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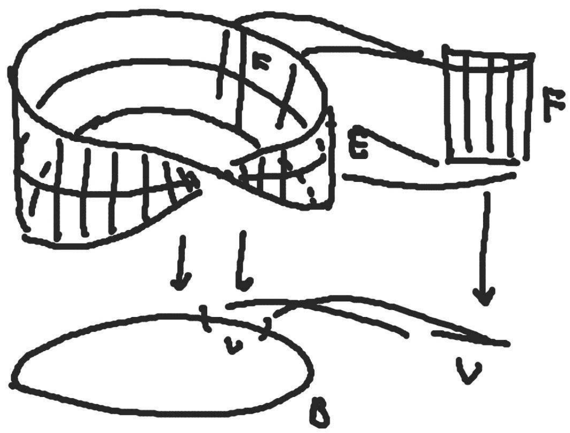
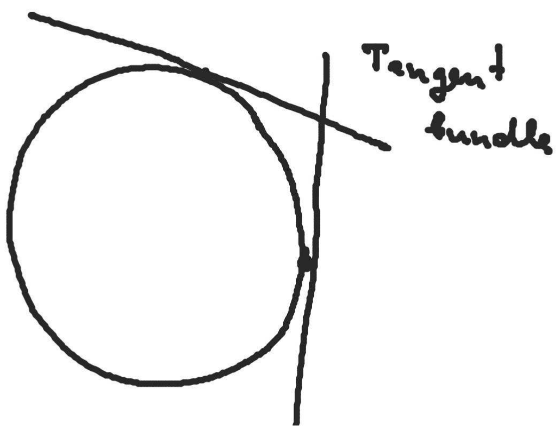
Combinatorial Fiber bundles

The question : How to encode fiber bundles by pure combinatorial data?

"Good spaces" $\leftrightsquigarrow$ simplicial complexes
"Manifolds" $\leftrightsquigarrow$ Combinatorial Manifolds (or "Brauer manifolds" etc...)
"Fiber Bundles" $\leftrightsquigarrow$???

Fiber Bundle $F \to E \to B$ with fiber F over the base space B is a map $E \to B$ which looks locally, in a neighborhood V of any point of B, like a trivial projection $F \times V \xrightarrow{\pi_2} V$.





The definition of fiber bundle is not constructive. Any map of good spaces can be triangulated and represented combinatorially by a map of simplicial complexes, but how to ensure the local triviality condition for simplicial maps – no classical good answer. This is a wild problem. The problem was discussed many times in geometric topology – K-theory, characteristic classes... (Whitney, Whitehead, Cohen, Rourke, Hatcher, MacPherson etc.)

Relatively simple encoding of fiber bundles with fiber – compact manifold F and base – compact polyhedron B

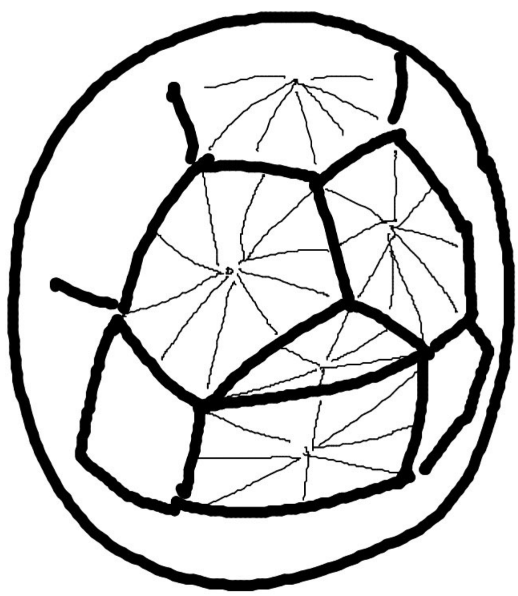
1. *Ball complex structures on manifold F.*

Ball complex structure on F is a covering of F $F = \bigcup_i D_i$ by embeddeded *closed* balls of different dimensions such that

