Chapter I Simplicial sets

This chapter introduces the basic elements of the homotopy theory of simplicial sets. Technically, the purpose is twofold: to prove that the category of simplicial sets has a homotopical structure in the sense that it admits the structure of a closed model category (Theorem 11.3), and to show that the resulting homotopy theory is equivalent in a strong sense to the ordinary homotopy theory of topological spaces (Theorem 11.4). Insofar as simplicial sets are algebraically defined, and the corresponding closed model structure is combinatorial in nature, we obtain an algebraic, combinatorial model for standard homotopy theory.

The substance of Theorem 11.3 is that we can find three classes of morphisms within the simplicial set category $\mathbf{S}$, called cofibrations, fibrations and weak equivalences, and then demonstrate that the following list of properties is satisfied:

CM1: $\mathbf{S}$ is closed under all finite limits and colimits.

CM2: Suppose that the following diagram commutes in $\mathbf{S}$:

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "h"'] & & Y \arrow[dl, "f"] \\ & Z. & \end{tikzcd}$$

If any two of f, g and h are weak equivalences, then so is the third.

CM3: If f is a retract of g and g is a weak equivalence, fibration or cofibration, then so is f.

CM4: Suppose that we are given a commutative solid arrow diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ V \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where i is a cofibration and p is a fibration. Then the dotted arrow exists, making the diagram commute, if either i or p is also a weak equivalence.

CM5: Any map $f : X \to Y$ may be factored:

(a) $f = p \cdot i$ where p is a fibration and i is a trivial cofibration, and

(b) $f = q \cdot j$ where q is a trivial fibration and j is a cofibration.

The fibrations in the simplicial set category are the Kan fibrations, which are defined by a lifting property that is analogous to the notion of a Serre fibration. The cofibrations are the monomorphisms, and the weak equivalences are morphisms which induce homotopy equivalences of CW-complexes after passage to topological spaces. We shall begin to investigate the consequences of this list of axioms in subsequent chapters — they are the basis of a great deal of modern homotopy theory.

Theorem 11.3 and Theorem 11.4 are due to Quillen [75], but the development given here is different: the results are proved simultaneously, and their proofs jointly depend fundamentally on Quillen's later result that the realization of a Kan fibration is a Serre fibration [76]. The category of simplicial sets is historically the first full algebraic model for homotopy theory to have been found, but the verification of its closed model structure is still the most difficult proof of abstract homotopy theory. These theorems and their proofs effectively summarize all of the classical homotopy theory of simplicial sets, as developed mostly by Kan in the 1950's. Kan's work was a natural outgrowth of the work of Eilenberg and Mac Lane on singular homology theory, and is part of a thread of ideas that used to be called "combinatorial homotopy theory" and which can be traced back to the work of Poincaré at the beginning of the twentieth century.

We give here, in the proof of the main results and the development leading to them, a comprehensive introduction to the homotopy theory of simplicial sets. Simplicial sets are defined, with examples, in Section 1, the functorial relationship with topological spaces via realization and the singular functor is described in Section 2, and we start to describe the combinatorial homotopical structure (Kan fibrations and Kan complexes) in Section 3. We introduce the Gabriel-Zisman theory of anodyne extensions in Section 4: this is the obstruction-theoretic machine that trivializes many potential difficulties related to the function complexes of Section 5, the notion of simplicial homotopy in Section 6, and the discussion of simplicial homotopy groups for Kan complexes in Section 7. The fundamental groupoid for a Kan complex is introduced in Section 8, by way of proving a major result about composition of simplicial sets maps which induce isomorphisms in homotopy groups (Theorem 8.2). This theorem, along with a lifting property result for maps which are simultaneously Kan fibrations and homotopy groups isomorphisms (Theorem 7.10 — later strengthened in Theorem 11.2), is used to demonstrate in Section 9 (Theorem 9.1) that the collection of Kan complexes and maps between them satisfies the axioms for a category of fibrant objects in the sense of Brown [15]. This is a first axiomatic approximation to the desired closed model structure, and is the platform on which the relation with standard homotopy theory is constructed with the introduction of minimal fibrations in Section 10. The basic ideas there are that every Kan fibration has a "minimal model" (Proposition 10.3 and Lemma 10.4), and the Gabriel-Zisman result that minimal fibrations induce Serre fibrations after realization (Theorem 10.9). It is then a relatively simple matter to show that the realization of a Kan fibration is a Serre fibration (Theorem 10.10).

The main theorems are proved in the final section, but Section 10 is the heart of the matter from a technical point of view once all the definitions and elementary properties have been established. We have not heard of a proof of Theorem 11.3 or Theorem 11.4 that avoids minimal fibrations. The minimality concept is very powerful wherever it appears (any rational homotopy theorist would agree), but not much has yet been made of it from a formal point of view.

1. Basic definitions.

Let $\mathbf{\Delta}$ be the category of finite ordinal numbers, with order-preserving maps between them. More precisely, the objects for $\mathbf{\Delta}$ consist of elements $\mathbf{n}$, $n \geq 0$, where $\mathbf{n}$ is a string of relations

$$0 \to 1 \to 2 \to \cdots \to n$$

(in other words $\mathbf{n}$ is a totally ordered set with $n+1$ elements). A morphism $\theta : \mathbf{m} \to \mathbf{n}$ is an order-preserving set function, or alternatively just a functor. We usually commit the abuse of saying that $\mathbf{\Delta}$ is the *ordinal number category*.

A *simplicial set* is a contravariant functor $X : \mathbf{\Delta}^{op} \to \mathbf{Sets}$, where $\mathbf{Sets}$ is the category of sets.

Example 1.1. There is a standard covariant functor

$$\underset{\mathbf{n} \mapsto |\Delta^n|}{\mathbf{\Delta} \to \mathbf{Top}}.$$

The topological *standard n-simplex* $|\Delta^n| \subset \mathbb{R}^{n+1}$ is the space

$$|\Delta^n| = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} | \sum_{i=0}^{n} t_i = 1, t_i \geq 0\},$$

with the subspace topology. The map $\theta_* : |\Delta^n| \to |\Delta^m|$ induced by $\theta : \mathbf{n} \to \mathbf{m}$ is defined by

$$\theta_*(t_0, \dots, t_m) = (s_0, \dots, s_n),$$

where

$$s_i = \begin{cases} 0 & \theta^{-1} = \emptyset \\ \sum_{j\in\theta^{-1}(i)} t_j & \theta^{-1}(i) \neq \emptyset \end{cases}$$

One checks that $\theta \mapsto \theta_*$ is indeed a functor (exercise). Let T be a topological space. The *singular set* $S(T)$ is the simplicial set given by

$$\mathbf{n} \mapsto \hom(|\Delta^n|, T).$$

This is the object that gives the singular homology of the space T.

Among all of the functors $\mathbf{m} \to \mathbf{n}$ appearing in $\mathbf{\Delta}$ there are special ones, namely

$$\begin{array}{lll} d^i : \mathbf{n-1} \to \mathbf{n} & 0 \leq i \leq n & \text{(cofaces)} \\ s^j : \mathbf{n+1} \to \mathbf{n} & 0 \leq j \leq n & \text{(codegeneracies)} \end{array}$$

where, by definition,

$$d^i(0 \to 1 \to \cdots \to n-1) = (0 \to 1 \to \cdots \to i-1 \to i+1 \to \cdots \to n)$$

(ie. compose $i-1 \to i \to i+1$, giving a string of arrows of length $n-1$ in $\mathbf{n}$), and

$$s^j(0 \to 1 \to \cdots \to n+1) = (0 \to 1 \to \cdots \to j \xrightarrow{1} j \to \cdots \to n)$$

(insert the identity 1_j in the j^{th} place, giving a string of length $n+1$ in $\mathbf{n}$). It is an exercise to show that these functors satisfy a list of identities as follows, called the *cosimplicial identities*:

$$(1.2) \qquad \begin{cases} d^j d^i = d^i d^{j-1} & \text{if } i < j \\ s^j d^i = d^i s^{j-1} & \text{if } i < j \\ s^j d^j = 1 = s^j d^{j+1} & \\ s^j d^i = d^{i-1} s^j & \text{if } i > j+1 \\ s^j s^i = s^i s^{j+1} & \text{if } i \leq j \end{cases}$$

The maps d^j, s^i and these relations can be viewed as a set of generators and relations for $\mathbf{\Delta}$ (see [66]). Thus, in order to define a simplicial set Y, it suffices to write down sets Y_n, $n \geq 0$ (sets of n-simplices) together with maps

$$\begin{array}{lll} d_i : Y_n \to Y_{n-1}, & 0 \leq i \leq n & \text{(faces)} \\ s_j : Y_n \to Y_{n+1}, & 0 \leq j \leq n & \text{(degeneracies)} \end{array}$$

satisfying the *simplicial identities*:

$$
\begin{cases}
d_i d_j = d_{j-1} d_i & \text{if } i < j \\
d_i s_j = s_{j-1} d_i & \text{if } i < j \\
d_j s_j = 1 = d_{j+1} s_j & \\
d_i s_j = s_j d_{i-1} & \text{if } i > j+1 \\
s_i s_j = s_{j+1} s_i & \text{if } i \leq j
\end{cases} \tag{1.3}
$$

This is the classical way to write down the data for a simplicial set Y.

From a simplicial set Y, one may construct a simplicial abelian group $\mathbb{Z}Y$ (ie. a contravariant functor $\mathbf{\Delta}^{op} \to \mathbf{Ab}$), with $\mathbb{Z}Y_n$ set equal to the free abelian group on Y_n. $\mathbb{Z}Y$ has associated to it a chain complex, called its *Moore complex* and also written $\mathbb{Z}Y$, with

$$\mathbb{Z}Y_0 \xleftarrow{\partial} \mathbb{Z}Y_1 \xleftarrow{\partial} \mathbb{Z}Y_2 \leftarrow \dots \qquad \text{and}$$

$$\partial = \sum_{i=0}^{n} (-1)^i d_i$$

in degree n. Recall that the integral singular homology groups $H_*(X;\mathbb{Z})$ of the space X are defined to be the homology groups of the chain complex $\mathbb{Z}SX$.

EXAMPLE 1.4. Suppose that $\mathcal{C}$ is a (small) category. The *classifying space* (or nerve) $B\mathcal{C}$ of $\mathcal{C}$ is the simplicial set with

$$B\mathcal{C}_n = \hom_{\mathbf{cat}}(\mathbf{n}, \mathcal{C}),$$

where $\hom_{\mathbf{cat}}(\mathbf{n}, \mathcal{C})$ denotes the set of functors from $\mathbf{n}$ to $\mathcal{C}$. In other words an n-simplex is a string

$$a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} a_n$$

of composeable arrows of length n in $\mathcal{C}$.

We shall see later that there is a topological space $|Y|$ functorially associated to every simplicial set Y, called the realization of Y. The term "classifying space" for the simplicial set $B\mathcal{C}$ is therefore something of an abuse – one really means that $|B\mathcal{C}|$ is the classifying space of $\mathcal{C}$. Ultimately, however, it does not matter; the two constructions are indistinguishable from a homotopy theoretic point of view.

EXAMPLE 1.5. If G is a group, then G can be identified with a category (or groupoid) with one object $*$ and one morphism $g : * \to *$ for each element g of G, and so the classifying space BG of G is defined. Moreover $|BG|$ is an Eilenberg-Mac Lane space of the form $K(G,1)$, as the notation suggests; this is now the standard construction.

EXAMPLE 1.6. Suppose that $\mathcal{A}$ is an exact category, like the category $\mathcal{P}(R)$ of finitely generated projective modules on a ring R (see [78]). Then $\mathcal{A}$ has associated to it a category $Q\mathcal{A}$. The objects of $Q\mathcal{A}$ are those of $\mathcal{A}$. The arrows of $Q\mathcal{A}$ are equivalence classes of diagrams

$$\bullet \twoheadleftarrow \bullet \rightarrowtail \bullet$$

where both arrows are parts of exact sequences of $\mathcal{A}$, and composition is represented by pullback. Then $K_{i-1}(\mathcal{A}) := \pi_i|BQ\mathcal{A}|$ defines the K-groups of $\mathcal{A}$ for $i \geq 1$; in particular $\pi_i|BQ\mathcal{P}(R)| = K_{i-1}(R)$, the i^{th} *algebraic K-group* of the ring R.

EXAMPLE 1.7. The *standard n-simplex, simplicial* Δ^n in the simplicial set category $\mathbf{S}$ is defined by

$$\Delta^n = \hom_{\mathbf{\Delta}}(\ ,\mathbf{n}).$$

In other words, Δ^n is the contravariant functor on $\mathbf{\Delta}$ which is represented by $\mathbf{n}$.

A *map $f : X \to Y$ of simplicial sets* (or, more simply, a *simplicial map*) is the obvious thing, namely a natural transformation of contravariant set-valued functors defined on $\mathbf{\Delta}$. $\mathbf{S}$ will denote the resulting category of simplicial sets and simplicial maps.

The Yoneda Lemma implies that simplicial maps $\Delta^n \to Y$ classify n-simplices of Y in the sense that there is a natural bijection

$$\hom_{\mathbf{S}}(\Delta^n, Y) \cong Y_n$$

between the set Y_n of n-simplices of Y and the set $\hom_{\mathbf{S}}(\Delta^n, Y)$ of simplicial maps from Δ^n to Y (see [66], or better yet, prove the assertion as an exercise). More precisely, write $\iota_n = 1_{\mathbf{n}} \in \hom_{\mathbf{\Delta}}(\mathbf{n},\mathbf{n})$. Then the bijection is given by associating the simplex $\varphi(\iota_n) \in Y_n$ to each simplicial map $\varphi : \Delta^n \to Y$. This means that each simplex $x \in Y_n$ has associated to it a unique simplicial map $\iota_x : \Delta^n \to Y$ such that $\iota_x(\iota_n) = x$. One often writes $x = \iota_x$, since it's usually convenient to confuse the two.

Δ^n contains subcomplexes $\partial\Delta^n$ (*boundary* of Δ^n) and Λ^n_k, $0 \leq k \leq n$ (k^{th} *horn*, really the cone centred on the k^{th} vertex). $\partial\Delta^n$ is the smallest subcomplex of

Δ^n containing the faces $d_j(\iota_n)$, $0 \leq j \leq n$ of the standard simplex ι_n. One finds that

$$\partial \Delta^n_j = \begin{cases} \Delta^n_j & \text{if } 0 \leq j \leq n-1, \\ \text{iterated degeneracies of elements of } \Delta^n_k, \\ \qquad 0 \leq k \leq n-1, \quad \text{if } j \geq n. \end{cases}$$

It is a standard convention to write $\partial \Delta^0 = \emptyset$, where $\emptyset$ is the "unique" simplicial set which consists of the empty set in each degree. $\emptyset$ is an initial object for the simplicial set category $\mathbf{S}$.

The k^{th} horn $\Lambda^n_k \subset \Delta^n$ $(n \geq 1)$ is the subcomplex of Δ^n which is generated by all faces $d_j(\iota_n)$ *except* the k^{th} face $d_k(\iota_n)$. One could represent Λ^2_0, for example, by the picture

$$\begin{tikzcd} & 0 \arrow[dl] \arrow[dr] & \\ 1 & & 2 \end{tikzcd} \quad \subset \quad \begin{tikzcd} & 0 \arrow[dl] \arrow[dr] & \\ 1 \arrow[rr] & & 2 \end{tikzcd} \quad = \quad \Delta^2.$$

2. Realization.

Let **Top** denote the category of topological spaces. To go further, we have to get serious about the realization functor $|\ | : \mathbf{S} \to \mathbf{Top}$. There is a quick way to construct it which uses the *simplex category* $\mathbf{\Delta} \downarrow X$ of a simplicial set X. The objects of $\mathbf{\Delta} \downarrow X$ are the maps $\sigma : \Delta^n \to X$, or simplices of X. An arrow of $\mathbf{\Delta} \downarrow X$ is a commutative diagram of simplicial maps

$$\begin{tikzcd} \Delta^n \arrow[d, "\theta"'] \arrow[dr, "\sigma"] & \\ \Delta^m \arrow[r, "\tau"'] & X \end{tikzcd}$$

Observe that θ is induced by a unique ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$.

LEMMA 2.1. *There is an isomorphism*

$$X \cong \varinjlim_{\substack{\Delta^n \to X \\ \text{in } \mathbf{\Delta} \downarrow X}} \Delta^n.$$

PROOF: The proof is easy; it is really the observation that any functor $\mathcal{C} \to \mathbf{Sets}$, which is defined on a small category $\mathcal{C}$, is a colimit of representable functors.

The *realization* $|X|$ of a simplicial set X is defined by the colimit

$$|X| = \varinjlim_{\substack{\Delta^n \to X \\ \text{in } \mathbf{\Delta} \downarrow X}} |\Delta^n|.$$

in the category of topological spaces. $|X|$ is seen to be functorial in simplicial sets X, by using the fact that any simplicial map $f : X \to Y$ induces a functor $f_* : \mathbf{\Delta} \downarrow X \to \mathbf{\Delta} \downarrow Y$ in the obvious way, by composition with f.

PROPOSITION 2.2. *The realization functor is left adjoint to the singular functor in the sense that there is an isomorphism*

$$\hom_{\mathbf{Top}}(|X|, Y) \cong \hom_{\mathbf{S}}(X, SY)$$

which is natural in simplicial sets X and topological spaces Y.

PROOF: There are isomorphisms

$$\begin{aligned} \hom_{\mathbf{Top}}(|X|, Y) &\cong \varprojlim_{\Delta^n \to X} \hom_{\mathbf{Top}}(|\Delta^n|, Y) \\ &\cong \varprojlim_{\Delta^n \to X} \hom_{\mathbf{S}}(\Delta^n, S(Y)) \\ &\cong \hom_{\mathbf{S}}(X, SY)). \end{aligned}$$

■

Note that $\mathbf{S}$ has all colimits and the realization functor $|\ |$ preserves them, since it has a right adjoint.

