofibrations and p is a fibration. Let F denote the fibre of the fibration p over the image of the base point x in $\tilde{Z}$. Then Corollary 9.18 implies that the induced map $X \to F$ is a weak equivalence. Now consider the diagram

$$\begin{tikzcd} \tilde{Y} \times_{\tilde{Z}} P\tilde{Z} \arrow[r, "pr_R"] \arrow[d, "pr_L"'] & P\tilde{Z} \arrow[d, "\pi"] \\ \tilde{Y} \arrow[r, "p"'] & \tilde{Z}, \end{tikzcd}$$

where $P\tilde{Z}$ is the standard path space for the Kan complex $\tilde{Z}$ and the base point x, and π is the canonical fibration. Then the map $y : * \to \tilde{Y}$ is a weak equivalence, so that the inclusion $\Omega\tilde{Z} \to \tilde{Y} \times_{\tilde{Z}} P\tilde{Z}$ of the fibre of the fibration pr_L is a weak equivalence, by properness, as is the inclusion $F \to \tilde{Y} \times_{\tilde{Z}} P\tilde{Z}$ of the fibre of pr_R. In summary, we have constructed weak equivalences

$$X \xrightarrow{\simeq} F \xrightarrow{\simeq} \tilde{Y} \times_{\tilde{Z}} P\tilde{Z} \xleftarrow{\simeq} \Omega\tilde{Z}.$$

This collection of ideas indicates that it makes sense to define the loop space of a connected simplicial set X to be the loops $\Omega\tilde{X}$ of a fibrant model $\tilde{X}$ for X — the loop space of X is therefore an example of a total right derived functor, in the sense of Section II.7.

Here is a clutch of results that illustrates the formal similarities between homotopy cartesian diagrams and pullbacks:

LEMMA 9.20.

(1) *Suppose that*

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d] & Y \arrow[d] \\ Z \arrow[r, "\beta"'] & W \end{tikzcd}$$

is a commutative diagram in the simplicial set category such that the maps α and β are weak equivalences. Then the diagram is homotopy cartesian.

(2) *Suppose given a commutative diagram*

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathbf{I}"] & X_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathbf{II}"] & X_3 \arrow[d] \\ Y_1 \arrow[r] & Y_2 \arrow[r] & Y_3 \end{tikzcd}$$

in $\mathbf{S}$. *Then*

(a) *if the diagrams* $\mathbf{I}$ *and* $\mathbf{II}$ *are homotopy cartesian then so is the composite diagram* $\mathbf{I} + \mathbf{II}$

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] & X_3 \arrow[d] \\ Y_1 \arrow[r] & Y_3, \end{tikzcd}$$

(b) *if the diagrams* $\mathbf{I} + \mathbf{II}$ *and* $\mathbf{II}$ *are homotopy cartesian, then* $\mathbf{I}$ *is homotopy cartesian.*

The proof of this lemma is left to the reader as an exercise.

The stream of results and definitions beginning with Corollary 9.12 for simplicial sets holds much more generally for arbitrary proper simplicial model categories, in view of the following result (and its dual):

LEMMA 9.21. *Let $\mathcal{C}$ be a proper simplicial model category. Consider a diagram*

$$\begin{tikzcd} D_1 \arrow[d] & C_1 \arrow[l, "j_1"'] \arrow[r] \arrow[d] & X_1 \arrow[d] \\ D_2 & C_2 \arrow[l, "j_2"] \arrow[r] & X_2 \end{tikzcd}$$

where j_1 and j_2 are cofibrations and the three vertical maps are weak equivalences. Then the map $D_1 \cup_{C_1} X_1 \to D_2 \cup_{C_2} X_2$ is a weak equivalence.

PROOF: We use the proper hypothesis to reduce to the case $C_1 = C_2$ and $X_1 = X_2$ and then argue appropriately.

First notice that there is a pushout diagram

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] & X_2 \arrow[d] \\ D_1 \cup_{C_1} X_1 \arrow[r] & D_1 \cup_{C_1} X_2. \end{tikzcd}$$

Since $X_1 \to D_1 \cup_{C_1} X_1$ is a cofibration (because j_1 is) and $X_1 \to X_2$ is a weak equivalence, properness implies $D_1 \cup_{C_1} X_1 \to D_1 \cup_{C_1} X_2$ is a weak equivalence. Thus we may assume $X_1 = X_2 = X$.