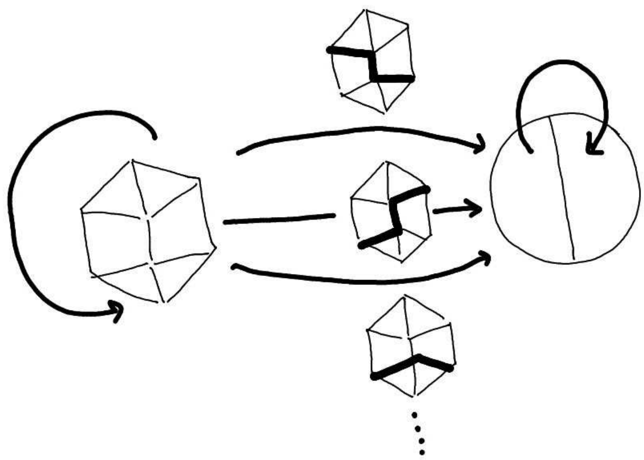
• relint$D_i \cap$ relint$D_j = \emptyset$ if $i \neq j$

• the boundary of every ball D_i is a union of balls of smaller dimension.

The poset of balls of a ball complex defines manifold F. The order complex of this poset is just a combinatorial manifold – a "baricentric subdivison" of the complex.



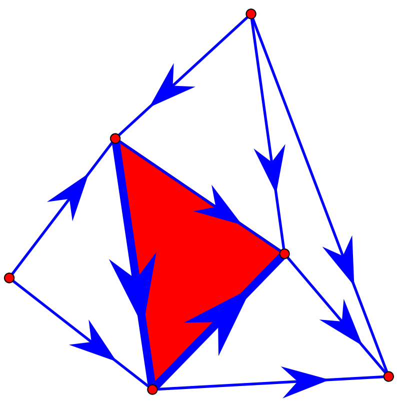
Let $\mathcal{D}^1 = \{D_i^1\}$ and $\mathcal{D}^2 = \{D_j^2\}$ be two ball complex structures on F. Define **"combinatorial assembly morphism"** as a map of *sets* of balls $\mathcal{D}^1 \xrightarrow{\xi} \mathcal{D}^2$ such that $\exists$ a homeomorphism $F \xrightarrow{g} F$: $\forall i$: $g(D_i^1) \subseteq \xi(D_i^1)$



The composition of combinatorial assemblies is a combinatorial assembly. So for the manifold F we got a category $\mathbf{Assembly}(F)$ with objects – ball complexes on F and morphisms – combinatorial assemblies.

We claim that fiber bundles with the base – polyhedron B and fiber – compact manifold F are encoded by the "colorings" of B using $\mathrm{Assembly}(F)$

The coloring of B by $\mathbf{Assembly}(F)$ is a triangulation T of B, $|T| = B$ + assigning for every vertex of T a ball complex from $\mathrm{Ob}\mathbf{Assembly}(F)$ and for any edge of T a combinatorial assembly from $\mathrm{Mor}\mathbf{Assembly}(X)$ in such a way that the diagram of assemblies coming from 2-skeleton of T is commutative.



$\mathbf{Assembly}(F)$ - coloring of B is something like a singular combinatorial connection of a bundle which is known only at the vertices of a triangulation of B

Theorems:

A) Any $\mathbf{Assembly}(F)$ - coloring of B defines canonically triangulated fiber bundle on B with a fiber F. (L. Anderson, N.M. ....)

B) Any fiber bundle on B with a fiber F comes (up to isomorphism) from some $\mathbf{Assembly}(F)$ coloring of B.

C) Two $\mathbf{Assembly}(F)$-colorings of B defines isomorphic bundles iff they are concordant.

The compact version of A) B) C) modulo geometric topology nonsense:

Theorem (N.M., 2007 http://arxiv.org/abs/0708.4039)

$B\mathbf{Assembly}(F) \approx BPL(F)$.