PROPOSITION 2.3. *$|X|$ is a CW-complex for each simplicial set X.*

PROOF: Define the n^{th} *skeleton* $\mathrm{sk}_n X$ of X be the subcomplex of X which is generated by the simplices of X of degree $\leq n$. Then

$$X = \bigcup_{n \geq 0} \mathrm{sk}_n X,$$

and there are pushout diagrams

$$\begin{CD} \bigsqcup_{x \in NX_n} \partial\Delta^n @>>> \mathrm{sk}_{n-1} X \\ @VVV @VVV \\ \bigsqcup_{x \in NX_n} \Delta^n @>>> \mathrm{sk}_n X \end{CD}$$

of simplicial sets, where $NX_n \subset X_n$ is the set of *non-degenerate simplices* of degree n. In other words,

$NX_n = \{x \in X_n | x \text{ is not of the form } s_i y \text{ for any } 0 \le i \le n-1 \text{ and } y \in X_{n-1}\}.$

The realization of Δ^n is the space $|\Delta^n|$, since $\mathbf{\Delta} \downarrow \Delta^n$ has terminal object $1 : \Delta^n \to \Delta^n$. Furthermore, one can show that there is a coequalizer

$$\bigsqcup_{0 \le i < j \le n} \Delta^{n-2} \rightrightarrows \bigsqcup_{i=0}^{n} \Delta^{n-1} \to \partial\Delta^n$$

given by the relations $d^j d^i = d^i d^{j-1}$ if $i < j$ (exercise), and so there is a coequalizer diagram of spaces

$$\bigsqcup_{0 \le i < j \le n} |\Delta^{n-2}| \rightrightarrows \bigsqcup_{i=0}^{n} |\Delta^{n-1}| \to |\partial\Delta^n|$$

Thus, the induced map $|\partial\Delta^n| \to |\Delta^n|$ maps $|\partial\Delta^n|$ onto the $(n-1)$-sphere bounding $|\Delta^n|$. It follows that $|X|$ is a filtered colimit of spaces $|\operatorname{sk}_n X|$ where $|\operatorname{sk}_n X|$ is obtained from $|\operatorname{sk}_{n-1} X|$ by attaching n-cells according to the pushout diagram

$$\begin{tikzcd}
\bigsqcup_{x \in NX_n} |\partial\Delta^n| \arrow[r] \arrow[d, hook] & |\operatorname{sk}_{n-1} X| \arrow[d, hook] \\
\bigsqcup_{x \in NX_n} |\Delta^n| \arrow[r] & |\operatorname{sk}_n X|.
\end{tikzcd}$$ ∎

In particular $|X|$ is a *compactly generated Hausdorff space*, and so the realization functor takes values in the category **CGHaus** of all such. We shall interpret $|\ |$ as such a functor. Here is the reason:

PROPOSITION 2.4. *The functor* $|\ | : \mathbf{S} \to \mathbf{CGHaus}$ *preserves finite limits.*

We won't get into the general topology involved in proving this result; a demonstration is given in [33]. Proposition 2.4 avoids the problem that $|X \times Y|$ may not be homeomorphic to $|X| \times |Y|$ in general in the ordinary category of topological spaces, in that it implies that

$$|X \times Y| \cong |X| \times_{Ke} |Y|$$

(Kelley space product = product in **CGHaus**). We lose no homotopical information by working **CGHaus** since, for example, the definition of homotopy groups of a CW-complex does not see the difference between **Top** and **CGHaus**.

3. Kan complexes.

Recall the "presentation"

$$\bigsqcup_{0\le i<j\le n} \Delta^{n-2} \rightrightarrows \bigsqcup_{i=0}^{n} \Delta^{n-1} \to \partial\Delta^n$$

of $\partial\Delta^n$ that was mentioned in the last section. There is a similar presentation for Λ^n_k.

Lemma 3.1. *The "fork" defined by the commutative diagram*

$$\begin{tikzcd}
\Delta^{n-2} \arrow[r, "d^{j-1}"] \arrow[d, "in_{i<j}"'] & \Delta^{n-1} \arrow[d, "in_i"] \arrow[dr, "d^i"] & \\
\bigsqcup_{0\le i<j\le n} \Delta^{n-2} \arrow[r, shift left] \arrow[r, shift right] & \bigsqcup_{i\ne k} \Delta^{n-1} \arrow[r] & \Lambda^n_k \\
\Delta^{n-2} \arrow[r, "d^i"'] \arrow[u, "in_{i<j}"] & \Delta^{n-1} \arrow[u, "in_j"'] \arrow[ur, "d^j"'] &
\end{tikzcd}$$

is a coequalizer in $\mathbf{S}$.

Proof: There is a coequalizer

$$\bigsqcup_{i<j} \Delta^{n-1} \times_{\Lambda^n_k} \Delta^{n-1} \rightrightarrows \bigsqcup_{\substack{i\ne k\\ 0\le i\le n}} \Delta^{n-1} \to \Lambda^n_k.$$

But the fibre product $\Delta^{n-1} \times_{\Lambda^n_k} \Delta^{n-1}$ is isomorphic to

$$\Delta^{n-1} \times_{\Delta^n} \Delta^{n-1} \cong \Delta^{n-2}$$

since the diagram

$$\begin{tikzcd}
\mathbf{n-2} \arrow[r, "d^{j-1}"] \arrow[d, "d^i"'] & \mathbf{n-1} \arrow[d, "d^i"] \\
\mathbf{n-1} \arrow[r, "d^j"'] & \mathbf{n}
\end{tikzcd}$$

is a pullback in $\mathbf{\Delta}$. In effect, the totally ordered set $\{0 \dots \hat{\imath} \dots \hat{\jmath} \dots n\}$ is the intersection of the subsets $\{0 \dots \hat{\imath} \dots n\}$ and $\{0 \dots \hat{\jmath} \dots n\}$ of $\{0 \dots n\}$, and this poset is isomorphic to $\mathbf{n-2}$. ∎

The notation $\{0 \dots \hat{\imath} \dots n\}$ means that i isn't there.

Corollary 3.2. *The set* $\hom_{\mathbf{S}}(\Lambda^n_k, X)$ *of simplicial set maps from* Λ^n_k *to* X *is in bijective correspondence with the set of* n*-tuples* $(y_0, \dots, \hat{y}_k, \dots, y_n)$ *of* $(n-1)$*-simplices* y_i *of* X *such that* $d_i y_j = d_{j-1} y_i$ *if* $i < j$*, and* $i, j \neq k$.

We can now start to describe the internal homotopy theory carried by $\mathbf{S}$. The central definition is that of a fibration of simplicial sets. A map $p : X \to Y$ of simplicial sets is said to be a *fibration* if for every commutative diagram of simplicial set homomorphisms

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook, "i"'] & X \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

there is a map $\theta : \Delta^n \to X$ (dotted arrow) making the diagram commute. i is the obvious inclusion of Λ^n_k in Δ^n.

This requirement was called the *extension condition* at one time (see [58], [67], for example), and fibrations were (and still are) called *Kan fibrations*. The condition amounts to saying that if $(x_0 \dots \hat{x}_k \dots x_n)$ is an n-tuple of simplices of X such that $d_i x_j = d_{j-1} x_i$ if $i < j$, $i, j \neq k$, and there is an n-simplex y of Y such that $d_i y = p(x_i)$, then there is an n-simplex x of X such that $d_i x = x_i$, $i \neq k$, and such that $p(x) = y$. It is usually better to formulate it in terms of diagrams.

The same language may be used to describe Serre fibrations: a continuous map of spaces $f : T \to U$ is said to be a *Serre fibration* if the dotted arrow exists in each commutative diagram of continuous maps

$$\begin{tikzcd} {|\Lambda^n_k|} \arrow[r] \arrow[d, hook] & T \arrow[d, "f"] \\ {|\Delta^n|} \arrow[r] \arrow[ur, dashed] & U \end{tikzcd}$$

making it commute. By adjointness 2.2, all such diagrams may be identified with diagrams

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & S(T) \arrow[d, "S(f)"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & S(U), \end{tikzcd}$$

so that $f : T \to U$ is a Serre fibration if and only if $S(f) : S(T) \to S(U)$ is a (Kan) fibration. This is partial motivation for the definition of fibration of simplicial sets. The simplicial set $|\Lambda^n_k|$ is a strong deformation retract of $|\Delta^n|$, so that we've proved

LEMMA 3.3. *For each space X, the map $S(X) \to *$ is a fibration.*

$*$ is different notation for the simplicial set Δ^0. It consists of a singleton set in each degree, and is therefore a terminal object in the category of simplicial sets.

A *fibrant* simplicial set (or *Kan complex*) is a simplicial set Y such that the canonical map $Y \to *$ is a fibration. Alternatively, Y is a Kan complex if and only if one of the following equivalent conditions is met:

K1: Every map $\alpha : \Lambda^n_k \to Y$ may be extended to a map defined on Δ^n in the sense that there is a commutative diagram

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\alpha"] \arrow[d, hook] & Y \\ \Delta^n \arrow[ur] & \end{tikzcd}$$

K2: For each n-tuple of $(n-1)$-simplices $(y_0 \dots \hat{y}_k \dots y_n)$ of Y such that $d_i y_j = d_{j-1} y_i$ if $i < j$, $i, j \neq k$, there is an n-simplex y such that $d_i y = y_i$.

The standard examples of fibrant simplicial sets are singular complexes, as we've seen, as well as classifying spaces BG of groups G, and simplicial groups. A *simplicial group* H is a simplicial object in the category of groups; this means that H is a contravariant functor from $\mathbf{\Delta}$ to the category $\mathbf{Grp}$ of groups. I generally reserve the symbol e for the identities of the groups H_n, for all $n \geq 0$.

LEMMA 3.4 (MOORE). *The underlying simplicial set of any simplicial group H is fibrant.*

PROOF: Suppose that $(x_0, \dots, x_{k-1}, x_{\ell-1}, x_\ell, \dots, x_n)$, $\ell \geq k+2$, is a family of $(n-1)$-simplices of H which is compatible in the sense that $d_i x_j = d_{j-1} x_i$ for $i < j$ whenever the two sides of the equation are defined. Suppose that there is an n-simplex y of H such that $d_i y = x_i$ for $i \leq k-1$ and $i \geq \ell$. Then the family

$$(e, \dots, \overset{k-1}{e}, \overset{\ell-1}{x_{\ell-1} d_{\ell-1}(y^{-1})}, e, \dots, e)$$

is compatible, and $d_i(s_{\ell-2}(x_{\ell-1} d_{\ell-1} y^{-1}) y) = x_i$ for $i \leq k-1$ and $i \geq \ell - 1$. This is the inductive step in the proof of the Proposition. ■

Recall that a *groupoid* is a category in which every morphism is invertible. Categories associated to groups as above are obvious examples, so that the following result specializes to the assertion that classifying spaces of groups are Kan complexes.

LEMMA 3.5. *Suppose that G is a groupoid. Then BG is fibrant.*

PROOF: If $\mathcal{C}$ is a small category, then its nerve $B\mathcal{C}$ is a 2-coskeleton in the sense that the set of simplicial maps $f : X \to B\mathcal{C}$ is in bijective correspondence with commutative (truncated) diagrams

$$\begin{tikzcd}
X_2 \arrow[r, "f_2"] \arrow[d, shift right] & B\mathcal{C}_2 \arrow[d, shift right] \\
X_1 \arrow[r, "f_1"] \arrow[u, shift right] \arrow[d, shift right] & B\mathcal{C}_1 \arrow[u, shift right] \arrow[d, shift right] \\
X_0 \arrow[r, "f_0"] \arrow[u, shift right] & B\mathcal{C}_0 \arrow[u, shift right]
\end{tikzcd}$$

in which the vertical maps are the relevant simplicial structure maps. It suffices to prove this for $X = \Delta^n$ since X is a colimit of simplices. But any simplicial map $f : \Delta^n \to B\mathcal{C}$ can be identified with a functor $f : \mathbf{n} \to \mathcal{C}$, and this functor is completely specified by its action on vertices (f_0), and morphisms (f_1), and the requirement that f respects composition (f_2, and $d_i f_2 = f_1 d_i$). Another way of saying this is that a simplicial map $X \to B\mathcal{C}$ is completely determined by its restriction to $\mathrm{sk}_2 X$.

The inclusion $\Lambda^n_k \subset \Delta^n$ induces an isomorphism

$$\mathrm{sk}_{n-2} \Lambda^n_k \cong \mathrm{sk}_{n-2} \Delta^n.$$

To see this, observe that every simplex of the form $d_i d_j \iota_n$, $i < j$, is a face of some $d_r \iota_n$ with $r \neq k$: if $k \neq i, j$ use $d_i(d_j \iota_n)$, if $k = i$ use $d_k(d_j \iota_n)$, and if $k = j$ use $d_i d_k \iota_n = d_{k-1}(d_i \iota_n$. It immediately follows that the extension problem

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\alpha"] \arrow[d, hook] & BG \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

is solved if $n \geq 4$, for in that case $\mathrm{sk}_2 \Lambda^n_k = \mathrm{sk}_2 \Delta^n$.

Suppose that $n = 3$, and consider the extension problem

$$\begin{tikzcd} \Lambda^3_0 \arrow[r, "\alpha"] \arrow[d, hook] & BG \\ \Delta^3 \arrow[ur, dashed] & \end{tikzcd}$$

Then $\mathrm{sk}_1 \Lambda^3_0 = \mathrm{sk}_1 \Delta^3$ and so we are entitled to write $\alpha_1 : a_0 \to a_1$, $\alpha_2 : a_1 \to a_2$ and $\alpha_3 : a_2 \to a_3$ for the images under the simplicial map α of the 1-simplices $0 \to 1$, $1 \to 2$ and $2 \to 3$, respectively. Write $x : a_1 \to a_3$ for the image of $1 \to 3$ under α. Then the boundary of $d_0 \iota_3$ maps to the graph

$$\begin{tikzcd} a_1 \arrow[rr, "\alpha_2"] \arrow[dr, "x"'] & & a_2 \arrow[dl, "\alpha_3"] \\ & a_3 & \end{tikzcd}$$

in the groupoid G under α, and this graph bounds a 2-simplex of BG if and only if $x = \alpha_3 \alpha_2$ in G. But the images of the 2-simplices $d_2 \iota_3$ and $d_1 \iota_3$ under α

together determine a commutative diagram

$$\begin{tikzcd}
& a_0 \arrow[dl, "\alpha_1"'] \arrow[dr, "\alpha_2\alpha_1"] \arrow[dd, "\alpha_3(\alpha_2\alpha_1)"] & \\
a_1 \arrow[dr, "x"'] & & a_2 \arrow[dl, "\alpha_3"] \\
& a_3 &
\end{tikzcd}$$

in G, so that

$$x\alpha_1 = \alpha_3(\alpha_2\alpha_1),$$

and $x = \alpha_3\alpha_2$, by right cancellation. It follows that the simplicial map $\alpha : \Lambda^3_0 \to BG$ extends to $\partial\Delta^3 = \mathrm{sk}_2\,\Delta^3$, and the extension problem is solved.

The other cases corresponding to the inclusions $\Lambda^3_i \subset \Delta^3$ are similar.

If $n = 2$, then, for example, a simplicial map $\alpha : \Lambda^2_0 \to BG$ can be identified with a diagram

$$\begin{tikzcd}
& a_0 \arrow[dl, "\alpha_1"'] \arrow[dr, "x"] & \\
a_1 & & a_2
\end{tikzcd}$$

and α can be extended to a 2-simplex of BG if and only if there is an arrow $\alpha_2 : a_1 \to a_2$ of G such that $\alpha_2\alpha_1 = x$. But obviously $\alpha_2 = x\alpha_1^{-1}$ does the trick. The other cases in dimension 2 are similar. ■

The standard n-simplex $\Delta^n = B\mathbf{n}$ fails to be fibrant for $n \geq 2$, precisely because the last step in the proof of Lemma 3.5 fails in that case.

4. Anodyne extensions.

The homotopy theory of simplicial sets is based on the definition of fibration given above. Originally, all of the work done in this area was expressed in terms of the extension condition, and some rather grisly-looking combinatorics was involved. The Gabriel-Zisman theory of anodyne extensions [33] gives a way to short-circuit most of the pain.

A class M of (pointwise) monomorphisms of $\mathbf{S}$ is said to be *saturated* if the following conditions are satisfied:

A: All isomorphisms are in M.

B: M is closed under pushout in the sense that, in a pushout square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & C \arrow[d, "i_*"] \\ B \arrow[r] & B \cup_A C, \end{tikzcd}$$

if $i \in M$ then so is i_* (Exercise: Show that i_* is monic).

C: Each retract of an element of M is in M. This means that, given a diagram

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "i'"] \arrow[rr, bend left, "1"] & A \arrow[r] \arrow[d, "i"] & A' \arrow[d, "i'"] \\ B' \arrow[r] \arrow[rr, bend right, "1"'] & B \arrow[r] & B', \end{tikzcd}$$

if i is in M then i' is in M.

D: M is closed under countable compositions and arbitrary direct sums, meaning respectively that:

D1: Given

$$A_1 \xrightarrow{i_1} A_2 \xrightarrow{i_2} A_3 \xrightarrow{i_3} \ldots$$

with $i_j \in M$, the canonical map $A_1 \to \varinjlim A_i$ is in M.

D2: Given $i_j : A_j \to B_j$ in M, $j \in I$, the map

$$\sqcup\, i_j : \bigsqcup_{j \in I} A_j \to \bigsqcup_{j \in I} B_j$$

is in M.

A map $p : X \to Y$ is said to have the *right lifting property* (RLP is the standard acronym) with respect to a class of monomorphisms M if in every solid arrow diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

with $i \in M$ the dotted arrow exists making the diagram commute.

LEMMA 4.1. *The class M_p of all monomorphisms which have the left lifting property (LLP) with respect to a fixed simplicial map $p : X \to Y$ is saturated.*

PROOF: (trivial) For example, we prove the axiom **B**. Suppose given a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & C \arrow[r] \arrow[d, "j"'] & X \arrow[d, "p"] \\ B \arrow[r] & B \cup_A C \arrow[r] & Y, \end{tikzcd}$$

where the square on the left is a pushout. Then there is a map $\theta : B \to X$ such that the "composite" diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, "\theta"] & Y \end{tikzcd}$$

commutes. But then θ induces the required lifting $\theta_* : B \cup_A C \to X$ by the universal property of the pushout. ■

The *saturated class M_B generated by a class of monomorphisms B* is the intersection of all saturated classes containing B. One also says that M_B is the *saturation* of B.