Next notice that there is a pushout diagram

$$\begin{tikzcd} C_1 \arrow[r] \arrow[d, "j_1"'] & C_2 \arrow[d] \\ D_1 \arrow[r] & D_1 \cup_{C_1} C_2. \end{tikzcd}$$

Since j_1 is a cofibration and $C_1 \to C_2$ is a weak equivalence, properness implies $D_1 \to D_1 \cup_{C_1} C_2$ is a weak equivalence. Now

$$D_1 \cup_{C_1} X \cong (D_1 \cup_{C_1} C_2) \cup_{C_2} X$$

and we are reduced to considering the diagram

$$\begin{tikzcd} D_1 \cup_{C_1} C_2 \arrow[d] & C_2 \arrow[l] \arrow[r] \arrow[d, "="] & X \arrow[d, "="] \\ D_1 & C_2 \arrow[l] \arrow[r] & X. \end{tikzcd}$$

Note that $C_2 \to D_1 \cup_{C_1} C_2$ is a cofibration and $D_1 \cup_{C_1} C_2 \to D_2$ is a weak equivalence. Thus we may assume $C_1 = C_2 = C$.

Now factor $D_1 \to D_2$ as

$$D_1 \xrightarrow{j} Z \xrightarrow{q} D_2$$

where j is a trivial cofibration and q is a fibration. Then q is a trivial fibration, and the composite $C \to Z$ is a cofibration. There is a pushout diagram

$$\begin{tikzcd} D_1 \arrow[r] \arrow[d, "j"'] & D_1 \cup_C X \arrow[d] \\ Z \arrow[r] & Z \cup_C X. \end{tikzcd}$$

Since j is a trivial cofibration, so is $D_1 \cup_C X \to Z \cup_C X$. We will argue that $Z \cup_C X \to D_2 \cup_C X$ is a homotopy equivalence and we'll be done. To see this last claim, note that there is a left $s : D_2 \to Z$ solving the lifting problem

$$\begin{tikzcd} C \arrow[r] \arrow[d] & Z \arrow[d, "q"] \\ D_2 \arrow[r, "="'] \arrow[ur, dashed] & D_2. \end{tikzcd}$$

Then $qs = 1_{D_2}$ and sq and 1_Z are both solutions to the lifting problem

$$\begin{tikzcd} C \arrow[r] \arrow[d] & Z \arrow[d, "q"] \\ Z \arrow[r, "q"'] \arrow[ur, dashed] & D_2 \end{tikzcd}$$

so sq and 1_Z are homotopic under C. Thus $Z \cup_C X \to D_2 \cup_C X$ has a section induced from s and is a homotopy equivalence. Now invoke Lemma 4.1. ∎

The dual of Lemma 9.21 is an obvious generalization of the statement of Corollary 9.12. The proofs of the statements in 9.15 – 9.18 and Lemma 9.20 are formal consequences of properness for the simplicial set category and Corollary 9.12, and can therefore be promoted to arbitrary proper simplicial model categories $\mathcal{C}$. Thus, we are entitled to say that a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ W \arrow[r] & Z \end{tikzcd}$$

in $\mathcal{C}$ is *homotopy cartesian* if for any factorization

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "i"'] & & Z \\ & \tilde{Y} \arrow[ur, "p"'] & \end{tikzcd}$$

of f into a weak equivalence i followed by a fibration p the induced map

$$X \xrightarrow{i_*} W \times_Z \tilde{Y}$$

is a weak equivalence. Then the standard calculus of homotopy cartesian diagrams in simplicial sets obtains in this extended context.