The construction of the fiber bundle by $\mathrm{Assembly}(F)$ coloring uses a simple construction well known in simplicial topology by the names *iterated mapping cone*, *homotopy colimit*, *Grothendieck construction*, *double bar-construction*. Consider a poset P considered as a category and a functor $P \xrightarrow{\mathcal{A}} \mathbf{Posets}$ to the category of *all* posets and poset maps. Then form a new poset

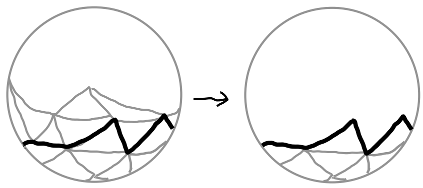
hocolim $\mathcal{A} = \{(p,x)|p \in P, x \in \mathcal{A}(p)\}$, $(p,x) \leq (q,y)$ iff $p \underset{P}{\leq} q$ and $\mathcal{A}(p \leq q)(x) \underset{\mathcal{A}(q)}{\leq} y$. Projection on the fisrt component gives a canonical poset map hocolim $\mathcal{A} \xrightarrow{\pi_1} P$. Passing to simplicial order complexes $\Delta()$ of the posets we will get a simplicial map $\Delta(\text{hocolim } \mathcal{A}) \xrightarrow{\Delta(\pi_1)} \Delta(P)$

The $\mathbf{Assembly}(F)$coloring of some triangulation T of the polyhedron B induces a linear order on the vertices of any k-simplex σ of T and a functor $[k] \xrightarrow{\mathcal{A}_\sigma} \mathbf{Posets}$ sending vertices to posets of balls of corresponding ball complex on F and edges to assembly morphisms as a poset morphisms. Passing to simplicial maps $\Delta(\text{hocolim } \mathcal{A}_\sigma) \xrightarrow{\Delta(\pi_1)} \Delta([k]) = \sigma$ we got a simplicial map for every simplex, which are neaturally pasted together into canonical simplicial map $E \xrightarrow{T}$ constructed by coloring. The fact that this map is a fiber bundle with fiber F is just an intensive application of Alexander trick.

R^n - **bundles**

There is a problem with the most classical bundles – real vector bundles since R^n is non-compact an has no finite ball complex structure. But one can easily compactify R^n by a point at infinity. There is a theory by that states than one can correctly fiber-wise compactify entire R^n fiber bundles. So that the isomorphism classes of R^n fiber bundles are in one-to one correspondence with S^n fiber bundles having one distinguished section (∞-section) or having two everywhere different distinguished sections (∞ and 0-sections).

Consider the category $\mathbf{Assembly}_n$ with objects – ball complexes on S^n with distinguished n-dimensional ball and morphism – combinatorial assemblies sending the distinguished ball to the distinguished ball.