Consider the following three classes of monomorphisms:

$\mathbf{B_1}$:= the set of all inclusions $\Lambda^n_k \subset \Delta^n$, $0 \leq k \leq n$, $n > 0$

$\mathbf{B_2}$:= the set of all inclusions

$$(\Delta^1 \times \partial\Delta^n) \cup (\{e\} \times \Delta^n) \subset (\Delta^1 \times \Delta^n), \qquad e = 0, 1$$

$\mathbf{B_3}$:= the set of all inclusions

$$(\Delta^1 \times Y) \cup (\{e\} \times X) \subset (\Delta^1 \times X),$$

where $Y \subset X$ is an inclusion of simplicial sets, and $e = 0, 1$.

PROPOSITION 4.2. *The classes* $\mathbf{B_1}$, $\mathbf{B_2}$ *and* $\mathbf{B_3}$ *have the same saturation.*

$M_{\mathbf{B_1}}$ is called the class of *anodyne extensions.*

COROLLARY 4.3. *A fibration is a map which has the right lifting property with respect to all anodyne extensions.*

PROOF OF PROPOSITION 4.2: We shall show only that $M_{\mathbf{B_2}} = M_{\mathbf{B_1}}$; it is relatively easy to see that $M_{\mathbf{B_2}} = M_{\mathbf{B_3}}$, since a simplicial set X can be built up from a subcomplex Y by attaching cells. To show that $M_{\mathbf{B_2}} \subset M_{\mathbf{B_1}}$, observe that any saturated set is closed under finite composition. The simplicial set $\Delta^n \times \Delta^1$ has non-degenerate simplices

$$h_j : \Delta^{n+1} \to \Delta^n \times \Delta^1, \qquad 0 \leq j \leq n,$$

where the h_j may be identified with the strings of morphisms

$$\begin{array}{ccccccc}
(0,0) \longrightarrow (0,1) \longrightarrow \dots \longrightarrow & (0,j) & & \\
& \downarrow & & \\
& (1,j) & \longrightarrow \dots \longrightarrow (1,n)
\end{array}$$

of length $n+1$ in $\mathbf{n} \times \mathbf{1}$ (anything longer must have a repeat). One can show that there are commutative diagrams

(4.4)
$$\begin{tikzcd}
\Delta^n \arrow[r, "d^0"] & \Delta^{n+1} \arrow[d, "h_0"] \\
\Delta^n \times \{1\} \arrow[u, "\cong"] \arrow[r, hook] & \Delta^n \times \Delta^1
\end{tikzcd}
\qquad
\begin{tikzcd}
\Delta^n \arrow[r, "d^{n+1}"] & \Delta^{n+1} \arrow[d, "h_n"] \\
\Delta^n \times \{0\} \arrow[u, "\cong"] \arrow[r, hook] & \Delta^n \times \Delta^1
\end{tikzcd}$$

$$
(4.5) \qquad \begin{cases}
\begin{array}{ccc}
\Delta^n & \xrightarrow{d^i} & \Delta^n \\
{\scriptstyle h_{j-1}}\downarrow & & \downarrow{\scriptstyle h_j} \\
\Delta^{n-1}\times\Delta^1 & \xrightarrow[d^i\times 1]{} & \Delta^n\times\Delta^1
\end{array} & \text{if } i<j \\[2ex]
\begin{array}{ccc}
\Delta^n & \xrightarrow{d^{j+1}} & \Delta^{n+1} \\
{\scriptstyle d^{j+1}}\downarrow & & \downarrow{\scriptstyle h_{j+1}} \\
\Delta^{n+1} & \xrightarrow[h_j]{} & \Delta^n\times\Delta^1
\end{array} & \\[2ex]
\begin{array}{ccc}
\Delta^n & \xrightarrow{d^i} & \Delta^{n+1} \\
{\scriptstyle h_j}\downarrow & & \downarrow{\scriptstyle h_j} \\
\Delta^{n-1}\times\Delta^1 & \xrightarrow[d^{i-1}\times 1]{} & \Delta^n\times\Delta^1
\end{array} & \text{if } i>j+1.
\end{cases}
$$

Moreover $d_{j+1}h_j \notin \partial\Delta^n\times\Delta^1$ for $j\geq 0$ since it projects to ι_n under the projection map $\Delta^n\times\Delta^1\to\Delta^n$. Finally, $d_{j+1}h_j$ is not a face of h_i for $j\geq i+1$ since it has vertex $(0,j)$.

Let $(\Delta^n\times\Delta^1)^{(i)}$, $i\geq 1$ be the smallest subcomplex of $\Delta^n\times\Delta^1$ containing $\partial\Delta^n\times\Delta^1$ and the simplices $h_0,\dots,h_i$. Then $(\Delta^n\times\Delta^1)^{(n)}=\Delta^n\times\Delta^1$ and there is a sequence of pushouts, each having the form

$$
\begin{array}{ccc}
\Lambda^{n+1}_{i+2} & \xrightarrow{(d_0h_{i+1},\dots,\widehat{d_{i+2}h_{i+1}},\dots,d_{n+1}h_{i+1})} & (\Delta^n\times\Delta^1)^{(i)} \\
\downarrow & & \downarrow \\
\Delta^{n+1} & \xrightarrow[h_{i+1}]{} & (\Delta^n\times\Delta^1)^{(i+1)}
\end{array}
\qquad n-1\geq i\geq -1
$$

by the observation above.

To see that $M_{\mathbf{B_1}}\subset M_{\mathbf{B_2}}$, suppose that $k<n$, and construct the functors

$$\mathbf{n}\xrightarrow{i}\mathbf{n}\times\mathbf{1}\xrightarrow{r_k}\mathbf{n},$$

where $i(j)=(j,1)$ and r_k is defined by the diagram

$$
\begin{array}{ccccccccccccccc}
0 & \to & 1 & \to & \dots & \to & k-1 & \to & k & \to & k & \to & \dots & \to & k \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
0 & \to & 1 & \to & \dots & \to & k-1 & \to & k & \to & k+1 & \to & \dots & \to & n
\end{array}
$$

in $\mathbf{n}$. Then $r \cdot i = 1_{\mathbf{n}}$, and r and i induce a retraction diagram of simplicial set maps

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d, hook] & (\Lambda^n_k \times \Delta^1) \cup (\Delta^n \times \{0\}) \arrow[r] \arrow[d, hook] & \Lambda^n_k \arrow[d, hook] \\
\Delta^n \arrow[r] & \Delta^n \times \Delta^1 \arrow[r] & \Delta^n
\end{tikzcd}$$

(apply the classifying space functor B). It follows that the inclusion $\Lambda^n_k \subset \Delta^n$ is in $M_{\mathbf{B_2}}$ if $k < n$.

Similarly, if $k > 0$, then the functor $v_k : \mathbf{n} \times \mathbf{1} \to \mathbf{n}$ defined by the diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 1 \arrow[r] \arrow[d] & \dots \arrow[r] & k \arrow[r] \arrow[d] & k+1 \arrow[r] \arrow[d] & \dots \arrow[r] & n \arrow[d] \\
k \arrow[r] & k \arrow[r] & \dots \arrow[r] & k \arrow[r] & k+1 \arrow[r] & \dots \arrow[r] & n
\end{tikzcd}$$

may be used to show that the inclusion $\Lambda^n_k \subset \Delta^n$ is a retract of

$$(\Lambda^n_k \times \Delta^1) \cup (\Delta^n \times \{1\}).$$

Thus, $\Lambda^n_k \subset \Delta^n$ is in the class $M_{\mathbf{B_2}}$ for all n and k. ■

Corollary 4.6. *Suppose that $i : K \hookrightarrow L$ is an anodyne extension and that $Y \hookrightarrow X$ is an arbitrary inclusion. Then the induced map*

$$(K \times X) \cup (L \times Y) \to (L \times X)$$

is an anodyne extension.

Proof: The set of morphisms $K' \to L'$ such that

$$(K' \times X) \cup (L' \times Y) \to (L' \times X)$$

is anodyne is a saturated set. Consider the inclusion

$$(\Delta^1 \times Y') \cup (\{e\} \times X') \subset (\Delta^1 \times X') \qquad (Y' \subset X')$$

and the induced inclusion

$$\begin{tikzcd}
(((\Delta^1 \times Y') \cup (\{e\} \times X')) \times X) \cup ((\Delta^1 \times X') \times Y) \arrow[r, hook] \arrow[d, "\cong"'] & ((\Delta^1 \times X') \times X) \arrow[d, "\cong"] \\
(((\Delta^1 \times ((Y' \times X) \cup (X' \times Y))) \cup (\{e\} \times (X' \times X)) & \Delta^1 \times (X' \times X)
\end{tikzcd}$$

This inclusion is anodyne, and so the saturated set in question contains all anodyne morphisms. ■

5. Function complexes.

Let X and Y be simplicial sets. The *function complex* $\mathbf{Hom}(X,Y)$ is the simplicial set defined by

$$\mathbf{Hom}(X,Y)_n = \hom_{\mathbf{S}}(X \times \Delta^n, Y).$$

If $\theta : \mathbf{m} \to \mathbf{n}$ is an ordinal number map, then the induced function

$$\theta^* : \mathbf{Hom}(X,Y)_n \to \mathbf{Hom}(X,Y)_m$$

is defined by

$$(X \times \Delta^n \xrightarrow{f} Y) \mapsto (X \times \Delta^m \xrightarrow{1\times\theta} X \times \Delta^n \xrightarrow{f} Y).$$

In other words, one thinks of $X \times \Delta^n$ as a cosimplicial "space" in the obvious way.

There is an evaluation map

$$ev : X \times \mathbf{Hom}(X,Y) \to Y$$

defined by $(x,f) \mapsto f(x,\iota_n)$. To show, for example, that ev commutes with face maps d_j, one has to check that

$$f \cdot (1 \times d^j)(d_j x, \iota_{n-1}) = d_j f(x, \iota_n).$$

But

$$f \cdot (1 \times d^j)(d_j x, \iota_{n-1}) = f(d_j x, d_j \iota_n) = d_j f(x, \iota_n).$$

More generally, ev commutes with all simplicial structure maps and is thus a simplicial set map which is natural in X and Y.

PROPOSITION 5.1 (EXPONENTIAL LAW). *The function*

$$ev_* : \hom_{\mathbf{S}}(K, \mathbf{Hom}(X,Y)) \to \hom_{\mathbf{S}}(X \times K, Y),$$

which is defined by sending the simplicial map $g : K \to \mathbf{Hom}(X,Y)$ to the composite

$$X \times K \xrightarrow{1\times g} X \times \mathbf{Hom}(X,Y) \xrightarrow{ev} Y,$$

is a bijection which is natural in K, X and Y.

PROOF: The inverse of ev_* is the map

$$\hom_{\mathbf{S}}(X \times K, Y) \to \hom_{\mathbf{S}}(K, \mathbf{Hom}(X, Y))$$

defined by sending $g : X \times K \to Y$ to the map $g_* : K \to \mathbf{Hom}(X.Y)$, where, for $x \in K_n$, $g_*(x)$ is the composite

$$X \times \Delta^n \xrightarrow{1 \times \iota_x} X \times K \xrightarrow{g} Y. \qquad \blacksquare$$

The relation between function complexes and the homotopy theory of simplicial sets is given by

PROPOSITION 5.2. *Suppose that $i : K \hookrightarrow L$ is an inclusion of simplicial sets and $p : X \to Y$ is a fibration. Then the map*

$$\mathbf{Hom}(L, X) \xrightarrow{(i^*, p_*)} \mathbf{Hom}(K, X) \times_{\mathbf{Hom}(K,Y)} \mathbf{Hom}(L, Y),$$

which is induced by the diagram

$$\begin{tikzcd}
\mathbf{Hom}(L, X) \arrow[r, "p_*"] \arrow[d, "i^*"'] & \mathbf{Hom}(L, Y) \arrow[d, "i^*"] \\
\mathbf{Hom}(K, X) \arrow[r, "p_*"'] & \mathbf{Hom}(K, Y),
\end{tikzcd}$$

is a fibration.

PROOF: Diagrams of the form

$$\begin{tikzcd}
\Lambda^n_k \arrow[rr] \arrow[d, hook] & & \mathbf{Hom}(L, X) \arrow[d, "(i^*, p_*)"] \\
\Delta^n \arrow[r] & \mathbf{Hom}(K, X) \times_{\mathbf{Hom}(K,Y)} \mathbf{Hom}(L, Y) &
\end{tikzcd}$$

may be identified with diagrams

$$\begin{tikzcd}
(\Lambda^n_k \times L) \cup_{(\Lambda^n_k \times K)} (\Delta^n \times K) \arrow[r] \arrow[d, hook, "j"'] & X \arrow[d, "p"] \\
\Delta^n \times L \arrow[r] & Y
\end{tikzcd}$$

by the Exponential Law (Proposition 5.1). But j is an anodyne extension by 4.6, so the required lifting exists. $\blacksquare$

COROLLARY 5.3.

(1) *If $p : X \to Y$ is a fibration, then so is $p_* : \mathbf{Hom}(K, X) \to \mathbf{Hom}(K, Y)$*
(2) *If X is fibrant then the induced map $i^* : \mathbf{Hom}(L, X) \to \mathbf{Hom}(K, X)$ is a fibration.*

PROOF:

(1) The diagram

$$\begin{tikzcd}
\mathbf{Hom}(K, Y) \arrow[r, "1"] \arrow[d] & \mathbf{Hom}(K, Y) \arrow[d] \\
* \arrow[r] & *
\end{tikzcd}$$

is a pullback, and the following commutes:

$$\begin{tikzcd}
\mathbf{Hom}(K, X) \arrow[dr] \arrow[drr, "p_*"] \arrow[ddr] & & \\
& \mathbf{Hom}(K, Y) \arrow[r, "1"'] \arrow[d] & \mathbf{Hom}(K, Y) \arrow[d] \\
& \mathbf{Hom}(\emptyset, X) \arrow[r, "p_*"'] \arrow[d, equal] & \mathbf{Hom}(\emptyset, Y) \arrow[d, equal] \\
& * & *
\end{tikzcd}$$

for a uniquely determined choice of map $\mathbf{Hom}(K, Y) \to \mathbf{Hom}(\emptyset, X)$.

(2) There is a commutative diagram

$$\begin{tikzcd}
\mathbf{Hom}(L, X) \arrow[dr] \arrow[drr] \arrow[ddr, "i^*"'] & & \\
& \mathbf{Hom}(K, X) \arrow[r] \arrow[d, "1"'] & \mathbf{Hom}(L, *) = * \arrow[d] \\
& \mathbf{Hom}(K, X) \arrow[r] & \mathbf{Hom}(K, *) = *
\end{tikzcd}$$

where the inner square is a pullback. ∎

6. Simplicial homotopy.

Let $f, g : K \to X$ be simplicial maps. We say that there is a *simplicial homotopy* $f \xrightarrow{\simeq} g$ from f to g if there is a commutative diagram

$$\begin{tikzcd}
K \times \Delta^0 = K \arrow[d, "1 \times d^1"'] \arrow[dr, "f"] & \\
K \times \Delta^1 \arrow[r, "h"] & X \\
K \times \Delta^0 = K \arrow[u, "1 \times d^0"] \arrow[ur, "g"'] &
\end{tikzcd}$$

The map h is called a *homotopy*.

It's rather important to note that the commutativity of the diagram defining the homotopy h implies that $h(x,0) = f(x)$ and $h(x,1) = g(x)$ for all simplices $x \in K$. We have given a definition of homotopy which is intuitively correct elementwise – it is essentially the reverse of the definition that one is usually tempted to write down in terms of face (or coface) maps.

Suppose $i : L \subset K$ denotes an inclusion and that the restrictions of f and g to L coincide. We say that there is a simplicial homotopy from f to g, (rel L) and write $f \xrightarrow{\simeq} g$ (rel L), if the diagram exists above, and the following commutes as well:

$$\begin{tikzcd}
K \times \Delta^1 \arrow[r, "h"] & X \\
L \times \Delta^1 \arrow[u, hook, "i \times 1"] \arrow[r, "pr_L"'] & L \arrow[u, "\alpha"']
\end{tikzcd}$$

where $\alpha = f|_L = g|_L$, and pr_L is projection onto the left factor (pr_R will denote projection on the right). A homotopy of the form

$$L \times \Delta^1 \xrightarrow{pr_L} L \xrightarrow{\alpha} X$$

is called a *constant homotopy* (at α).

The homotopy relation may fail to be an equivalence relation in general. Consider the maps $\iota_0, \iota_1 : \Delta^0 \rightrightarrows \Delta^n$, $(n \geq 1)$, which classify the vertices 0 and 1, respectively, of Δ^n. There is a simplex $[0,1] : \Delta^1 \to \Delta^n$ determined by these vertices, and so $\iota_0 \xrightarrow{\simeq} \iota_1$ (alternatively, $0 \xrightarrow{\simeq} 1$). But there is no 1-simplex which could give a homotopy $\iota_1 \xrightarrow{\simeq} \iota_0$, since $0 \leq 1$. This observation provides a second means (see Lemma 3.5) of seeing that Δ^n not fibrant, since we can prove

LEMMA 6.1. *Suppose that X is a fibrant simplicial set. Then simplicial homotopy of vertices $x : \Delta^0 \to X$ of X is an equivalence relation.*

PROOF: There is a homotopy $x \xrightarrow{\simeq} y$ if and only if there is a 1-simplex v of X such that $d_1 v = x$ and $d_0 v = y$ (alternativley $\partial v = (y, x)$; in general the *boundary* $\partial\sigma$ of an n-simplex σ is denoted by $\partial\sigma = (d_0\sigma, \dots, d_n\sigma)$). But then the equation $\partial(s_0 x) = (x, x)$ gives the reflexivity of the homotopy relation.