We close with a further application of categories of cofibrant objects structures. Let $\mathcal{C}$ be a fixed choice of simplicial model category having an adequate supply of colimits. Suppose that β is a limit ordinal, and say that a *cofibrant β-sequence* in $\mathcal{C}$ is a functor $X : \mathbf{Seq}(\beta) \to \mathcal{C}$, such that all objects X_i are cofibrant, each map $X_i \to X_{i+1}$ is a cofibration, and $X_t = \varinjlim_{i<t} X_i$ for all limit ordinals $t < \beta$. The cofibrant β-sequences, with ordinary natural transformations between them, form a category which will be denoted by $\mathcal{C}_\beta$. Say that a map $f : X \to Y$ in $\mathcal{C}_\beta$ is a weak equivalence if all its components $f : X_i \to Y_i$ are weak equivalences of $\mathcal{C}$, and say that $g : A \to B$ is a cofibration of $\mathcal{C}_\beta$ if the maps $g : A_i \to B_i$ are cofibrations of $\mathcal{C}$, as are all induced maps $B_i \cup_{A_i} A_{i+1} \to B_{i+1}$.

LEMMA 9.22. *Let $\mathcal{C}$ be a simplicial model category having all filtered colimits. With these definitions, the category $\mathcal{C}_\beta$ of cofibrant β-sequences in $\mathcal{C}$ satisfies the axioms for a category of cofibrant objects.*

PROOF: Suppose that $A \to B \to C$ are cofibrations of $\mathcal{C}_\beta$. To show that the composite $A \to C$ is a cofibration, observe that the canonical map $C_i \cup_{A_i} A_{i+1} \to C_{i+1}$ has a factorization

$$\begin{tikzcd} C_i \cup_{A_i} A_{i+1} \arrow[rr] \arrow[dr] & & C_{i+1} \\ & C_i \cup_{B_i} B_{i+1} \arrow[ur] & \end{tikzcd}$$

and there is a pushout diagram

$$\begin{tikzcd} B_i \cup_{A_i} A_{i+1} \arrow[r] \arrow[d] & B_{i+1} \arrow[d] \\ C_i \cup_{A_i} A_{i+1} \arrow[r] & C_i \cup_{B_i} B_{i+1}. \end{tikzcd}$$

Suppose that

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & B \arrow[d, "i_*"] \\ C \arrow[r] & D \end{tikzcd}$$

is a pushout diagram of $\mathbf{Seq}(\beta)$-diagrams in $\mathcal{C}$, where A, B and C are cofibrant β-sequences and the map i is a cofibration of same. We show that D is a cofibrant β-sequence and that i_* is a cofibration by observing that there are pushouts

$$\begin{tikzcd} C_i \cup_{A_i} A_{i+1} \arrow[r] \arrow[d] & D_i \cup_{B_i} B_{i+1} \arrow[d] \\ C_{i+1} \arrow[r] & D_{i+1}, \end{tikzcd}$$

and that the maps $D_i \to D_i \cup_{B_i} B_{i+1}$ are cofibrations since B is a cofibrant β-sequence.

Suppose that A is a cofibrant β-sequence, and let K be a simplicial set. Then the functor $A \otimes K : \mathbf{Seq}(\beta) \to \mathcal{C}$ is defined by $(A \otimes K)_i = A_i \otimes K$. The functor $X \mapsto X \otimes K$ preserves cofibrations and filtered colimits of $\mathcal{C}$, so that $A \otimes K$ is a cofibrant β-sequence. Furthermore, if $K \to L$ is a cofibration of $\mathbf{S}$ then the induced map $A \otimes K \to A \otimes L$ is a cofibration of $\mathcal{C}_\beta$: the proof is an instance of **SM7**. It follows that the obvious diagram

$$\begin{tikzcd} A \otimes \partial\Delta^1 \arrow[r] \arrow[d] & A \otimes \Delta^0 \\ A \otimes \Delta^1 \arrow[ur] & \end{tikzcd}$$

is a candidate for the cylinder object required by the category of cofibrant objects structure for the category $\mathcal{C}_\beta$. ∎

Lemma 9.23. *Suppose that $\mathcal{C}$ is a simplicial model category having all filtered colimits. Suppose that $f : A \to B$ is a cofibration and a weak equivalence of $\mathcal{C}_\beta$. Then the induced map*

$$f_* : \varinjlim_{i<\beta} A_i \to \varinjlim_{i<\beta} B_i$$

is a trivial cofibration of $\mathcal{C}$.

