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Combinatorial Fiber bundles

The question: How to encode fiber bundles by pure combinatorial data?

"Good spaces" $\leftrightarrow$ simplicial complexes
"Manifolds" $\leftrightarrow$ Combinatorial Manifolds (or "Brauer manifolds" etc...)
"Fiber Bundles" $\leftrightarrow$ ???
Fiber Bundle $F \to E \to B$ with fiber $F$ over the base space $B$ is a map $E \to B$ which looks locally, in a neighborhood $V$ of any point of $B$, like a trivial projection $F \times V \overset{\pi_2}{\longrightarrow} V$. 
The definition of fiber bundle is not constructive. Any map of good spaces can be triangulated and represented combinatorially by a map of simplicial complexes, but how to ensure the local triviality condition for simplicial maps – no classical good answer. This is a wild problem. The problem was discussed many times in geometric topology – K-theory, characteristic classes... (Whitney, Whitehead, Cohen, Rourke, Hatcher, MacPherson etc. )
Relatively simple encoding of fiber bundles with fiber – compact manifold $F$ and base – compact polyhedron $B$

1. *Ball complex structures on manifold $F$.* Ball complex structure on $F$ is a covering of $F = \bigcup_i D_i$ by embedded closed balls of different dimensions such that
   - $\text{relint} D_i \cap \text{relint} D_j = \emptyset$ if $i \neq j$
   - the boundary of every ball $D_i$ is a union of balls of smaller dimension.
The poset of balls of a ball complex defines manifold \( F \). The order complex of this poset is just a combinatorial manifold – a "baricentric subdivision" of the complex.
Let $\mathcal{D}^1 = \{D^1_i\}$ and $\mathcal{D}^2 = \{D^2_j\}$ be two ball complex structures on $F$. Define "combinatorial assembly morphism" as a map of sets of balls $\mathcal{D}^1 \xrightarrow{\xi} \mathcal{D}^2$ such that $\exists$ a homeomorphism $F \xrightarrow{g} F : \forall i : g(D^1_i) \subseteq \xi(D^1_i)$

The composition of combinatorial assemblies is a combinatorial assembly. So for the manifold $F$ we got a category $\text{Assembly}(F)$ with objects – ball complexes on $F$ and morphisms – combinatorial assemblies.
We claim that fiber bundles with the base – polyhedron $B$ and fiber – compact manifold $F$ are encoded by the "colorings" of $B$ using $\text{Assembly}(F)$.

The coloring of $B$ by $\text{Assembly}(F)$ is a triangulation $T$ of $B$, $|T| = B +$ assigning for every vertex of $T$ a ball complex from $\text{ObAssembly}(F)$ and for any edge of $T$ a combinatorial assembly from $\text{MorAssembly}(X)$ in such a way that the diagram of assemblies coming from 2-skeleton of $T$ is commutative.
Assembly\((F)\) - coloring of \(B\) is something like a singular combinatorial connection of a bundle which is known only at the vertices of a triangulation of \(B\)

**Theorems:**

A) Any Assembly\((F)\) - coloring of \(B\) defines canonically triangulated fiber bundle on \(B\) with a fiber \(F\). (L. Anderson, N.M. ....)

B) Any fiber bundle on \(B\) with a fiber \(F\) comes (up to isomorphism) from some Assembly\((F)\)-coloring of \(B\).

C) Two Assembly\((F)\)-colorings of \(B\) defines isomorphic bundles iff they are concordant.