Suppose that $\partial v_2 = (y, x)$ and $\partial v_0 = (z, y)$. Then $d_0 v_2 = d_1 v_0$, and so v_0 and v_2 determine a map $(v_0, \ , v_2) : \Lambda^2_1 \to X$ in the obvious way. Choose a lifting

$$\begin{tikzcd} \Lambda^2_1 \arrow[r, "{(v_0,\ ,v_2)}"] \arrow[d, hook] & X. \\ \Delta^2 \arrow[ur, "\theta"'] & \end{tikzcd}$$

Then

$$\begin{aligned} \partial(d_1\theta) &= (d_0 d_1 \theta, d_1 d_1 \theta) \\ &= (d_0 d_0 \theta, d_1 d_2 \theta) \\ &= (z, x), \end{aligned}$$

and so the relation is transitive. Finally, given $\partial v_2 = (y, x)$, set $v_1 = s_0 x$. Then $d_1 v_1 = d_1 v_2$ and so v_1 and v_2 define a map $(\ , v_1, v_2) : \Lambda^2_0 \to X$. Choose an extension

$$\begin{tikzcd} \Lambda^2_0 \arrow[r, "{(\ ,v_1,v_2)}"] \arrow[d, hook] & X. \\ \Delta^2 \arrow[ur, "\theta'"'] & \end{tikzcd}$$

Then

$$\begin{aligned}\partial(d_0\theta') &= (d_0 d_0\theta', d_1 d_0\theta') \\ &= (d_0 d_1\theta', d_0 d_2\theta') \\ &= (x, y),\end{aligned}$$

and the relation is symmetric. ■

COROLLARY 6.2. *Suppose X is fibrant and that $L \subset K$ is an inclusion of simplicial sets. Then*

(a) *homotopy of maps $K \to X$ is an equivalence relation, and*
(b) *homotopy of maps $K \to X$ (rel L) is an equivalence relation.*

PROOF: (a) is a special case of (b), with $L = \emptyset$. But homotopy of maps $K \to X$ (rel L) corresponds to homotopy of vertices in the fibres of the Kan fibration

$$i^* : \mathbf{Hom}(K, X) \to \mathbf{Hom}(L, X)$$

via the Exponential Law 5.1. ■

7. Simplicial homotopy groups.

Let X be a fibrant simplicial set and let $v \in X_0$ be a vertex of X. Define $\pi_n(X, v)$, $n \geq 1$, to be the set of homotopy classes of maps $\alpha : \Delta^n \to X$ (rel $\partial\Delta^n$) for maps α which fit into diagrams

$$\begin{tikzcd} \Delta^n \arrow[r, "\alpha"] & X \\ \partial\Delta^n \arrow[u, hook] \arrow[r] & \Delta^0. \arrow[u, "v"'] \end{tikzcd}$$

One often writes $v : \partial\Delta^n \to X$ for the composition

$$\partial\Delta^n \to \Delta^0 \xrightarrow{v} X.$$

Define $\pi_0(X)$ to be the set of homotopy classes of vertices of X. $\pi_0(X)$ is the set of *path components* of X. The simplicial set X is said to be *connected* if $\pi_0(X)$ is trivial (ie. a one-element set). We shall write $[\alpha]$ for the homotopy class of α, in all contexts.

Suppose that $\alpha, \beta : \Delta^n \to X$ represent elements of $\pi_n(X, v)$. Then the simplices

$$\begin{cases} v_i = v, & 0 \le i \le n-2, \\ v_{n-1} = \alpha, & \text{and} \\ v_{n+1} = \beta \end{cases}$$

satisfy $d_i v_j = d_{j-1} v_i$ if $i < j$ and $i, j \neq n$, since all faces of all simplices v_i map through the vertex v. Thus, the v_i determine a simplicial set map $(v_0, \dots, v_{n-1}, \ , v_{n+1}) : \Lambda^{n+1}_n \to X$, and there is an extension

$$\begin{tikzcd} \Lambda^{n+1}_n \arrow[r, "{(v_0, \dots, v_{n-1},\ , v_{n+1})}"] \arrow[d, hook] & X. \\ \Delta^{n+1} \arrow[ur, "\omega"'] & \end{tikzcd}$$

Observe that

$$\begin{aligned} \partial(d_n\omega) &= (d_0 d_n \omega, \dots, d_{n-1} d_n \omega, d_n d_n \omega) \\ &= (d_{n-1} d_0 \omega, \dots, d_{n-1} d_{n-1} \omega, d_n d_{n+1} \omega) \\ &= (v, \dots, v), \end{aligned}$$

and so $d_n\omega$ represents an element of $\pi_n(X, v)$.

Lemma 7.1. *The homotopy class of $d_n\omega$ (rel $\partial\Delta^n$) is independent of the choices of representatives of $[\alpha]$ and $[\beta]$ and of the choice of ω.*

Proof: Suppose that h_{n-1} is a homotopy $\alpha \xrightarrow{\simeq} \alpha'$ (rel $\partial\Delta^n$) and h_{n+1} is a homotopy $\beta \xrightarrow{\simeq} \beta'$ (rel $\partial\Delta^n$). Suppose further that

$$\partial\omega = (v, \dots, v, \alpha, d_n\omega, \beta)$$

and

$$\partial\omega' = (v, \dots, v, \alpha', d_n\omega', \beta').$$

Then there is a map

$$(\Delta^{n+1} \times \partial\Delta^1) \cup (\Lambda^{n+1}_n \times \Delta^1) \xrightarrow{(\omega', \omega, (v, \dots, h_{n-1},\ , h_{n+1}))} X$$

which is determined by the data. Choose an extension

$$\begin{tikzcd}
(\Delta^{n+1} \times \partial\Delta^1) \cup (\Lambda^{n+1}_n \times \Delta^1) \arrow[r, "{(\omega', \omega, (v, \dots, h_{n-1}, \ , h_{n+1}))}"] \arrow[d, hook] & X. \\
\Delta^{n+1} \times \Delta^1 \arrow[ur, "w"'] &
\end{tikzcd}$$

Then the composite

$$\Delta^n \times \Delta^1 \xrightarrow{d^n \times 1} \Delta^{n+1} \times \Delta^1 \xrightarrow{w} X$$

is a homotopy $d_n\omega \xrightarrow{\simeq} d_n\omega'$ (rel $\partial\Delta^n$). ∎

It follows from Lemma 7.1 that the assignment

$$([\alpha], [\beta]) \mapsto [d_n\omega], \quad \text{where} \quad \partial\omega = (v, \dots, v, \alpha, d_n\omega, \beta),$$

gives a well-defined pairing

$$m : \pi_n(X, v) \times \pi_n(X, v) \to \pi_n(X, v).$$

Let $e \in \pi_n(X, v)$ be the homotopy class which is represented by the constant map

$$\Delta^n \to \Delta^0 \xrightarrow{v} X.$$

THEOREM 7.2. *With these definitions, $\pi_n(X, v)$ is a group for $n \geq 1$, which is abelian if $n \geq 2$.*

PROOF: We shall demonstrate here that the $\pi_n(X, v)$ are groups; the abelian property for the higher homotopy groups will be proved later.

It is easily seen (exercise) that $\alpha \cdot e = e \cdot \alpha = \alpha$ for any $\alpha \in \pi_n(X, v)$, and that the map $\pi_n(X, v) \to \pi_n(X, v)$ induced by left multiplication by α is bijective. The result follows, then, if we can show that the multiplication in $\pi_n(X, v)$ is associative in general and abelian if $n \geq 2$.

To see that the multiplication is associative, let $x, y, z : \Delta^n \to X$ represent elements of $\pi_n(X, v)$. Choose $(n+1)$-simplices $\omega_{n-1}, \omega_{n+1}, \omega_{n+2}$ such that

$$\begin{aligned}
\partial\omega_{n-1} &= (v, \dots, v, x, d_n\omega_{n-1}, y), \\
\partial\omega_{n+1} &= (v, \dots, v, d_n\omega_{n-1}, d_n\omega_{n+1}, z), \quad \text{and} \\
\partial\omega_{n+2} &= (v, \dots, v, y, d_n\omega_{n+2}, z).
\end{aligned}$$

Then there is a map

$$\Lambda^{n+2}_n \xrightarrow{(v,\dots,v,\omega_{n-1},\ ,\omega_{n+1},\omega_{n+2})} X$$

which extends to a map $u : \Delta^{n+2} \to X$. But then

$$\partial(d_n\omega) = (v, \dots, v, x, d_n\omega_{n+1}, d_n\omega_{n+2}),$$

and so

$$\begin{aligned}
([x][y])[z] &= [d_n\omega_{n-1}][z] \\
&= [d_n\omega_{n+1}] \\
&= [d_n d_n u] \\
&= [x][d_n\omega_{n+1}] \\
&= [x]([y][z]).
\end{aligned}$$

■

In order to prove that $\pi_n(X, v)$ abelian for $n \geq 2$, it is most instructive to show that there is a loop-space ΩX such that $\pi_n(X, v) \cong \pi_{n-1}(\Omega X, v)$ and then to show that $\pi_i(\Omega X, v)$ is abelian for $i \geq 1$. This is accomplished with a series of definitions and lemmas, all of which we'll need in any case. The first step is to construct the long exact sequence of a fibration.

Suppose that $p : X \to Y$ is a Kan fibration and that F is the fibre over a vertex $* \in Y$ in the sense that the square

$$\begin{tikzcd}
F \arrow[r, "i"] \arrow[d] & X \arrow[d, "p"] \\
\Delta^0 \arrow[r, "*"'] & Y
\end{tikzcd}$$

is cartesian. Suppose that v is a vertex of F and that $\alpha : \Delta^n \to Y$ represents an element of $\pi_n(Y, *)$. Then in the diagram

$$\begin{tikzcd}
\Lambda^n_0 \arrow[r, "(\ ,v,\dots,v)"] \arrow[d, hook] & X \arrow[d, "p"] \\
\Delta^n \arrow[r, "\alpha"'] \arrow[ur, "\theta"] & Y,
\end{tikzcd}$$

the element $[d_0\theta] \in \pi_{n-1}(F, v)$ is independent of the choice of θ and representative of $[\alpha]$. The resulting function

$$\partial : \pi_n(Y, *) \to \pi_{n-1}(F, v)$$

is called the *boundary map*.

Lemma 7.3.

(a) *The boundary map* $\partial : \pi_n(Y, *) \to \pi_{n-1}(F, v)$ *is a group homomorphism if* $n > 1$.

(b) *The sequence*

$$\cdots \to \pi_n(F, v) \xrightarrow{i_*} \pi_n(X, v) \xrightarrow{p_*} \pi_n(Y, *) \xrightarrow{\partial} \pi_{n-1}(F, v) \to \dots$$
$$\dots \xrightarrow{p_*} \pi_1(Y, *) \xrightarrow{\partial} \pi_0(F) \xrightarrow{i_*} \pi_0(X) \xrightarrow{p_*} \pi_0(Y)$$

is exact in the sense that kernel equals image everywhere. Moreover, there is an action of $\pi_1(Y, *)$ *on* $\pi_0(F)$ *such that two elements of* $\pi_0(F)$ *have the same image under* i_* *in* $\pi_0(X)$ *if and only if they are in the same orbit for the* $\pi_1(Y, *)$*-action.*

Most of the proof of Lemma 7.3 is easy, once you know

Lemma 7.4. *Let* $\alpha : \Delta^n \to X$ *represent an element of* $\pi_n(X, v)$. *Then* $[\alpha] = e$ *if and only if there is an* $(n+1)$*-simplex* ω *of* X *such that* $\partial\omega = (v, \dots, v, \alpha)$.

The proof of Lemma 7.4 is an exercise.

Proof of Lemma 7.3: (a) To see that $\partial : \pi_n(Y, *) \to \pi_{n-1}(F, v)$ is a homomorphism if $n \geq 2$, suppose that we are given diagrams

$$\begin{tikzcd} \Lambda^n_0 \arrow[r, "v"] \arrow[d, hook] & X \arrow[d, "p"] \\ \Delta^n \arrow[ur, "\theta_i"] \arrow[r, "\alpha_i"'] & Y \end{tikzcd} \qquad i = n-1, n, n+1,$$

where the α_i represent elements of $\pi_n(Y, *)$. Suppose that there is an $(n+1)$-simplex ω such that

$$\partial\omega = (*, \dots, *, \alpha_{n-1}, \alpha_n, \alpha_{n+1}).$$

Then there is a commutative diagram

$$\begin{tikzcd} \Lambda^{n+1}_0 \arrow[r, "(\ {,} v{,} \dots{,} v{,} \theta_{n-1}{,} \theta_n{,} \theta_{n+1})"] \arrow[d, hook] & X \arrow[d, "p"] \\ \Delta^{n+1} \arrow[ur, "\gamma"] \arrow[r, "\omega"'] & Y, \end{tikzcd}$$

and

$$\begin{aligned}\partial(d_0\gamma) &= (d_0d_0\gamma, d_1d_0\gamma, \dots, d_nd_0\gamma)\\ &= (d_0d_1\gamma, d_0d_2\gamma, \dots, d_0d_{n-1}\gamma, d_0d_n\gamma, d_0d_{n+1}\gamma)\\ &= (v, \dots, v, d_0\theta_{n-1}, d_0\theta_n, d_0\theta_{n+1})\end{aligned}$$

Thus $[d_0\theta_n] = [d_0\theta_{n-1}][d_0\theta_{n+1}]$, and so $\partial([\alpha_{n-1}][\alpha_{n+1}]) = \partial[\alpha_{n-1}]\partial[\alpha_{n+1}]$ in $\pi_{n-1}(F, v)$.

(b) We shall show that the sequence

$$\pi_n(X, v) \xrightarrow{p*} \pi_n(Y, *) \xrightarrow{\partial} \pi_{n-1}(F, v)$$

is exact; the rest of the proof is an exercise. The composite is trivial, since in the diagram

$$\begin{tikzcd} \Lambda^n_0 \arrow[r, "v"] \arrow[d, hook] & X \arrow[d, "p"] \\ \Delta^n \arrow[r, "p\alpha"'] \arrow[ur, "\alpha"] & Y \end{tikzcd}$$

with $[\alpha] \in \pi_n(X, v)$, we find that $d_0\alpha = v$. On the other hand, suppose that $\gamma : \Delta^n \to Y$ represents a class $[\gamma]$ such that $\partial[\gamma] = e$. Choose a diagram

$$\begin{tikzcd} \Lambda^n_0 \arrow[r, "v"] \arrow[d, hook] & X \arrow[d, "p"] \\ \Delta^n \arrow[r, "\gamma"'] \arrow[ur, "\theta"] & Y \end{tikzcd}$$

so that $[d_0\theta] = \partial[\gamma]$. But then there is a simplex homotopy

$$\Delta^{n-1} \times \Delta^1 \xrightarrow{h_0} F$$

giving $d_0\theta \simeq v$ (rel $\partial\Delta^n$). Thus, there is a diagram

$$\begin{tikzcd} (\Delta^n \times 1) \cup (\partial\Delta^n \times \Delta^1) \arrow[r, "{(\theta, (h_0, v, \dots, v))}"] \arrow[d, hook] & X. \\ \Delta^n \times \Delta^1 \arrow[ur, "h"'] & \end{tikzcd}$$

Moreover $p \cdot h$ is a homotopy $\gamma \simeq p \cdot (h \cdot d^1)$ (rel $\partial\Delta^n$). ∎

Now for some definitions. For a Kan complex X and a vertex $*$ of X, the *path space* PX is defined by requiring that the following square is a pullback.

$$\begin{tikzcd} PX \arrow[r, "pr"] \arrow[d] & \mathbf{Hom}(\Delta^1, X) \arrow[d, "(d^0)^*"] \\ \Delta^0 \arrow[r, "*"'] & \mathbf{Hom}(\Delta^0, X) \cong X. \end{tikzcd}$$

Furthermore, the map $\pi : PX \to X$ is defined to be the composite

$$PX \xrightarrow{pr} \mathbf{Hom}(\Delta^1, X) \xrightarrow{(d^1)^*} \mathbf{Hom}(\Delta^0, X) \cong X.$$

The maps $(d^\epsilon)^*$ are fibrations for $\epsilon = 0, 1$, by 5.3. In particular, PX is fibrant.

Lemma 7.5. *$\pi_i(PX, v)$ is trivial for $i \geq 0$ and all vertices v, and π is a fibration.*

Proof: $d^\epsilon : \Delta^0 \to \Delta^1$ is an anodyne extension, and so $(d^0)^*$ has the right lifting property with respect to all maps $\partial\Delta^n \subset \Delta^n$, $n \geq 0$, (see the argument in 5.2). Thus, the map $PX \to \Delta^0 = *$ has the right lifting property with respect to all such maps. Any two vertices of PX are homotopic, by finding extensions

$$\begin{tikzcd} \partial\Delta^1 \arrow[r] \arrow[d, hook] & PX. \\ \Delta^1 \arrow[ur, dashed] & \end{tikzcd}$$

If $\alpha : \Delta^n \to PX$ represents an element of $\pi_n(PX, v)$, then there is a commutative diagram

$$\begin{tikzcd} \partial\Delta^{n+1} \arrow[r, "{(v, \dots, v, \alpha)}"] \arrow[d, hook] & PX, \\ \Delta^{n+1} \arrow[ur, dashed] & \end{tikzcd}$$

and so $[\alpha] = e$ in $\pi_n(PX, v)$. Finally, the map π sits inside the pullback diagram

$$\begin{tikzcd} PX \arrow[r] \arrow[dd, "\pi"'] & \mathbf{Hom}(\Delta^1, X) \arrow[d, "i^*"] \\ & \mathbf{Hom}(\partial\Delta^1, X) \arrow[d, "\cong"] \\ X \arrow[r, "(*, 1_X)"'] & X \times X \end{tikzcd}$$

and so π is a fibration since i^* is, by 5.3. ■

Define the *loop space* ΩX to be the fibre of $\pi : PX \to X$ over the base point $*$. A simplex of ΩX is a simplicial map $f : \Delta^n \times \Delta^1 \to X$ such that the restriction of f to $\Delta^n \times \partial\Delta^1$ maps into $*$. Now we can prove

Lemma 7.6. *$\pi_i(\Omega X, *)$ is abelian for $i \geq 1$.*

Proof: $\pi_n(\Omega X, *)$, as a set, consists of homotopy classes of maps of the form

$$\begin{tikzcd} \Delta^n \times \Delta^1 \arrow[r, "\alpha"] & X, \\ (\partial\Delta^n \times \Delta^1) \cup (\Delta^n \times \partial\Delta^1) \arrow[u, hook] \arrow[ur, "*"'] & \end{tikzcd}$$

rel the boundary $(\partial\Delta^n \times \Delta^1) \cup (\Delta^n \times \partial\Delta^1)$. Show that $\pi_n(\Omega X, *)$ has a second multiplication $[\alpha] \star [\beta]$ (in the 1-simplex direction) such that $[*]$ is an identity for this multiplication and that $\star$ and the original multiplication satisfy the *interchange law*

$$([\alpha_1] \star [\beta_1])([\alpha_2] \star [\beta_2]) = ([\alpha_1][\alpha_2]) \star ([\beta_1][\beta_2]).$$

It follows that $[\alpha][\beta] = [\alpha] \star [\beta]$, and that both multiplications are abelian. ■

Corollary 7.7. *Suppose that X is fibrant. Then $\pi_i(X, *)$ is abelian if $i \geq 2$.*

The proof of Theorem 7.2 is now complete.