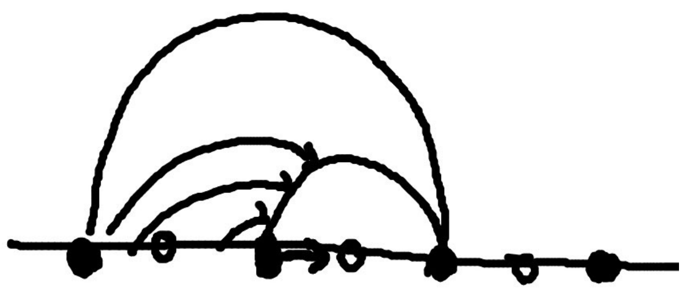


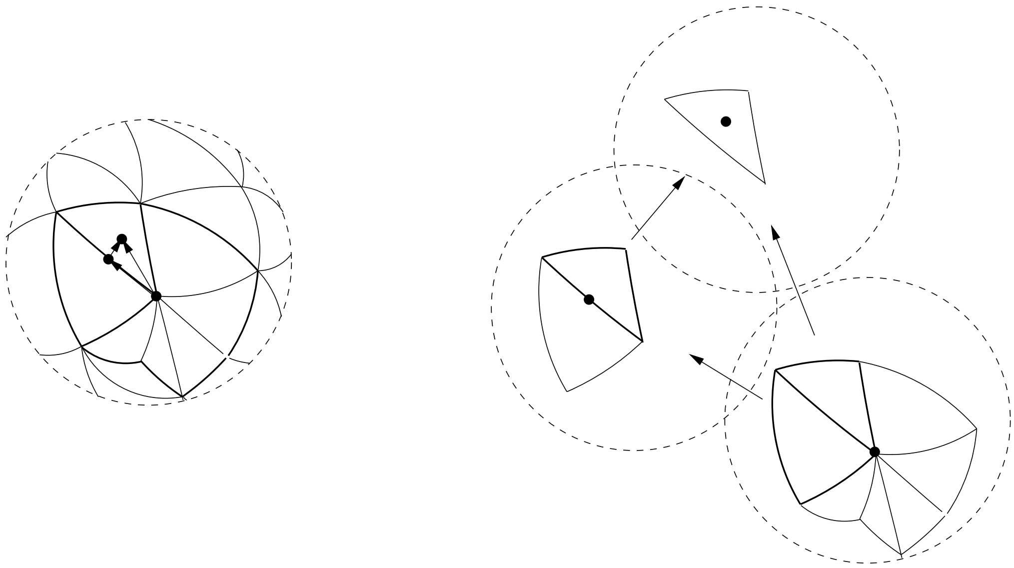
Theorems:

A) Any $\mathbf{Assembly}_n$ - coloring of polyhedron B defines canonically triangulated fiber S^n-bundle on B with distinguished section.
B) Any vector bundle on B with a fiber R^n comes (up to isomorphism) from some $\mathbf{Assembly}_n$-coloring of B.
C) Two $\mathbf{Assembly}_n$-colorings of B defines isomorphic bundles iff they are concordant.
The compact version of A) B) C) modulo geometric topology nonsense: $B\mathbf{Assembly}_n \approx BPL_n$. Mention that for $n = 1, 2, 3, 4$ $BPL_n \approx BO(n)$.