Proof: Suppose given a diagram

$$\begin{tikzcd} \varinjlim_{i<\beta} A_i \arrow[r] \arrow[d, "f_*"'] & X \arrow[d, "p"] \\ \varinjlim_{i<\beta} B_i \arrow[r] & Y \end{tikzcd}$$

where p is a trivial fibration of $\mathcal{C}$. We construct a compatible family of lifts

$$\begin{tikzcd} A_i \arrow[r] \arrow[d, "f_i"'] & X \arrow[d, "p"] \\ B_i \arrow[r] \arrow[ur, "\theta_i"] & Y \end{tikzcd} \tag{9.24}$$

as follows:

1) Let θ_s be the map induced by all θ_i for $i < s$ at limit ordinals $s < \beta$.
2) Given a lifting θ_i as in diagram (9.24), form the induced diagram

$$\begin{tikzcd} B_i \cup_{A_i} A_{i+1} \arrow[r, "\theta_*"] \arrow[d, "f_*"'] & X \arrow[d, "p"] \\ B_{i+1} \arrow[r] \arrow[ur, dashed, "\theta_{i+1}"] & Y. \end{tikzcd}$$

The map f_* is a trivial cofibration of $\mathcal{C}$, since f is a cofibration and a weak equivalence of $\mathcal{C}_\beta$, so the indicated lift θ_{i+1} exists. ■

COROLLARY 9.25. *Suppose that $\mathcal{C}$ is a simplicial model category having all filtered colimits, and that $f : X \to Y$ is a weak equivalence of cofibrant β-sequences in $\mathcal{C}$. Then the induced map*

$$f_* : \varinjlim_{i<\beta} X_i \to \varinjlim_{i<\beta} Y_i$$

is a weak equivalence of $\mathcal{C}$.

PROOF: We have it from Lemma 9.22 that $\mathcal{C}_\beta$ is a category of cofibrant objects, and Lemma 9.4 says that $f : X \to Y$ has a factorization $f = q \cdot j$, where j is a cofibration and q is left inverse to a trivial cofibration. Then j is a trivial cofibration since f is a weak equivalence, and so Lemma 9.23 implies that both j and p induce weak equivalences after taking filtered colimits. ■

The dual assertion for Corollary 9.25 is entertaining. Suppose, again, that β is a limit ordinal and that $\mathcal{C}$ is a simplicial model category having enough filtered inverse limits. Define a *fibrant β-tower* in $\mathcal{C}$ to be (contravariant) functor X :

$\mathbf{Seq}(\beta)^{op} \to \mathcal{C}$ such that each X_i is a fibrant object of $\mathcal{C}$, each map $X_{i+1} \to X_i$ is a fibration of $\mathcal{C}$, and $X_t = \varprojlim_{i<t} X_i$ for all limit ordinals $t < \beta$. Then the dual of Lemma 9.22 asserts that, for pointwise weak equivalences and a suitable definition of fibration, the category of fibrant β-towers in $\mathcal{C}$ has a category of fibrant objects structure. The dual of Lemma 9.23 asserts that the inverse limit functor takes trivial fibrations of fibrant β-towers to trivial fibrations of $\mathcal{C}$, and then we have

Lemma 9.26. *Suppose that $\mathcal{C}$ is a simplicial model category having all filtered inverse limits, and that $f : X \to Y$ is a weak equivalence of fibrant β-towers in $\mathcal{C}$. Then the induced map*

$$f_* : \varprojlim_{i<\beta} X_i \to \varprojlim_{i<\beta} Y_i$$

is a weak equivalence of $\mathcal{C}$.

For fibrant β-towers $X : \mathbf{Seq}(\beta)^{op} \to \mathbf{S}$ taking values in simplicial sets, one can take a different point of view, in a different language. In that case, fibrant β-towers are globally fibrant $\mathbf{Seq}(\beta)^{op}$-diagrams, and inverse limits and homotopy inverse limits coincide up to weak equivalence for globally fibrant diagrams, for all β. Homotopy inverse limits preserve weak equivalences, so inverse limits preserve weak equivalences of fibrant β-towers. Homotopy inverse limits and homotopy theories for categories of diagrams will be discussed in Chapters 6 and 7.