Let G be a group, and recall that the classifying space BG is fibrant, by 3.5. BG has exactly one vertex $*$. We can now show easily that BG is an Eilenberg-Mac Lane space.

Proposition 7.8. *There are natural isomorphisms*

$$\pi_i(BG,*) \cong \begin{cases} G & \text{if } i = 1, \\ 0 & \text{if } i \neq 1. \end{cases}$$

Proof: BG is a 2-coskeleton (see 3.5), and so $\pi_i(BG,*) = 0$ for $i \geq 2$, by 7.4. It is an elementary exercise to check that the identification $BG_1 = G$ induces an isomorphism of groups $\pi_1(BG,*) \xrightarrow{\cong} G$. $\pi_0(BG)$ is trivial, since BG has only one vertex. ∎

Suppose that $f : Y \to X$ is a map between fibrant simplicial sets. f is said to be a *weak equivalence* if

$$(7.9) \qquad \begin{cases} \text{for each vertex } y \text{ of } Y \text{ the induced map } f_* : \pi_i(Y,y) \to \pi_i(X,fy) \\ \qquad \text{is an isomorphism for } i \geq 1, \text{ and} \\ \text{the map } f_* : \pi_0(Y) \to \pi_0(X) \text{ is a bijection.} \end{cases}$$

Theorem 7.10. *A map $f : Y \to X$ between fibrant simplicial sets is a fibration and a weak equivalence if and only if f has the right lifting property with respect to all maps $\partial\Delta^n \subset \Delta^n$, $n \geq 0$.*

Proof: $(\Rightarrow)$ The simplicial homotopy $\Delta^n \times \Delta^1 \to \Delta^n$, given by the diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \arrow[r] & 0 \arrow[d] \\ 0 \arrow[r] & 1 \arrow[r] & \dots \arrow[r] & n \end{tikzcd}$$

in $\mathbf{n}$, contracts Δ^n onto the vertex 0. This homotopy restricts to a homotopy $\Lambda^n_0 \times \Delta^1 \to \Lambda^n_0$ which contracts Λ^n_0 onto 0.

Now suppose that we're given a diagram

D

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "\alpha"] \arrow[d, hook, "i"'] & X \arrow[d, "p"] \\ \Delta^n \arrow[r, "\beta"'] & Y. \end{tikzcd}$$

If there is a diagram

$$
\begin{tikzcd}
\partial\Delta^n \arrow[dr, "d^0"'] \arrow[drr, "\alpha"] \arrow[ddd, hook, "i"'] & & \\
& \partial\Delta^n \times \Delta^1 \arrow[r, "h"'] \arrow[d, hook, "i \times 1"'] & X \arrow[d, "p"] \\
& \Delta^n \times \Delta^1 \arrow[r, "g"] & Y \\
\Delta^n \arrow[ur, "d^0"] \arrow[urr, "\beta"'] & &
\end{tikzcd}
$$

such that the lifting exists in the diagram

$$
\begin{tikzcd}
\partial\Delta^n \arrow[r, "h \cdot d^1"] \arrow[d, hook] & X \arrow[d, "p"] \\
\Delta^n \arrow[r, "g \cdot d^1"'] \arrow[ur, "\theta"] & Y,
\end{tikzcd}
$$

then the lifting exists in the original diagram **D**. This is a consequence of the fact that there is a commutative diagram

$$
\begin{tikzcd}
\partial\Delta^n \arrow[r, "d^0"] \arrow[d, hook, "i"'] \arrow[rr, bend left, "\alpha"] & (\partial\Delta^n \times \Delta^1) \cup (\Delta^n \times \{1\}) \arrow[r, "(h,\theta)"] \arrow[d, hook] & X \arrow[d, "p"] \\
\Delta^n \arrow[r, "d^0"'] \arrow[rr, bend right, "\beta"'] & \Delta^n \times \Delta^1 \arrow[r, "g"'] \arrow[ur, "\gamma"] & Y
\end{tikzcd}
$$

Now, the contracting homotopy $H : \Lambda^n_0 \times \Delta^1 \to \Lambda^n_0$ determines a diagram

$$\begin{tikzcd}
\Lambda^n_0 \arrow[r, hook, "j"] \arrow[d, "d^0"'] & \partial\Delta^n \arrow[d, "\alpha"] \\
\Lambda^n_0 \times \Delta^1 \arrow[r, "h_1"] & X \\
\Lambda^n_0 \arrow[u, "d^1"] \arrow[r] & \Delta^0, \arrow[u, "\alpha(0)"']
\end{tikzcd}$$

where $h_1 = \alpha \cdot j \cdot H$. There is a diagram

$$\begin{tikzcd}
(\partial\Delta^n \times \{1\}) \cup (\Lambda^n_0 \times \Delta^1) \arrow[r, "{(\alpha, h_1)}"] \arrow[d, hook] & X \\
\partial\Delta^n \times \Delta^1 \arrow[ur, "h"'] &
\end{tikzcd}$$

since X is fibrant. Moreover, there is a diagram

$$\begin{tikzcd}
(\Delta^n \times \{1\}) \cup (\partial\Delta^n \times \Delta^1) \arrow[r, "{(\beta, ph)}"] \arrow[d, hook] & Y \\
\Delta^n \times \Delta^1 \arrow[ur, "g"'] &
\end{tikzcd}$$

since Y is fibrant. It therefore suffices to solve the problem for diagrams

D_1

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "{(x_0, *, \dots, *)}"] \arrow[d, hook] & X \arrow[d, "p"] \\
\Delta^n \arrow[r, "\omega"'] & Y
\end{tikzcd}$$

for some vertex $*$ $(= \alpha(0))$ of X, since the composite diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "d^1"] \arrow[d, hook, "i"'] & \partial\Delta^n \times \Delta^1 \arrow[r, "h"] \arrow[d, hook, "i \times 1"] & X \arrow[d, "p"] \\ \Delta^n \arrow[r, "d^1"'] & \Delta^n \times \Delta^1 \arrow[r, "g"'] & Y \end{tikzcd}$$

has this form. Then x_0 represents an element $[x_0]$ of $\pi_{n-1}(X, *)$ such that $p_*[x_0] = e$ in $\pi_{n-1}(Y, p*)$. Thus, $[x_0] = e$ in $\pi_{n-1}(X, *)$, and so the trivializing homotopy $h_0 : \Delta^{n-1} \times \Delta^1 \to X$ for X_0 determines a homotopy

$$h' = (h_0, *, \dots, *) : \partial\Delta^n \times \Delta^1 \to X.$$

But again there is a diagram

$$\begin{tikzcd} (\Delta^n \times \{1\}) \cup (\partial\Delta^n \times \Delta^1) \arrow[r, "{(\omega, ph)}"] \arrow[d, hook] & Y, \\ \Delta^n \times \Delta^1 \arrow[ur, "g'"'] & \end{tikzcd}$$

so it suffices to solve the lifting problem for diagrams

D₂

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "*"] \arrow[d, hook] & X \arrow[d, "p"] \\ \Delta^n \arrow[r, "\beta"'] & Y. \end{tikzcd}$$

p_* is onto, so $\beta \simeq p\alpha$ (rel $\partial\Delta^n$) via some homotopy $h'' : \Delta^n \times \Delta^1 \to Y$, and so there is a commutative diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "d^0"] \arrow[d, hook] & (\partial\Delta^n \times \Delta^1) \cup (\Delta^n \times \{0\}) \arrow[r, "{(*, \alpha)}"] \arrow[d, hook] & X \arrow[d, "p"] \\ \Delta^n \arrow[r, "d^0"'] & \Delta^n \times \Delta^1 \arrow[r, "h''"'] \arrow[ur, dashed] & Y. \end{tikzcd}$$

$\mathbf{D_2}$ is the composite of these two squares, and the lifting problem is solved.

($\Leftarrow$) Suppose that $p : X \to Y$ has the right lifting property with respect to all $\partial\Delta^n \subset \Delta^n$, $n \geq 0$. Then p has the right lifting property with respect to all inclusions $L \subset K$, and is a Kan fibration in particular. It is then easy to see that $p_* : \pi_0 X \to \pi_0 Y$ is a bijection. Also, if $x \in X$ is any vertex of X and F_x is the fibre over $p(x)$, then $F_x \to *$ has the right lifting property with respect to all $\partial\Delta^n \subset \Delta^n$, $n \geq 0$. Then F_x is fibrant, and $\pi_0(F_x) = *$ and $\pi_i(F_x, x) = 0$, $i \geq 1$, by the argument of Lemma 7.5. Thus, $p_* : \pi_i(X, x) \to \pi_i(Y, px)$ is an isomorphism for all $i \geq 1$. ∎

8. Fundamental groupoid.

Let X be a fibrant simplicial set. Provisionally, the fundamental groupoid $\pi_f X$ of X is a category having as objects all vertices of X. An arrow $x \to y$ in $\pi_f X$ is a homotopy class of 1-simplices $\omega : \Delta^1 \to X$ (rel $\partial\Delta^1$) where the diagram

$$\begin{tikzcd}
\Delta^0 \arrow[d, "d^1"'] \arrow[dr, "x"] & \\
\Delta^1 \arrow[r, "\omega"] & X \\
\Delta^0 \arrow[u, "d^0"] \arrow[ur, "y"'] &
\end{tikzcd}$$

commutes. If v_2 represents an arrow $x \to y$ of $\pi_f X$ and v_0 represents an arrow $y \to z$; then the composite $[v_0][v_2]$ is represented by $d_1\omega$, where ω is a 2-simplex such that the following diagram commutes

$$\begin{tikzcd}
\Lambda^2_1 \arrow[r, "{(v_0,\ ,v_2)}"] \arrow[d, hook] & X. \\
\Delta^2 \arrow[ur, "\omega"'] &
\end{tikzcd}$$

The fact that this is well-defined should be clear. The identity at x is represented by $s_0 x$. This makes sense because, if $v_2 : x \to y$ and $v_0 : y \to z$ then $\partial s_0 v_0 = (v_0, v_0, s_0 y)$, and $\partial(s_1 v_2) = (s_0 y, v_2, v_2)$. The associativity is proved as it was for π_1. In fact, $\pi_f X(x,x) = \pi_1(X,x)$ specifies the group of homomorphisms $\pi_f X(x,x)$ from x to itself in $\pi_f X$, by definition. By solving the lifting problem

$$\begin{tikzcd}
\Lambda^2_0 \arrow[r, "{(\ , s_0 x, v_2)}"] \arrow[d, hook] & X \\
\Delta^2 \arrow[ur, dashed, "\omega"'] &
\end{tikzcd}$$

for $v_2 : x \to y$, one finds a $v_0 : y \to x$ (namely $d_0\omega$) such that $[v_0][v_2] = 1_x$. But then $[v_2]$ is also epi since it has a right inverse by a similar argument. Thus $[v_2][v_0][v_2] = [v_2]$ implies $[v_2][v_0] = 1_y$, and so $\pi_f X$ really is a groupoid.

Now, let $\alpha : \Delta^n \to X$ represent an element of $\pi_n(X,x)$ and let $\omega : \Delta^1 \to X$ represent an element of $\pi_f X(x,y)$. Then there is a commutative diagram

$$\begin{tikzcd}
(\partial\Delta^n \times \Delta^1) \cup (\Delta^n \times \{0\}) \arrow[d, hook] \arrow[dr, "{(\omega \cdot pr_R, \alpha)}"] & \\
\Delta^n \times \Delta^1 \arrow[r, "h_{(\omega,\alpha)}"] & X, \\
\Delta^n \arrow[u, "d^0"] \arrow[ur, "\omega_*\alpha"'] &
\end{tikzcd}$$

and $\omega_*\alpha$ represents an element of $\pi_n(X,y)$.

Proposition 8.1. *The class $[\omega_*\alpha]$ is independent of the relevant choices of representatives. Moreover, $[\alpha] \mapsto [\omega_*\alpha]$ is a group homomorphism which is functional in $[\omega]$, and so the assignment $x \mapsto \pi_n(X,x)$ determines a functor on $\pi_f X$.*

Proof: We shall begin by establishing independence from the choice of representative for the class $[\omega]$. Suppose that $G : \omega \xrightarrow{\simeq} \eta$ (rel $\partial\Delta^1$) is a homotopy of paths from x to y. Then there is a 2-simplex σ of X such that

$$\partial\sigma = (s_0 y, \eta, \omega).$$

Find simplices of the form $h_{(\eta,\alpha)}$ and $h_{(\omega,\alpha)}$ according to the recipe given above, and let h_σ be the composite

$$\partial\Delta^n \times \Delta^2 \xrightarrow{pr} \Delta^2 \xrightarrow{\sigma} X.$$

There is a commutative diagram

$$\begin{tikzcd}
(\partial\Delta^n \times \Delta^2) \cup (\Delta^n \times \Lambda^2_0) \arrow[rr, "{(h_\sigma, (\ , h(\eta,\alpha), h(\omega,\alpha)))}"] \arrow[d, hook, "i"'] & & X, \\
\Delta^n \times \Delta^2 \arrow[urr, "\theta_1"'] & &
\end{tikzcd}$$

since the inclusion i is anodyne. Then the composite

$$\Delta^n \times \Delta^1 \xrightarrow{1 \times d^0} \Delta^n \times \Delta^2 \xrightarrow{\theta_1} X$$

is a homotopy from $\omega_*\alpha$ to $\eta_*\alpha$ (rel $\partial\Delta^n$), and so $[\omega_*\alpha] = [\eta_*\alpha]$ in $\pi_n(X, y)$.

Suppose that $H : \Delta^n \times \Delta^1 \to X$ gives $\alpha \xrightarrow{\simeq} \beta$ (rel $\partial\Delta^n$), and choose a homotopy $h(\omega, \beta) : \Delta^n \times \Delta^1 \to X$ as above. Let $h_{s_0\omega}$ denote the composite

$$\partial\Delta^n \times \Delta^2 \xrightarrow{pr} \Delta^2 \xrightarrow{s_0\omega} X.$$

Then there is a commutative diagram

$$\begin{tikzcd}
(\partial\Delta^n \times \Delta^2) \cup (\Delta^n \times \Lambda^2_1) \arrow[rr, "{(h_{s_0\omega}, (h(\omega,\beta),\ , H))}"] \arrow[d, hook, "j"'] & & X, \\
\Delta^n \times \Delta^2 \arrow[urr, "\gamma"'] & &
\end{tikzcd}$$

for some map γ, since the inclusion j is anodyne. But then the simplex given by the composite

$$\Delta^n \xrightarrow{d^0} \Delta^n \times \Delta^1 \xrightarrow{1 \times d^1} \Delta^n \times \Delta^2 \xrightarrow{\gamma} X$$

is a construction for both $\omega_*\alpha$ and $\omega_*\beta$, so that $[\omega_*\alpha] = [\omega_*\beta]$ in $\pi_n(X, y)$.

For the functoriality, suppose that $\omega : \Delta^1 \to X$ and $\eta : \Delta^1 \to X$ represent elements of $\pi_f X(x, y)$ and $\pi_f X(y, z)$ respectively, and choose a 2-simplex γ such that $\partial\gamma = (\eta, d_1\gamma, \omega)$. Then $[d_1\gamma] = [\eta]\cdot[\omega]$ in $\pi_f X$. Choose $h_{(\omega,\alpha)}$ and $h_{(\eta,\omega_*\alpha)}$ according to the recipe above. Then there is a diagram

$$\begin{tikzcd}
(\partial\Delta^n \times \Delta^2) \cup (\Delta^n \times \Lambda^2_1) \arrow[d, hook] \arrow[drr, "{(\gamma\circ pr, (h(\eta,\omega_*\alpha),\ , h(\omega,\alpha))}"] & & \\
\Delta^n \times \Delta^2 & & X, \\
\Delta^n \times \Delta^1 \arrow[u, "1\times d^1"] \arrow[urr, "\xi"'] & &
\end{tikzcd}$$

and hence a diagram

$$\begin{tikzcd}
(\partial\Delta^n \times \Delta^1) \cup (\Delta^n \times \{0\}) \arrow[d, hook] \arrow[dr, "{(h_{d_1\gamma}, \alpha)}"] & \\
\Delta^n \times \Delta^1 \arrow[r, "\xi"] & X, \\
\Delta^n \arrow[u, "d^1"] \arrow[ur, "\eta_*\omega_*\alpha = (d_1\gamma)_*\alpha"'] &
\end{tikzcd}$$

where $h_{d_1\gamma}$ is the composite

$$\partial\Delta^n \times \Delta^1 \xrightarrow{pr} \Delta^1 \xrightarrow{d_1\gamma} X.$$

The statement that ω_* is a group homomorphism is easily checked. ∎

THEOREM 8.2. *Suppose that the following is a commutative triangle of simplicial set maps:*

$$\begin{tikzcd}
X \arrow[rr, "g"] \arrow[dr, "h"'] & & Y \\
& Z \arrow[ur, "f"'] &
\end{tikzcd}$$

with X, Y, and Z fibrant. If any two of f, g, or h are weak equivalences, then so is the third.