tangent bundle of a combinatorial manifold

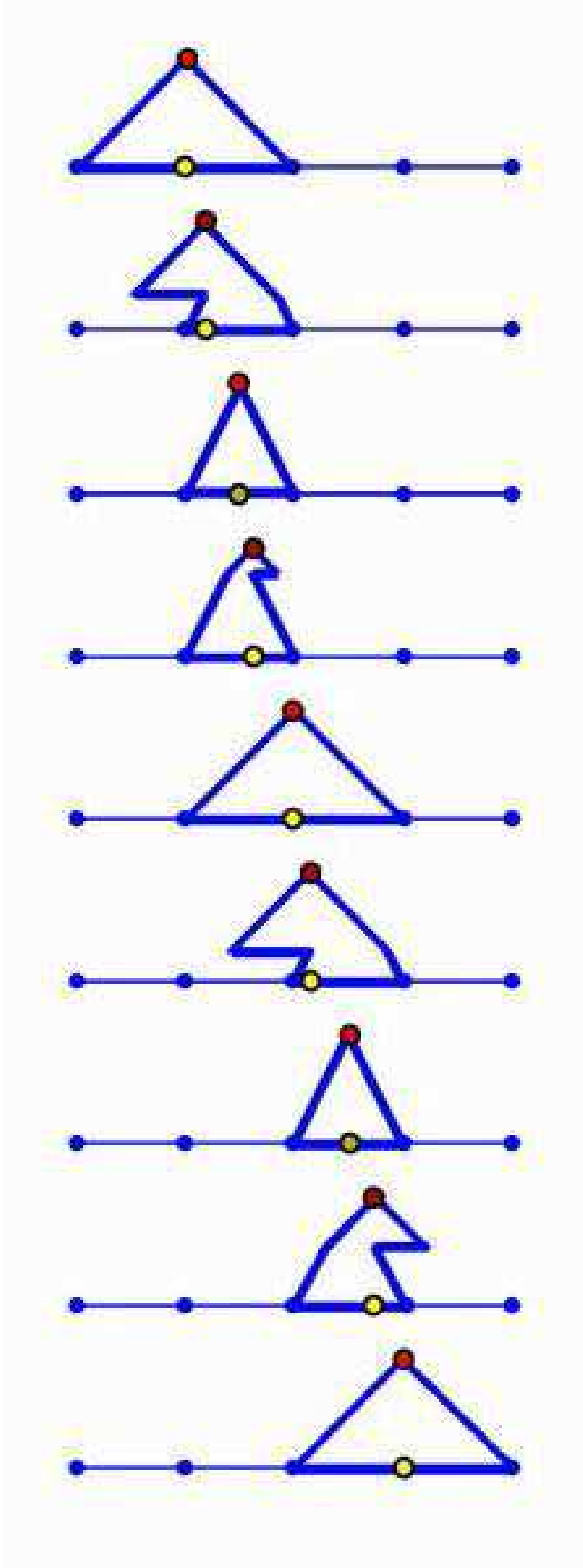
Consider n-dimensional combinatorial manifold M let M' be a first barycentric subdivision of M. Then we can canonically associate a $\mathbf{Assembly}_n$ coloring $\mathbf{Assembly}_n$ of M'. Vertices of M' are numbered by simplices of M. We associate to $v_\sigma \in M'$ the star of σ in M with attached by bounding $n-1$ sphere a new distingushed n-ball. When we walk by edges of M' – the natural assembly happiens.