The compact version of A) B) C) modulo geometric topology nonsense:

**Theorem (N.M., 2007)**

http://arxiv.org/abs/0708.4039

\[ BA_{\text{Assembly}}(F) \approx BPL(F). \]
The construction of the fiber bundle by Assembly\((F)\) coloring uses a simple construction well known in simplicial topology by the names \textit{iterated mapping cone}, \textit{homotopy colimit}, \textit{Grothendieck construction}, \textit{double bar-construction}. Consider a poset \(P\) considered as a category and a functor \(P \xrightarrow{A} \text{Posets}\) to the category of all posets and poset maps. Then form a new poset
\[
hocolim A = \{(p, x) | p \in P, x \in A(p)\}, (p, x) \leq (q, y) \text{ iff } p \leq q \text{ and } A(p \leq q)(x) \leq A(q) y.
\]
Projection on the first component gives a canonical poset map \(hocolim A \xrightarrow{\pi_1} P\).
Passing to simplicial order complexes \(\Delta()\) of the posets we will get a simplicial map
\[
\Delta(hocolim A) \xrightarrow{\Delta(\pi_1)} \Delta(P)
\]
The Assembly\((F)\)coloring of some triangulation \(T\) of the polyhedron \(B\) induces a linear order on the vertices of any \(k\)-simplex \(\sigma\) of \(T\) and a functor \([k] \xrightarrow{\mathcal{A}_\sigma} \text{Posets}\) sending vertices to posets of balls of corresponding ball complex on \(F\) and edges to assembly morphisms as a poset morphisms. Passing to simplicial maps \(\Delta(\text{hocolim} \mathcal{A}_\sigma) \xrightarrow{\Delta(\pi_1)} \Delta([k]) = \sigma\) we got a simplicial map for every simplex, which are naturally pasted together into canonical simplicial map \(E \xrightarrow{T} \text{constructed by coloring}\). The fact that this map is a fiber bundle with fiber \(F\) is just an intensive application of Alexander trick.
$\mathbb{R}^n$ - bundles

There is a problem with the most classical bundles – real vector bundles since $\mathbb{R}^n$ is non-compact and has no finite ball complex structure. But one can easily compactify $\mathbb{R}^n$ by a point at infinity. There is a theory by that states than one can correctly fiber-wise compactify entire $\mathbb{R}^n$ fiber bundles. So that the isomorphism classes of $\mathbb{R}^n$ fiber bundles are in one-to one correspondence with $S^n$ fiber bundles having one distinguished section ($\infty$-section) or having two everywhere different distinguished sections ($\infty$ and 0-sections).
Consider the category \textbf{Assembly}_n with objects – ball complexes on $S^n$ with distinguished n-dimensional ball and morphism – combinatorial assemblies sending the distinguished ball to the distinguished ball.

\begin{center}
\includegraphics[width=0.2\textwidth]{diagram.png}
\end{center}

**Theorems:**

A) Any \textbf{Assembly}_n - coloring of polyhedron $B$ defines canonically triangulated fiber $S^n$-bundle on $B$ with distinguished section.

B) Any vector bundle on $B$ with a fiber $R^n$ comes (up to isomorphism) from some \textbf{Assembly}_n-coloring of $B$.

C) Two \textbf{Assembly}_n-colorings of $B$ defines isomorphic bundles iff they are concordant.

The compact version of A) B) C) modulo geometric topology nonsense: $B\textbf{Assembly}_n \approx BPL_n$. Mention that for $n = 1, 2, 3, 4$ $BPL_n \approx BO(n)$.
tangent bundle of a combinatorial manifold

Consider $n$-dimensional combinatorial manifold $M$ let $M'$ be a first barycentric subdivision of $M$. Then we can canonically associate a $\text{Assembly}_n$ coloring $\text{Assembly}_n$ of $M'$. Vertices of $M'$ are numbered by simplices of $M$. We associate to $v_{\sigma} \in M'$ the star of $\sigma$ in $M$ with attached by bounding $n - 1$ sphere a new distinguished $n$-ball. When we walk by edges of $M'$ – the natural assembly happens.
Theorem

The $\text{Assembly}_n$- coloring $\text{Assembly}_n$ of $M'$ defines canonically triangulated Kuiper-Lashoff compactification of tangent bundle on $M$. The total space of this bundle is a combinatorial manifold.
A fiber of the tangent bundle on a 2-manifold