PROOF: There is one non-trivial case, namely to show that f is a weak equivalence if g and h are. This is no problem at all for π_0. Suppose $y \in Y$ is a vertex. We must show that $f_* : \pi_n(Y, y) \to \pi_n(Z, fy)$ is an isomorphism. y may not be in the image of g, but there is an $x \in X$ and a path $\omega : y \to gx$ since $\pi_0(g)$ is epi. But then there is a diagram

$$\begin{tikzcd}
\pi_n(Y,y) \arrow[r, "{[\omega]_*}"] \arrow[d, "f_*"'] & \pi_n(Y,gx) \arrow[d, "f_*"] & \pi_n(X,x). \arrow[l, "g_*"'] \arrow[dl, "h_*"] \\
\pi_n(Z,fy) \arrow[r, "{[f\omega]_*}"'] & \pi_n(Z,fgx) &
\end{tikzcd}$$

The maps g_*, h_*, $[\omega]_*$, and $[f\omega]_*$ are isomorphisms, and so both of the maps labelled f_* are isomorphisms. ∎

There are three competing definitions for the fundamental groupoid of an arbitrary simplicial set X. The most obvious choice is the classical fundamental groupoid $\pi|X|$ of the realization of X; in the notation above, this is $\pi_f S|X|$. Its objects are the elements of $|X|$, and its morphisms are homotopy classes of paths in $|X|$. The second choice is the model GP_*X of Gabriel and Zisman. GP_*X is the free groupoid associated to the *path category* P_*X of X. The path category has, as objects, all the vertices (elements of X_0) of X. It is generated, as a category, by the 1-simplices of X, subject to the relation that, for each 2-simplex σ of X, the diagram

$$\begin{tikzcd}
v_0 \arrow[rr, "d_2\sigma"] \arrow[dr, "d_1\sigma"'] & & v_1 \arrow[dl, "d_0\sigma"] \\
& v_2 &
\end{tikzcd}$$

commutes. The free groupoid $G(\mathbf{\Delta} \downarrow X)$ associated to the simplex category $\mathbf{\Delta} \downarrow X$ is also a good model. We shall see later on, in Section 12 that $\pi|X|$, GP_*X and $G(\mathbf{\Delta} \downarrow X)$ are all naturally equivalent, once we have developed the techniques for doing so.

9. Categories of fibrant objects.

Let $\mathbf{S}_f$ be the full subcategory of the simplicial set category whose objects are the Kan complexes. $\mathbf{S}_f$ has all finite products. We have two distinguished classes of maps in $\mathbf{S}_f$, namely the fibrations (defined by the lifting property) and the weak equivalences (defined via simplicial homotopy groups). A *trivial fibration* $p : X \to Y$ in $\mathbf{S}_f$ is defined to be a map which is both a fibration and a weak equivalence. A *path object* for $X \in \mathbf{S}_f$ is a commutative diagram

$$\begin{tikzcd} & X^I \arrow[d, "{(d_0, d_1)}"] \\ X \arrow[ur, "s"] \arrow[r, "\Delta"'] & X \times X \end{tikzcd}$$

where s is a weak equivalence and (d_0, d_1) is a fibration. The maps d_0 and d_1 are necessarily trivial fibrations. Any Kan complex X has a natural choice of path object, namely the diagram

$$\begin{tikzcd} & \mathbf{Hom}(\Delta^1, X) \arrow[d] \arrow[dd, bend left=60, "{(d_0, d_1)}"] \\ & \mathbf{Hom}(\partial\Delta^1, X) \arrow[d, "\cong"] \\ X \arrow[uur, "s"] \arrow[r] & X \times X \end{tikzcd}$$

where s is the map

$$X \cong \mathbf{Hom}(\Delta^0, X) \xrightarrow{(s^0)^*} \mathbf{Hom}(\Delta^1, X).$$

$(s^0)^* = s$ is a weak equivalence; in effect, it is a right inverse for the map

$$\mathbf{Hom}(\Delta^1, X) \xrightarrow{(d^0)^*} \mathbf{Hom}(\Delta^0, X),$$

and $(d^0)^*$ is a trivial fibration, by 7.10, since it has the right lifting property with respect to all inclusions $\partial\Delta^n \subset \Delta^n$, $n \geq 0$. $(d^0)^*$ is isomorphic to one of the components of the map $\mathbf{Hom}(\Delta^1, X) \to X \times X$.

The following list of properties of $\mathbf{S}_f$ is essentially a recapitulation of things that we've seen:

(A) Suppose given a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "h"'] & & Y \arrow[dl, "f"] \\ & Z. & \end{tikzcd}$$

If any two of f, g and h are weak equivalences, then so is the third.

(B) The composite of two fibrations is a fibration. Any isomorphism is a fibration.

(C) Pullback diagrams of the form

$$\begin{tikzcd} Z \times_Y X \arrow[r] \arrow[d, "pr"'] & X \arrow[d, "p"] \\ Z \arrow[r] & Y \end{tikzcd}$$

exist in the case where p is a fibration. Furthermore, pr is a fibration which is trivial if p is trivial.

(D) For any object X there is at least one path space X^I.

(E) For any object X, the map $X \to *$ is a fibration.

Recall that we proved **(A)** outright in 8.2. **(B)** is an easy exercise. **(C)** holds because fibrations and trivial fibrations are defined by lifting properties, by 7.10. **(D)** was discussed above, and **(E)** isn't really worth mentioning.

Following K. Brown's thesis [15] (where the notion was introduced), a category $\mathcal{C}$ which has all finite products and has distinguished classes of maps called fibrations and weak equivalences which together satisfy axioms **(A)** – **(E)** is called a *category of fibrant objects* (for a homotopy theory). We've proved:

THEOREM 9.1. $\mathbf{S}_f$ *is a category of fibrant objects for a homotopy theory.*

Other basic examples for us are the category **CGHaus** of compactly generated Hausdorff spaces, and the category **Top** of topological spaces. In fact, more is true. The fibrations of **CGHaus** are the Serre fibrations, and the weak equivalences are the weak homotopy equivalences. A map $i : U \to V$ in **CGHaus** is said to be a *cofibration* if it has the left lifting property with respect to all trivial fibrations.

PROPOSITION 9.2. *The category* **CGHaus** *and these three classes of maps satisfy the following list of axioms:*

CM1: **CGHaus** *is closed under all finite limits and colimits.*

CM2: *Suppose that the following diagram commutes in* **CGHaus***:*

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "h"'] & & Y \arrow[dl, "f"] \\ & Z. & \end{tikzcd}$$

If any two of f, g and h are weak equivalences, then so is the third.

CM3: *If f is a retract of g and g is a weak equivalence, fibration or cofibration, then so is f.*

CM4: *Suppose that we are given a commutative solid arrow diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ V \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where i is a cofibration and p is a fibration. Then the dotted arrow exists, making the diagram commute, if either i or p is also a weak equivalence.

CM5: *Any map $f : X \to Y$ may be factored:*

(a) *$f = p \cdot i$ where p is a fibration and i is a trivial cofibration, and*

(b) *$f = q \cdot j$ where q is a trivial fibration and j is a cofibration.*

PROOF: The category **CGHaus** has all small limits and colimits, giving **CM1** (see [66, p.182]). This fact is also used to prove the factorization axioms **CM5**; this is the next step.

The map $p : X \to Y$ is a Serre fibration if and only if it has the right lifting property with respect to all inclusions $j : |\Lambda^n_k| \to |\Delta^n|$. Each such j is necessarily a cofibration. Now consider all diagrams

$$\textbf{D} \qquad \begin{tikzcd} {|\Lambda^{n_D}_{k_D}|} \arrow[r, "\alpha_D"] \arrow[d, hook] & X \arrow[d, "f"] \\ {|\Delta^{n_D}|} \arrow[r, "\beta_D"'] & Y \end{tikzcd}$$

and form the pushout

$$\begin{tikzcd} \bigsqcup_D |\Lambda^{n_D}_{k_D}| \arrow[r, "(\alpha_D)"] \arrow[d, hook] & X = X_0 \arrow[d, "i_1"'] \arrow[dr, "f = f_0"] & \\ \bigsqcup_D |\Delta^{n_D}| \arrow[r] & X_1 \arrow[r, "f_1"'] & Y. \end{tikzcd}$$

Then we obtain a factorization

$$f = f_0 = f_1 \cdot i_1,$$

where i_1 is a cofibration since it's a pushout of such, and also a weak equivalence since it is a pushout of a map which has a strong deformation retraction. We repeat the process by considering all diagrams.

$$\begin{tikzcd} {|\Lambda^{n_D}_{k_D}|} \arrow[r, "\alpha_D"] \arrow[d, hook] & X_1 \arrow[d, "f_1"] \\ {|\Delta^{n_D}|} \arrow[r, "\beta_D"'] & Y \end{tikzcd}$$

and so on. Thus, we obtain a commutative diagram

$$\begin{tikzcd} X = X_0 \arrow[r, "i_1"] \arrow[dr, "f_0"'] & X_1 \arrow[r, "i_2"] \arrow[d, "f_1"'] & X_2 \arrow[r] \arrow[dl, "f_2"] & \ldots, \\ & Y & & \end{tikzcd}$$

which induces a diagram

$$\begin{tikzcd}
X_0 \arrow[rr, "\tau_0"] \arrow[dr, "f = f_0"'] & & \varinjlim X_i \arrow[dl, "f_\infty"] \\
& Y &
\end{tikzcd}$$

But now τ_0 has the left lifting property with respect to all trivial fibrations, so it's a cofibration. Moreover, τ_0 is a weak equivalence since any compact subset of $\varinjlim X_i$ lies in some finite stage X_i, and all the $X_i \to X_{i+1}$ are weak equivalences. Finally, f_∞ is a fibration; in effect, for each diagram

$$\begin{tikzcd}
{|\Lambda^n_k|} \arrow[r, "\alpha"] \arrow[d, hook] & \varinjlim X_i \arrow[d, "f_\infty"] \\
{|\Delta^n|} \arrow[r, "\beta"'] & Y,
\end{tikzcd}$$

there is an index i and a map α_i making the following diagram commute:

$$\begin{tikzcd}
{|\Lambda^n_k|} \arrow[rr, "\alpha"] \arrow[dr, "\alpha_i"] \arrow[dd, hook] & & \varinjlim X_i \arrow[dd, "f_\infty"] \\
& X_i \arrow[ur] \arrow[dr, "f_i"] & \\
{|\Delta^n|} \arrow[rr, "\beta"'] & & Y.
\end{tikzcd}$$

But then

$$\begin{tikzcd}
{|\Lambda^n_k|} \arrow[r, "\alpha_i"] \arrow[d, hook] & X_i \arrow[d, "f_i"] \\
{|\Delta^n|} \arrow[r, "\beta"'] & Y
\end{tikzcd}$$

is one of the diagrams defining f_{i+1} and there is a diagram

$$\begin{tikzcd}
{|\Lambda^n_k|} \arrow[rr, "\alpha_i"] \arrow[dd, hook] & & X_i \arrow[dl] \arrow[dd, "f_i"] \\
& X_{i+1} \arrow[dr] & \\
{|\Delta^n|} \arrow[ur, "\theta_i"] \arrow[rr, "\beta"'] & & Y
\end{tikzcd}$$

which defines the lifting.

The other lifting property is similar, using

Lemma 9.4. *The map $p : X \to Y$ is a trivial fibration if and only if p has the right lifting property with respect to all inclusions $|\partial\Delta^n| \subset |\Delta^n|$.*

The proof is an exercise.

Quillen calls this proof a *small object argument* [75]. **CM4** is really a consequence of this argument as well. What we showed, in effect, was that any map $f : X \to Y$ has a factorization $f = p \cdot i$ such that p is a fibration and i is a weak equivalence which has the left lifting property with respect to all fibrations.

Suppose now that we have a diagram

$$\begin{tikzcd}
U \arrow[r, "\alpha"] \arrow[d, "i"'] & X \arrow[d, "p"] \\
V \arrow[r, "\beta"'] \arrow[ur, dashed] & Y,
\end{tikzcd}$$

where p is a fibration and i is a trivial cofibration. We want to construct the dotted arrow (giving the non-trivial part of **CM4**). Then there is a diagram

$$\begin{tikzcd}
U \arrow[r, "j"] \arrow[d, "i"'] & W \arrow[d, "\pi"] \\
V \arrow[r, "1_V"'] \arrow[ur, dashed, "r"] & V,
\end{tikzcd}$$

where j is a weak equivalence which has the left lifting property with respect to all fibrations, and π is a (necessarily trivial) fibration. Thus, the dotted arrow r exists. But then i is a retract of j, and so i has the same lifting property. All of the other axioms are trivial, and so the proof of Proposition 9.2 is complete.

A *closed model category* is a category $\mathcal{C}$, together with three classes of maps called cofibrations, fibrations and weak equivalences, such that the axioms **CM1** – **CM5** are satisfied. Proposition 9.2 is the statement that **CGHaus** has the structure of a closed model category.

Proposition 9.5. **CGHaus** *is a category of fibrant objects for a homotopy theory. In fact the subcategory of fibrant objects in any closed model category $\mathcal{C}$ is a category of fibrant objects for a homotopy theory.*

Proof: **(E)** is part of the definition. For **(D)**, the map $\Delta : X \to X \times X$ may be factored

$$\begin{tikzcd} & X^I \arrow[d, "{(d_0, d_1)}"] \\ X \arrow[ur, "s"] \arrow[r, "\Delta"'] & X \times X, \end{tikzcd}$$

where s is a trivial cofibration and (d_0, d_1) is a fibration. For **(B)** and **(C)**, we prove:

Lemma 9.6.

(a) *A map $f : X \to Y$ in $\mathcal{C}$ has the right lifting property with respect to all cofibrations (respectively trivial cofibrations) if and only if f is a trivial fibration (respectively fibration).*

(b) *$U \to V$ in $\mathcal{C}$ has the left lifting property with respect to all fibrations (respectively trivial fibratons) if and only if i is a trivial cofibration (respectively cofibration).*

Proof: We'll show that $f : X \to Y$ has the right lifting property with respect to all cofibrations if and only if f is a trivial fibration. The rest of the proof is an exercise.

Suppose that f has the advertised lifting property, and form the diagram

$$\begin{tikzcd} X \arrow[r, "1_X"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ V \arrow[r, "p"'] \arrow[ur, "r"] & Y, \end{tikzcd}$$

where i is a cofibration, p is a trivial fibration, and r exists by the lifting property. Then f is a retract of p and is therefore a trivial fibration. The reverse implication is **CM4**. ∎

Finally, since fibrations (respectively trivial fibrations) are those maps having the right lifting property with respect to all trivial cofibrations (respectively all fibrations), they are stable under composition and pullback and include all isomorphisms, yielding **(B)** and **(C)**. **(A)** is just **CM2**. This completes the proof of Proposition 9.5.

We shall see that the category of fibrant objects structure that we have displayed for $\mathbf{S}_f$ is the restriction of a closed model structure on the entire simplicial set category, as in the corollary above. This will be proved in the next section.

10. Minimal fibrations.

Minimal Kan complexes play roughly the same role in the homotopy theory of simplicial sets as minimal models play in rational homotopy theory (there ought to be an abstract theory of such things). Minimal Kan complexes appear as fibres of minimal fibrations; it turns out that minimal fibrations are exactly the right vehicle for relating the homotopy theories of $\mathbf{S}$ and $\mathbf{CGHaus}$.

A simplicial set map $q : X \to Y$ is said to be a *minimal fibration* if q is a fibration, and for every diagram

$$\begin{tikzcd} \partial\Delta^n \times \Delta^1 \arrow[r, "pr"] \arrow[d, hook] & \partial\Delta^n \arrow[d] \\ \Delta^n \times \Delta^1 \arrow[r, "h"] \arrow[d, "pr"'] & X \arrow[d, "q"] \\ \Delta^n \arrow[r] & Y \end{tikzcd} \tag{10.1}$$

the composites

$$\Delta^n \overset{d^0}{\underset{d^1}{\rightrightarrows}} \Delta^n \times \Delta^1 \xrightarrow{h} X$$

are equal. This means that, if two simplices x and y in X_n are fibrewise homotopic (rel $\partial\Delta^n$), then $x = y$.

Note that minimal fibrations are stable under base change.

More generally, write $x \simeq_p y$ if there is a diagram of the form (10.1) such that $h(\Delta^n \times 0) = x$, and $h(\Delta^n \times 1) = y$. The relation $\simeq_p$ is an equivalence relation (exercise).

LEMMA 10.2. *Suppose that x and y are degenerate r-simplices of a simplicial set X such that $\partial x = \partial y$. Then $x = y$.*

PROOF: (See also [67], p. 36.) Suppose that $x = s_m z$ and $y = s_n w$. If $m = n$, then

$$z = d_m x = d_m y = w,$$

and so $x = y$. Suppose that $m < n$. Then

$$z = d_m x = d_m s_n w = s_{n-1} d_m w,$$

and so

$$x = s_m s_{n-1} d_m w = s_n s_m d_m w.$$

Thus

$$s_m d_m w = d_n x = d_n y = w.$$

Therefore $x = s_n w = y$. ∎

Now we can prove:

PROPOSITION 10.3. *Let $p : X \to Y$ be a Kan fibration. Then p has a strong fibre-wise deformation retract $q : Z \to Y$ which is a minimal fibration.*

PROOF: Let $Z^{(0)}$ be the subcomplex of X which is generated by a choice of vertex in each p-class, and let $i^{(0)} : Z^{(0)} \subset X$ be the canonical inclusion. There is a map $r^{(0)} : \mathrm{sk}_0 X \to Z^{(0)}$ which is determined by choices of representatives. Moreover $pi^{(0)}r^{(0)} = p|_{\mathrm{sk}_0 X}$, and $j_0 \simeq i^{(0)}r^{(0)}$, where $j_0 : \mathrm{sk}_0 X \subset X$ is the obvious inclusion, via a homotopy $h_0 : \mathrm{sk}_0 X \times \Delta^1 \to X$ such that $h_0(x, 0) = x$ and $h_0(x, 1) = r^{(0)}(x)$, and h_0 is constant on simplices of $Z^{(0)}$. h_0 can be constructed fibrewise in the sense that $p \cdot h_0$ is constant, by using the homotopies implicit in the definition of $\simeq_p$. The subcomplex $Z^{(0)}$ has a unique simplex in each p-equivalence class that it intersects, by Lemma 10.2.