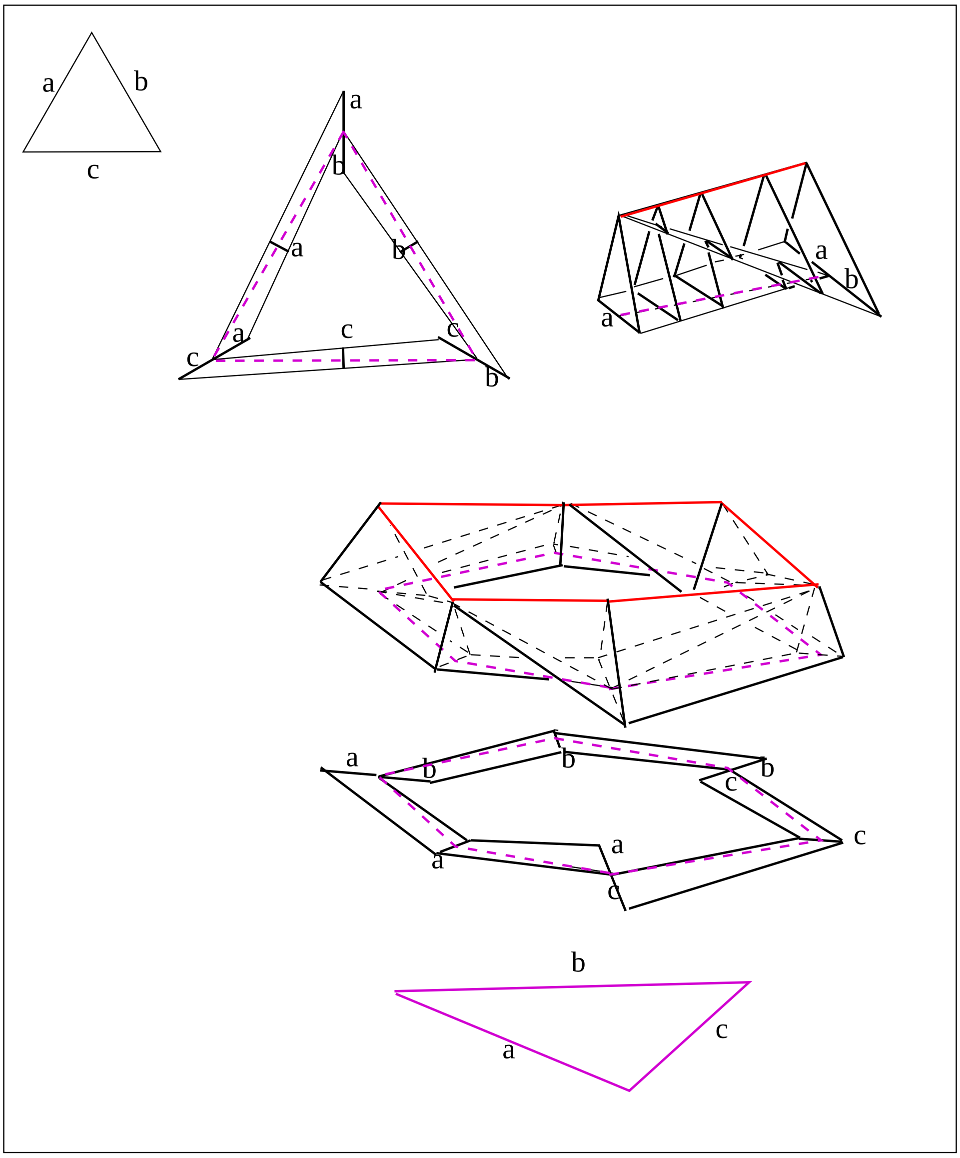




Theorem

The $\mathbf{Assembly}_n$- coloring $\mathbf{Assembly}_n$ of M' defines canonically triangulated Kuiper-Lashoff compactification of tangent bundle on M. The total space of this bundle is a combinatorial manifold.

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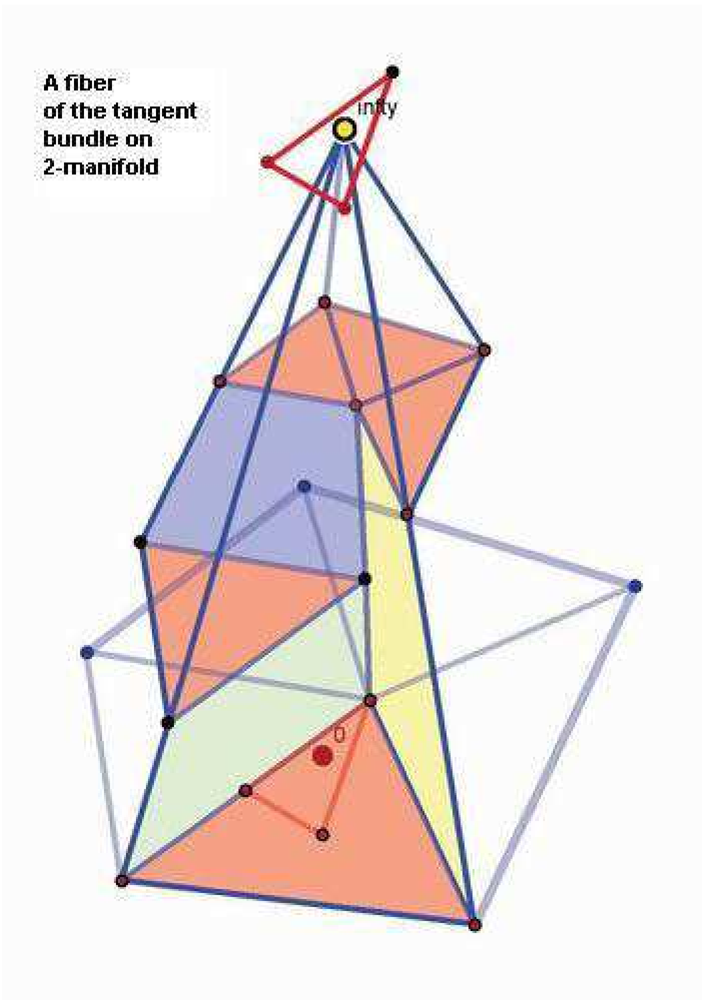
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