Let $Z^{(1)}$ be the subcomplex of X which is obtained by adjoining to $Z^{(0)}$ a representative for each homotopy class of 1-simplices x such that $\partial x \subset Z^{(0)}$ and x is not p-related to a 1-simplex of $Z^{(0)}$. Again, $Z^{(1)}$ has a unique simplex in each p-equivalence class that it intersects, by construction in degrees ≤ 1 and Lemma 10.2 in degrees > 1.

Let x be a non-degenerate 1-simplex of X. Then there is a commutative diagram

$$\begin{tikzcd}
(\Delta^1 \times \{0\}) \cup (\partial\Delta^1 \times \Delta^1) \arrow[rr, "{(x, h_0|_{\partial x})}"] \arrow[d, hook] & & X \arrow[d, "p"] \\
\Delta^1 \times \Delta^1 \arrow[r, "pr_L"'] \arrow[urr, "h_x"] & \Delta^1 \arrow[r, "px"'] & Y,
\end{tikzcd}$$

by the homotopy lifting property, where the constant homotopy is chosen for h_x if $x \in Z^{(1)}$. But then $\partial(h_x(\Delta^1 \times \{1\})) \subset Z^{(0)}$ and so $h_x(\Delta^1 \times \{1\},)$ is p-related to a unique 1-simplex $r^{(1)}(x)$ of $Z^{(1)}$ via some diagram

$$\begin{tikzcd}
\partial\Delta^1 \times \Delta^1 \arrow[r, "pr_L"] \arrow[d, hook] & \partial\Delta^1 \arrow[d, "\partial h_x(\Delta^1 \times \{1\})"] \\
\Delta^1 \times \Delta^1 \arrow[r, "g_x"] \arrow[d, "pr"'] & X \arrow[d, "p"] \\
\Delta^1 \arrow[r, "px"'] & Y,
\end{tikzcd}$$

where g_x is constant if $x \in Z^{(1)}$, $r^{(1)}(x) = g_x(\Delta^1 \times 1)$, and

$$g_x(\Delta^1 \times \{0\}) = h_x(\Delta^1 \times \{1\}).$$

This defines $r^{(1)} : \mathrm{sk}_1 X \to Z^{(1)}$.

We require a homotopy $h_1 : j_1 \simeq_p i^{(1)} r^{(1)}$, such that $i^{(1)} : Z^{(1)} \subset X$ and $j_1 : \mathrm{sk}_1 X \subset X$ are the obvious inclusions, and such that h_1 is consistent with h_0. We also require that the restriction of h_1 to $Z^{(1)}$ be constant. This is done for the simplex x by constructing a commutative diagram

$$\begin{tikzcd}
(\partial\Delta^1 \times \Delta^2) \cup (\Delta^1 \times \Lambda^2_1) \arrow[rr, "{(s_1 h_0, (g_x, \ , h_x))}"] \arrow[d, hook] & & X \arrow[d, "p"] \\
\Delta^1 \times \Delta^2 \arrow[r, "pr"'] \arrow[urr, "\theta_x"] & \Delta^1 \arrow[r, "px"'] & Y,
\end{tikzcd}$$

where the lifting θ_x is chosen to be the composite

$$\Delta^1 \times \Delta^2 \xrightarrow{pr_L} \Delta^1 \xrightarrow{x} X$$

if $x \in Z^{(1)}$. Then h_0 can be extended to the required homotopy $h_1 : j_1 \simeq_p i^{(1)} r^{(1)}$ by requiring that $h_1|_x = \theta_x \cdot (1 \times d^1)$.

Proceeding inductively gives $i : Z = \varinjlim Z^{(n)} \subset X$ and $r : X \to Z$ such that $1_X \simeq ir$ fibrewise, and such that $q : Z \to Y$ has the minimality property. Finally, q is a Kan fibration, since it is a retract of a Kan fibration. ∎

LEMMA 10.4. *Suppose that*

$$\begin{tikzcd}
Z \arrow[rr, "f"] \arrow[dr, "q"'] & & Z' \arrow[dl, "q'"] \\
& Y &
\end{tikzcd}$$

is a fibrewise homotopy equivalence of minimal fibrations q and q'. Then f is an isomorphism of simplicial sets.

To prove Lemma 10.4, one uses:

SUBLEMMA 10.5. *Suppose that two maps*

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "p"'] & & Z \arrow[dl, "q"] \\
& Y &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[rr, "g"] \arrow[dr, "p"'] & & Z \arrow[dl, "q"] \\
& Y &
\end{tikzcd}$$

are fibrewise homotopic, where g is an isomorphism and q is minimal. Then f is an isomorphism.

PROOF OF SUBLEMMA: Let the diagram

$$\begin{tikzcd} X \arrow[d, "d^0"'] \arrow[dr, "f"] & \\ X \times \Delta^1 \arrow[r, "h"] & Z \\ X \arrow[u, "d^1"] \arrow[ur, "g"'] & \end{tikzcd}$$

represent the homotopy. Suppose that $f(x) = f(y)$ for n-simplices x and y of X. Then inductively $d_i x = d_i y$, $0 \leq i \leq n$, and so the composites

$$\partial\Delta^n \times \Delta^1 \xrightarrow{i \times 1} \Delta^n \times \Delta^1 \xrightarrow{x \times 1} X \times \Delta^1 \xrightarrow{h} Z,$$

and

$$\partial\Delta^n \times \Delta^1 \xrightarrow{i \times 1} \Delta^n \times \Delta^1 \xrightarrow{y \times 1} X \times \Delta^1 \xrightarrow{h} Z,$$

are equal (to a map $h_* : \partial\Delta^n \times \Delta^1 \to Y$). Write h_x for the composite homotopy

$$\Delta^n \times \Delta^1 \xrightarrow{x \times 1} X \times \Delta^1 \xrightarrow{h} Y.$$

Then there is a commutative diagram

$$\begin{tikzcd} (\Delta^n \times \Lambda^2_2) \cup (\partial\Delta^n \times \Delta^2) \arrow[rr, "((h_x, h_y, \ ), s_0 h_*)"] \arrow[d, hook] & & Z \arrow[d, "q"] \\ \Delta^n \times \Delta^2 \arrow[r, "pr_L"'] \arrow[urr, "G"] & \Delta^n \arrow[r, "px = py"'] & Y, \end{tikzcd}$$

and the homotopy $G \cdot (1 \times d^2)$ shows that $x = y$. Thus, f is monic.

To see that f is epi, suppose inductively that $f : X_i \to Z_i$ is an isomorphism for $0 \leq i \leq n-1$, and let $x : \Delta^n \to Z$ be an n-simplex of Z. Then there is a commutative diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "(x_0, \dots, x_n)"] \arrow[d, hook] & X \arrow[d, "f"] \\ \Delta^n \arrow[r, "x"'] & Z \end{tikzcd}$$

by the inductive assumption, and so one can find a diagram

$$\begin{tikzcd}
(\partial\Delta^n \times \Delta^1) \cup (\Delta^n \times \{1\}) \arrow[rr, "{(h|_{(x_0,\dots,x_n)}, x)}"] \arrow[d, hook] & & Z \arrow[d, "q"] \\
\Delta^n \times \Delta^1 \arrow[r, "pr_L"'] \arrow[urr, "h_1"] & \Delta^n \arrow[r, "qx"'] & Y.
\end{tikzcd}$$

Then there is a diagram

$$\begin{tikzcd}
& & X \arrow[d, "g"] \\
\Delta^n \arrow[r, "d^1"'] \arrow[urr, "z"] & \Delta^n \times \Delta^1 \arrow[r, "h_1"'] & Z,
\end{tikzcd}$$

since g is epi. The restriction of z to $\partial\Delta^n$ is the composite $g \cdot (x_0, \dots, x_n)$, so that $\partial z = (x_0, \dots, x_n)$ since g is monic. Thus, there is a diagram

$$\begin{tikzcd}
(\Delta^n \times \Lambda^2_0) \cup (\partial\Delta^n \times \Delta^2) \arrow[rr, "{((\ , h_1, h_z), s_1 h|_{(x_0,\dots,x_n)})}"] \arrow[d, hook] & & Z \arrow[d, "q"] \\
\Delta^n \times \Delta^2 \arrow[r, "pr_L"'] \arrow[urr, "G'"] & \Delta^n \arrow[r, "qx"'] & Y.
\end{tikzcd}$$

Finally, the composite

$$\Delta^n \times \Delta^1 \xrightarrow{1 \times d^0} \Delta^n \times \Delta^2 \xrightarrow{G'} Z$$

is a fibrewise homotopy from $f(z)$ to x, and so $x = f(z)$. ■

Lemma 10.6. *Suppose given a Kan fibration p and pullback diagrams*

$$\begin{tikzcd}
f_i^{-1}p \arrow[r] \arrow[d, "p_i"'] & X \arrow[d, "p"] \\
A \arrow[r] & Y
\end{tikzcd} \qquad i = 0, 1.$$

Suppose further that there is a homotopy $h : f_0 \xrightarrow{\simeq} f_1$. Then there is a fibrewise homotopy equivalence

$$\begin{tikzcd} f_1^{-1}p \arrow[rr, "\simeq"] \arrow[dr, "p_1"'] & & f_2^{-1}p. \arrow[dl, "p_2"] \\ & A & \end{tikzcd}$$

Proof: Consider the diagrams of pullbacks

$$\begin{tikzcd} f_\epsilon^{-1}p \arrow[r, "x_\epsilon"] \arrow[d, "p_\epsilon"'] & h^{-1}p \arrow[r] \arrow[d, "p^h"] & X \arrow[d, "p"] \\ A \arrow[r, "d^\epsilon"'] & A \times \Delta^1 \arrow[r, "h"'] & Y \end{tikzcd} \qquad \epsilon = 0, 1.$$

Then there is a commutative diagram

$$\begin{tikzcd} f_0^{-1}p \arrow[r, "x_0"] \arrow[d, "d^0"'] & h^{-1}p \arrow[d, "p^h"] \\ f_0^{-1}p \times \Delta^1 \arrow[r, "p_0 \times 1"'] \arrow[ur, "\theta"] & A \times \Delta^1, \end{tikzcd}$$

by the homotopy lifting property. It follows that there is a diagram

$$\begin{tikzcd} f_0^{-1}p \arrow[r, "d^1"] \arrow[dr, "\theta_*"] \arrow[ddr, "p_0"'] & f_0^{-1}p \times \Delta^1 \arrow[dr, "\theta"] & \\ & f_1^{-1}p \arrow[r, "x_1"] \arrow[d, "p_1"] & h^{-1}p \arrow[d, "p^h"] \\ & A \arrow[r, "d^1"'] & A \times \Delta^1 \end{tikzcd}$$

and hence an induced map θ_* as indicated. Similarly there are diagrams

$$\begin{tikzcd}
f_1^{-1}p \arrow[r, "x_1"] \arrow[d, "d^1"'] & h^{-1}p \arrow[d, "p^h"] \\
f_1^{-1}p \times \Delta^1 \arrow[r, "p_1 \times 1"'] \arrow[ur, "\omega"] & A \times \Delta^1,
\end{tikzcd}$$

and

$$\begin{tikzcd}
f_1^{-1}p \arrow[r, "d^0"] \arrow[dr, "\omega_*"] \arrow[ddr, "p_1"'] & f_1^{-1}p \times \Delta^1 \arrow[dr, "\omega"] & \\
& f_0^{-1}p \arrow[r, "x_0"] \arrow[d, "p_0"] & h^{-1}p \arrow[d, "p^h"] \\
& A \arrow[r, "d^0"'] & A \times \Delta^1.
\end{tikzcd}$$

Form the diagram

$$\begin{tikzcd}
f_0^{-1}p \times \Lambda_0^2 \arrow[r, "{(\ ,\theta,\omega(\theta_* \times 1))}"] \arrow[d, hook] & h^{-1}p \arrow[d, "p^h"] \\
f_0^{-1}p \times \Delta^2 \arrow[r, "p_0 \times s^1"'] \arrow[ur, "\gamma"] & A \times \Delta^1,
\end{tikzcd}$$

by using the homotopy lifting property and the relations $d_1\theta = x_1\theta_* = d_1(\omega(\theta_* \times 1))$. Then there is a commutative diagram

$$\begin{tikzcd}
f_0^{-1}p \times \Delta^1 \arrow[r, "1 \times d^0"] \arrow[dr, "\gamma_*"] \arrow[dd, "pr_L"'] & f_0^{-1}p \times \Delta^2 \arrow[dr, "\gamma"] & \\
& f_0^{-1}p \arrow[r, "x_0"] \arrow[d, "p_0"'] & h^{-1}p \arrow[d, "p^h"] \\
f_0^{-1}p \arrow[r, "p_0"'] & A \arrow[r, "d^0"'] & A \times \Delta^1,
\end{tikzcd}$$

by the simplicial identities. Then $\gamma_* : \omega_*\theta_* \simeq 1$ is a fibrewise homotopy. There is a similar fibrewise homotopy $\theta_*\omega_* \simeq 1$. ∎

COROLLARY 10.7. *Suppose that $q : Z \to Y$ is a minimal fibration, and that $f_i : X \to Y$, $i = 0, 1$ are homotopic simplicial maps. Then there is a commutative diagram*

$$\begin{tikzcd} f_0^{-1}q \arrow[rr, "\cong"] \arrow[dr, "q_0"'] & & f_1^{-1}q. \arrow[dl, "q_1"] \\ & X & \end{tikzcd}$$

In particular, the pullbacks $f_0^{-1}q$ and $f_1^{-1}q$ are isomorphic.

COROLLARY 10.8. *Suppose that $q : Z \to Y$ is a minimal fibration with Y connected. Suppose that F is the fibre of q over a base point $*$ of Y. Then, for any simplex $\sigma : \Delta^n \to Y$ there is a commutative diagram*

$$\begin{tikzcd} F \times \Delta^n \arrow[rr, "\cong"] \arrow[dr, "pr_R"'] & & \sigma^{-1}q. \arrow[dl] \\ & \Delta^n & \end{tikzcd}$$

PROOF: Suppose that v and w are vertices of Y such that there is a 1-simplex z of Y with $\partial z = (v, w)$. Then the classifying maps $v : \Delta^0 \to Y$ and $w : \Delta^0 \to Y$ are homotopic, and so there is an isomorphism $F_v \cong F_w$ of fibres induced by the homotopy. In particular, there is an isomorphism $F_v \cong F$ for any vertex v of Y. Now let $i_0 : \Delta^0 \to \Delta^n$ be the map that picks out the vertex 0 of Δ^n. Finally, recall (see the proof of 7.10) that the composite

$$\Delta^n \to \Delta^0 \xrightarrow{i_0} \Delta^n$$

is homotopic to the identity on Δ^n. ∎

THEOREM 10.9 (GABRIEL-ZISMAN). *Suppose that $q : X \to Y$ is a minimal fibration. Then its realization $|q| : |X| \to |Y|$ is a Serre fibration.*

PROOF: It is enough to suppose that Y has only finitely many non-degenerate simplices, since the image of any continuous map $|\Delta^n| \to |Y|$ is contained in some finite subcomplex of $|Y|$. We may also suppose that Y is connected. The idea of the proof is to show that $|q| : |X| \to |Y|$ is locally trivial with fibre $|F|$, where F is the fibre over some base point $*$ of Y.

Now suppose that there is a pushout diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "\alpha"] \arrow[d, hook] & Z \arrow[d, hook] \\ \Delta^n \arrow[r, "\beta"'] & Y, \end{tikzcd}$$

where Z is subcomplex of Y with fewer non-degenerate simplices, and suppose that U is an open subset of $|Z|$ such that there is a fibrewise homeomorphism

$$\begin{tikzcd} U \times_{|Y|} |X| \arrow[rr, "\omega", "\cong"'] \arrow[dr] & & U \times |F|. \arrow[dl, "pr_L"] \\ & U & \end{tikzcd}$$

Let $U^1 = |\alpha|^{-1}(U) \subset |\partial\Delta^n|$. Then there is an induced fibrewise homeomorphism

$$\begin{tikzcd} U^1 \times_{|Y|} |X| \arrow[rr, "\omega", "\cong"'] \arrow[dr] & & U^1 \times |F|. \arrow[dl, "pr_L"] \\ & U^1 & \end{tikzcd}$$

On the other hand the simplicial fibrewise homeomorphism

$$\begin{tikzcd} \Delta^n \times F \arrow[rr, "\delta", "\cong"'] \arrow[dr, "pr_L"'] & & \Delta^n \times_Y X \arrow[dl] \\ & \Delta^n & \end{tikzcd}$$

induces a homeomorphism

$$\begin{tikzcd} V^1 \times F \arrow[rr, "\delta", "\cong"'] \arrow[dr, "pr_L"'] & & V^1 \times_{|Y|} |X| \arrow[dl] \\ & V^1 & \end{tikzcd}$$

over some open subset V^1 of $|\Delta^n|$ such that $V^1 \cap |\partial\Delta^n| = U^1$ and U^1 is a retract of V^1. The map δ restricts to a homeomorphism

$$\begin{tikzcd} U^1 \times F \arrow[rr, "\delta", "\cong"'] \arrow[dr, "pr_L"'] & & U^1 \times_{|Y|} |X| \arrow[dl] \\ & U^1 & \end{tikzcd}$$

over U^1. Now consider the fibrewise homeomorphism

$$\begin{tikzcd} U^1 \times |F| \arrow[rr, "\delta^{-1}\omega", "\cong"'] \arrow[dr, "pr_L"'] & & U^1 \times |F|. \arrow[dl, "pr_L"] \\ & U^1 & \end{tikzcd}$$

There is a homeomorphism

$$\begin{tikzcd} V^1 \times |F| \arrow[rr, "r^*(\delta^{-1}\omega)", "\cong"'] \arrow[dr, "pr_L"'] & & V^1 \times |F|. \arrow[dl, "pr_L"] \\ & V^1 & \end{tikzcd}$$

which restricts to $\delta^{-1}\omega$ over U^1. In effect $r^*(\delta^{-1}\omega)(v', f) = (v', \varphi(rv', f))$, where $\delta^{-1}\omega(w, f) = (w, \varphi(w, f))$, and this definition is "functorial". Thus, the fibrewise isomorphism

$$\begin{tikzcd} V^1 \times F \arrow[rr, "\delta r^*(\delta^{-1}\omega)", "\cong"'] \arrow[dr, "pr_L"'] & & V^1 \times_{|Y|} |X| \arrow[dl] \\ & V^1 & \end{tikzcd}$$

restricts to ω over U^1. It follows that there is a fibrewise homeomorphism

$$\begin{tikzcd} (V^1 \cup_{U^1} U) \times F \arrow[rr, "\cong"'] \arrow[dr, "pr_L"'] & & (V^1 \cup_{U^1} U) \times_{|Y|} |X| \arrow[dl] \\ & (V^1 \cup_{U^1} U) & \end{tikzcd}$$

over the open set. $V^1 \cup_{U^1} U$ of $|Y|$. ■

The following result of Quillen [76] is the key to both the closed model structure of the simplicial set category, and the relation between simplicial homotopy theory and ordinary homotopy theory. These results will appear in the next section.

THEOREM 10.10 (QUILLEN). *The realization of a Kan fibration is a Serre fibration.*

PROOF: Let $p : X \to Y$ be a Kan fibration. According to Proposition 10.3, one can choose a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "j"] \arrow[dr, "q"'] & X \arrow[r, "g"] \arrow[d, "p"] & Z \arrow[dl, "q"] \\ & Y & \end{tikzcd}$$

where q is a minimal fibrations, $gj = 1_Z$ and jg is fibrewise homotopic to 1_X. In view of Theorem 10.9, it clearly suffices to prove the following two results:

LEMMA 10.11. *$g : X \to Z$ has the right lifting property with respect to all $\partial\Delta^n \subset \Delta^n$, $n \geq 0$.*

LEMMA 10.12. *Suppose that $g : X \to Z$ has the right lifting property with respect to all $\partial\Delta^n \subset \Delta^n$, $n \geq 0$. Then $|g| : |X| \to |Z|$ is a Serre fibration.*

PROOF OF LEMMA 10.11: Suppose that the diagrams

$$\begin{tikzcd} X \times \Delta^1 \arrow[r, "h"] \arrow[d, "pr"'] & X \arrow[d, "p"] \\ X \arrow[r, "p"'] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[d, "d^0"'] \arrow[dr, "1_X"] & \\ X \times \Delta^1 \arrow[r, "h"'] & X \\ X \arrow[u, "d^1"] \arrow[r, "g"'] & Z \arrow[u, "j"'] \end{tikzcd}$$

represent the fibrewise homotopy, and suppose that the diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "u"] \arrow[d, hook, "i"'] & X \arrow[d, "g"] \\ \Delta^n \arrow[r, "v"'] & Z \end{tikzcd}$$

commutes. Then there are commutative diagrams

$$\begin{tikzcd} \partial\Delta^n \times \Delta^1 \arrow[r, "u \times 1"] \arrow[d, hook, "i \times 1"'] & X \times \Delta^1 \arrow[r, "h"] & X \arrow[d, "p"] \\ \Delta^n \times \Delta^1 \arrow[r, "pr_L"'] & \Delta^n \arrow[r, "qv"'] & Y \end{tikzcd}$$

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "u"] \arrow[d, hook, "i"'] & X \arrow[r, "g"] & Z \arrow[r, "j"] & X \arrow[d, "p"] \\ \Delta^n \arrow[rr, "v"'] & & Z \arrow[ur, "j"] \arrow[r, "q"'] & Y, \end{tikzcd}$$

and hence a diagram

$$\begin{tikzcd}
(\partial\Delta^n \times \Delta^1) \cup (\Delta^n \times \{0\}) \arrow[rr, "{(h(u \times 1), jv)}"] \arrow[d, hook] & & X \arrow[d, "p"] \\
\Delta^n \times \Delta^1 \arrow[r, "pr_L"'] \arrow[urr, "h_1"] & \Delta^n \arrow[r, "qv"'] & Y.
\end{tikzcd}$$

Let v_1 be the simplex classified by the composite

$$\Delta^n \xrightarrow{d^0} \Delta^n \times \Delta^1 \xrightarrow{h_1} X.$$

The idea of the proof is now to show that $gv_1 = v$. The diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "u"] \arrow[d, hook, "i"'] & X \\
\Delta^n \arrow[ur, "v_1"'] &
\end{tikzcd}$$

commutes, and consider the composite

$$\Delta^n \times \Delta^1 \xrightarrow{v_1 \times 1} X \times \Delta^1 \xrightarrow{h} X \xrightarrow{g} Z.$$

Then $gh(v_1 \times 1)$ is a homotopy $gjgv_1 = gv_1 \overset{\simeq}{\longrightarrow} gv_1$. Moreover, the homotopy on the boundary is $gh(u \times 1)$. It follows that there is a commutative diagram

$$\begin{tikzcd}
(\partial\Delta^n \times \Delta^2) \cup (\Delta^n \times \Lambda^2_2) \arrow[rr, "{(s_0(gh(u \times 1)), (gh_1, gh(v_1 \times 1),\ ))}"] \arrow[d, hook] & & Z \arrow[d, "q"] \\
\Delta^n \times \Delta^2 \arrow[r, "pr_L"'] \arrow[urr, "\xi"] & \Delta^n \arrow[r, "qv"'] & Y.
\end{tikzcd}$$

Then the diagram

$$\begin{tikzcd}
\partial\Delta^n \times \Delta^1 \arrow[r, "pr_L"] \arrow[d, hook, "i \times 1"'] & \partial\Delta^n \arrow[d, "gu"] \\
\Delta^n \times \Delta^1 \arrow[r, "\xi(1 \times d^2)"] \arrow[d, "pr_L"'] & Z \arrow[d, "q"] \\
\Delta^n \arrow[r, "qv"'] & Y
\end{tikzcd}$$

commutes, and so $gv_1 = gjv = v$ by the minimality of q. ■

PROOF OF LEMMA 10.12: Suppose that $f : X \to Y$ has the right lifting property with respect to all $\partial\Delta^n \subset \Delta^n$, $n \geq 0$, and hence with respect to all inclusions of simplicial sets. Then there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "1_X"] \arrow[d, "(1_X, f)"'] & X \arrow[d, "f"] \\ X \times Y \arrow[ur, "r"] \arrow[r, "pr_R"'] & Y, \end{tikzcd}$$

and so f is a retract of the projection $pr : X \times Y \to Y$. But then $|f|$ is a Serre fibration. ■

This also completes the proof of Theorem 10.10.

11. The closed model structure.

The results stated and proved in this section are the culmination of all of the hard work that we have done up to this point. We shall prove here that the entire simplicial set category **S** (not just the subcategory of fibrant objects) has a closed model structure, and that the resulting homotopy theory is equivalent to the ordinary homotopy theory of topological spaces. These are the central organizational theorems of simplicial homotopy theory.

PROPOSITION 11.1. *Suppose that X is a Kan complex. Then the canonical map $\eta_X : X \to S|X|$ is a weak equivalence in the sense that it induces an isomorphism in all possible simplicial homotopy groups.*

PROOF: Recall that $S|X|$ is also a Kan complex.

η_X induces an isomorphism in π_0: every map $v : |\Delta^0| \to |X|$ factors through the realization of a simplex $|\sigma| : |\Delta^n| \to |X|$ and so $S|X|$ is connected if $\pi_0 X = *$. On the other hand X is a disjoint union of its path components and $S|\ |$ preserves disjoint unions, so that $\pi_0 X \to \pi_0 S|X|$ is monic.

Suppose that we have shown that η_X induces an isomorphism

$$(\eta_X)_* : \pi_i(X, x) \xrightarrow{\cong} \pi_i(S|X|, \eta x)$$

for all choice of base points $x \in X$ and $i \leq n$. Then, using 10.10 for the path-loop fibration $\Omega X \to PX \to X$ determined by x (see the discussion following the proof of 7.3), one finds a commutative diagram

$$\begin{tikzcd} \pi_{n+1}(X,x) \arrow[r, "\eta_X"] \arrow[d, "\partial"', "\cong"] & \pi_{n+1}(S|X|,\eta x) \arrow[d, "\partial"] \\ \pi_n(\Omega X, x) \arrow[r, "\cong", "\eta_{\Omega X}"'] & \pi_n(S|\Omega X|, \eta x), \end{tikzcd}$$

and so we're done if we can show that PX and hence $S|PX|$ contracts onto its base point. But there is a diagram

$$\begin{tikzcd} (\Delta^0 \times \Delta^1) \cup (PX \times \partial\Delta^1) \arrow[r, "{(x,(1_{PX},x))}"] \arrow[d, hook] & PX \arrow[d, "t"] \\ PX \times \Delta^1 \arrow[r] \arrow[ur, "h"] & \Delta^0, \end{tikzcd}$$

and h exists because t has the right lifting property with respect to all $\partial\Delta^n \subset \Delta^n$, $n \geq 0$. ■

If X is a Kan complex and x is any vertex of X, then it follows from Proposition 11.1 and adjointness that η_X induces a canonical isomorphism

$$\pi_n(X,x) \cong \pi_n(|X|,x), \quad n \geq 1,$$

where the group on the right is the ordinary homotopy group of the space $|X|$. It follows that a map $f : X \to Y$ of Kan complexes is a (simplicial) weak equivalence if and only if the induced map $|f| : |X| \to |Y|$ is a topological weak equivalence. Thus, we are entitled to define a map $f : X \to Y$ of arbitrary simplicial sets to be a *weak equivalence* if the induced map $|f| : |X| \to |Y|$ is a weak equivalence of spaces. Our last major technical result leading to the closed model structure of $\mathbf{S}$ is

THEOREM 11.2. *Suppose that $g : X \to Y$ is a map between arbitrary simplicial sets. Then g is a Kan fibration and a weak equivalence if and only if g has the right lifting property with respect to all inclusions $\partial\Delta^n \subset \Delta^n$, $n \geq 0$.*

PROOF: Suppose that $g : X \to Y$ is a Kan fibration with the advertised lifting property. We have to show that $S|g| : S|X| \to S|Y|$ is a weak equivalence. Y is an arbitrary simplicial set, so we must define $\pi_0 Y$ to be the set of equivalence classes of vertices of Y for the relation generated by the vertex homotopy relation. In other words, $y \simeq z$ if and only if there is a string of vertices

$$y = y_0, y_1, \dots, y_n = z$$

and a string of 1-simplices

$$v_1, \dots, v_n$$

of X such that $\partial v_i = (y_{i-1}, y_i)$ or $\partial v_i = (y_i, y_{i-1})$ for $i = 1, \dots, n$. If Y is a Kan complex, then this definition of $\pi_0 Y$ coincides with the old definition. Moreover, the canonical map $\eta_Y : Y \to S|Y|$ induces an isomorphism $\pi_0 Y \xrightarrow{\cong} \pi_0 S|Y|$ for all simplicial sets Y. The lifting property implies that $g_* : \pi_0 X \to \pi_0 Y$ is an isomorphism, so that the induced map $\pi_0 S|X| \to \pi_0 S|Y|$ is an isomorphism as well. Finally, it suffices to show that the induced maps $\pi_i(S|X|, x) \to \pi_i(S|Y|, gx)$ of simplicial homotopy groups are isomorphisms for all vertices x of X and all $i \geq 1$. But Theorem 10.9 implies that the fibre of the fibration $S|g| : S|X| \to S|Y|$ over $g(x)$ is $S|F_x|$, where

$$\begin{tikzcd} F_x \arrow[r] \arrow[d] & X \arrow[d, "g"] \\ \Delta^0 \arrow[r, "gx"'] & Y \end{tikzcd}$$

is a pullback in the simplicial set category. F_x is a contractible Kan complex (see the corresponding argument for PX in 11.1), and so $S|F_x|$ is contractible as well. The result then follows from a long exact sequence argument.

For the reverse implication, it suffices (see the proof of Theorem 10.10) to assume that $g : X \to Y$ is a minimal fibration and a weak equivalence and then prove that it has the lifting property. We may also assume that Y is connected. Consider a diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "\alpha"] \arrow[d, hook] & X \arrow[d, "g"] \\ \Delta^n \arrow[r, "\beta"'] & Y \end{tikzcd}$$

and the induced diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "\alpha_*"] \arrow[d, hook] & \Delta^n \times_Y X \arrow[d] \\ \Delta^n \arrow[r, "1"'] & \Delta^n. \end{tikzcd}$$

It suffices to find a lifting for this last case. But there is a fibrewise isomorphism

$$\begin{tikzcd} \Delta^n \times_Y X \arrow[rr, "\cong"] \arrow[dr] & & \Delta^n \times F_y \arrow[dl, "pr_L"] \\ & \Delta^n & \end{tikzcd}$$

by 10.8, where F_y is the fibre over some vertex y of Y. Thus, it suffices to find a lifting of the following sort:

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, hook] & F_y. \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

But this can be done, since F_y is a Kan complex such that $\pi_0(S|F_y|)$ is trivial, and $\pi_i(S|F_y|, *) = 0$, $i \geq 1$ for any base point $*$, and $\eta : F_y \to S|F_y|$ is a weak equivalence by Proposition 11.1. ■

A *cofibration* of simplicial sets is an inclusion map.

THEOREM 11.3. *The simplicial set category* **S**, *together with the specified classes of Kan fibrations, cofibrations and weak equivalences, is a closed model category.*

PROOF: **CM1** is satisfied, since **S** is complete and cocomplete. **CM2** follows from **CM2** for **CGHaus**. **CM3** (the retract axiom) is trivial.

To prove the factorization axiom **CM5**, observe that a small object argument and the previous theorem together imply that any simplicial set map $f : X \to Y$ may be factored as:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\ & Z \arrow[ur, "p"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "j"'] & & Y, \\ & W \arrow[ur, "q"'] & \end{tikzcd}$$

where i is anodyne, p is a fibration, j is an inclusion, and q is a trivial fibration. The class of inclusions $i : U \to V$ of simplicial sets such that $|i| : |U| \to |V|$ is a trivial cofibration is saturated, by adjointness, and includes all $\Lambda^n_k \subset \Delta^n$. Thus all anodyne extensions are trivial cofibrations of $\mathbf{S}$. To prove **CM4** we must show that the lifting (dotted arrow) exists in any commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ V \arrow[r] \arrow[ur, dashed] & Y, \end{tikzcd}$$

where p is a fibration and i is a cofibration, and either i or p is trivial. The case where p is trivial is the previous theorem. On the other hand, if i is a weak equivalence, then there is a diagram

$$\begin{tikzcd} U \arrow[r, "j"] \arrow[d, "i"'] & Z \arrow[d, "p"] \\ V \arrow[r, "1_V"'] \arrow[ur, "s"] & V \end{tikzcd}$$

where j is anodyne and p is a (necessarily) trivial fibration, so that s exists. But then i is a retract of an anodyne extension, so i has the left lifting property with respect to all fibrations (compare the proof of 9.4). ∎

The homotopy category $\mathrm{Ho}(\mathbf{S})$ is obtained from $\mathbf{S}$ by formally inverting the weak equivalences. There are several ways to do this [75], [33], [15]. One may also form the category $\mathrm{Ho}(Top)$ by formally inverting the weak homotopy equivalences; this category is equivalent to the category of CW-complexes and ordinary homotopy classes of maps. For the same reason (see [75]), $\mathrm{Ho}(\mathbf{S})$ is equivalent to the category of Kan complexes and simplicial homotopy classes of maps. The realization functor preserve weak equivalences, by definition. One may use Theorem 10.10 (see the argument in 11.1) to show that the canonical map $\epsilon : |S(Y)| \to Y$ is a weak equivalence, for any topological space Y, and so the singular functor preserves weak equivalences as well. It follows that the realization and singular functors induce functors

$$\mathrm{Ho}(\mathbf{S}) \underset{S_*}{\overset{|\ |_*}{\rightleftarrows}} \mathrm{Ho}(\mathbf{Top})$$

of the associated homotopy categories.

THEOREM 11.4. *The realization and singular functors induce an equivalence of categories of* $\mathrm{Ho}(\mathbf{S})$ *with* $\mathrm{Ho}(\mathbf{Top})$.

PROOF: We have just seen that $\epsilon : |S(Y)| \to Y$ is a weak equivalence for all topological spaces Y. It remains to show that $\eta : X \to S|X|$ is a weak equivalence for all simplicial sets X. But η is a weak equivalence if X is a Kan complex, by 11.1, and every simplicial set is weakly equivalent to a Kan complex by **CM5**. The composite functor $S|\ |$ preserves weak equivalences. ∎

The original proof of Theorem 11.3 appears in [75], modulo some fiddling with axioms (see [77]). Theorem 11.4 has been known in some form or other since the late 1950's (see [67], [33], [59]).

Although it may now seem like a moot point, the function complex trick of Proposition 5.2 was a key step in the proof of Theorem 11.3. We can now amplify the statement of Proposition 5.2 as follows:

PROPOSITION 11.5. *The category* $\mathbf{S}$ *of simplicial sets satisfies the simplicial model axiom*

SM7: *Suppose that* $i : U \to V$ *is a cofibration and* $p : X \to Y$ *is a fibration. Then the induced map*

$$\mathbf{Hom}(V, X) \xrightarrow{(i^*, p_*)} \mathbf{Hom}(U, X) \times_{\mathbf{Hom}(U,Y)} \mathbf{Hom}(V, X)$$

is a fibration, which is trivial if either i *or* p *is trivial.*

PROOF: Use Proposition 5.2 and Theorem 11.2. ∎